A NOTE ON BLECHER’S CHARACTERIZATION OF HILBERT C*-MODULES

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Abstract. By a theorem of D. P. Blecher, a Hilbert C*-module \( V \) over a C*-algebra \( \mathcal{A} \) (faithfully and nondegenerately represented on a Hilbert space \( \mathcal{H} \)) is characterized by a certain Hilbert space \( \mathcal{H}_V \), such that \( V \) can be embedded in the algebra \( B(\mathcal{H}, \mathcal{H}_V) \) of bounded operators between \( \mathcal{H} \) and \( \mathcal{H}_V \). In this paper it is shown: 1. For a Hilbert C*-module over a C*-algebra of compact operators the Hilbert space \( \mathcal{H}_V \) coincides with a Hilbert subspace of the module, which characterizes all adjointable operators on the module. 2. For any Hilbert C*-module \( V \), its strict completion can be realized in the algebra \( B(\mathcal{H}, \mathcal{H}_V) \).

1. Introduction

The aim of this paper is to show two uses of the Hilbert space \( \mathcal{H}_V \) from the following characterization of Hilbert C*-modules:

**Theorem 1.1** (D. P. Blecher, [3]). Let \( \mathcal{A} \subseteq B(\mathcal{H}) \) be a nondegenerate C*-algebra and let \( V \) be a right Banach \( \mathcal{A} \)-module (and an operator space). \( V \) is a Hilbert \( \mathcal{A} \)-module (with the same norm-, resp. operator space structure) if and only if the following three conditions are satisfied:

(i) \( \mathcal{H}_V = V \otimes_{\mathcal{H}} \mathcal{H} \) is a (column) Hilbert space;
(ii) \( \phi : V \rightarrow B(\mathcal{H}, \mathcal{H}_V) \), defined by \( \phi(x)(\xi) = x \otimes \xi \), is a (complete) isometry;
(iii) \( \phi(x)^* \phi(x) \in \mathcal{A} \) for all \( x \in V \).

If this is the case, the (unique\(^1\)) inner product on \( V \) turning it into a Hilbert \( \mathcal{A} \)-module is given by

\[ \langle x | y \rangle = \phi(x)^* \phi(y). \]

\(^1\)E.C. Lance has shown in [7] that there is a 1-1 correspondence between norm and inner product of Hilbert \( \mathcal{A} \)-modules.
In this paper it is shown that Blecher’s Hilbert space $\mathcal{H}_V$ for Hilbert $C^*$-modules over $C^*$-algebras of compact operators coincides with a Hilbert subspace $V_\mathcal{H}$ of such a module $V$, known for several good properties. Further, it is shown that the $V$-strict completion $\mathcal{M}(V)$ of $V$, which is a generalization of the notion of a multiplier algebra from $C^*$-algebra theory to Hilbert $C^*$-modules, can be realized in $B(\mathcal{H}, \mathcal{H}_V)$ for any Hilbert $C^*$-module $V$.

Let us first shortly explain the objects in Blecher’s theorem (for more details, see [3]). The Hilbert space $V \otimes_A \mathcal{H}^c$ is the module tensor product $V \otimes_A \mathcal{H}^c$ of $V$ and the Hilbert column space $\mathcal{H}^c$, treated as a left $A$-module, completed with respect to the Haagerup norm. Elementary tensors in $V \otimes_A \mathcal{H}^c$ are denoted by $x \otimes_A \xi \in V \otimes \mathcal{H}^c$. $\mathcal{H}^c$ is isometric to the Hilbert space $\mathcal{H}$ (that’s why their elements are identified) equipped with an additional operator space structure, setting $\mathcal{H}^c = B(\mathcal{C}, \mathcal{H})$.

The inner product of the Hilbert space $\mathcal{H}_V$ from Blecher’s theorem is given on elementary tensors by

$$(x \otimes_A \xi \mid y \otimes_A \eta)_{\mathcal{H}_V} = (\langle x \mid y \rangle_{\mathcal{H}} \mid \xi \mid \eta)_{\mathcal{H}}.$$ 

and such that $V$ is complete with respect to the norm defined by $\| x \| = \sqrt{\langle x \mid x \rangle_{\mathcal{A}}}$. For example, any $C^*$-algebra $A$ is a Hilbert $A$-module, setting $\langle a \mid b \rangle = a^*b$.

Two classes of $(\mathcal{A})$-linear operators on $V$ shall be considered in this paper: the $C^*$-algebra $B_A(V)$ of adjointable (with respect to the Hilbert $C^*$-module inner product) maps and the $C^*$-algebra $K_A(V)$ of generalized compact operators (the norm-closure of the linear span of all operators $F_{x,y}$, $x,y \in V$, where $F_{x,y}(z) = x \langle y \mid z \rangle$). $K_A(V)$ is a closed two-sided ideal in $B_A(V)$.

For more details on Hilbert $C^*$-modules see [8].
Although Hilbert C*-modules are analogues of Hilbert spaces, many
Hilbert space properties cannot, in general, be transferred to Hilbert C*-modules, see e.g. [8] or [11]. A special class of Hilbert C*-modules for which most difficulties can be resolved is the class of Hilbert C-modules over a C*-algebra of (all) compact operators on a Hilbert space, see [1]. Every such module V contains a Hilbert subspace V_e which defines all adjointable operators on V, in the sense that the C*-algebras B_A(V) and B(V_e) are isomorphic, as are K_A(V) and K(V_e), via the restriction map T \mapsto T|_{V_e}. If A = \bigoplus_i K(H_i) is a general C*-algebra of compact operators and V a Hilbert A-module, then V can be decomposed as V = \bigoplus_i V_i, where V_i = V K(H_i). Then B_A(V) is isomorphic to \prod_i B((V_i)_{e_i}) and K_A(V) is isomorphic to \bigoplus_i K((V_i)_{e_i}), where e_i are minimal projections in K(H_i).

A known concept in the C*-algebra theory is the multiplier algebra M(A) of a C*-algebra A, which can be realized as a completion of A under a certain topology (called strict topology), for details see e.g. [11]. There is also a generalization of this concept for Hilbert C*-modules (for details, see [2]). For a Hilbert A-module V the V-strict topology is defined on any Hilbert B-module W which contains V in such a way that A is an essential ideal in B and V = W A (the set of all products of elements from W and A). A strict completion of a (full\(^3\)) Hilbert A-module V is such a module W which is V-strictly complete. It is proven in [2] that the strict completion of a Hilbert A-module V is the Hilbert M(A)-module B_A(A, V) (consisting of all adjointable maps from A to V).

Throughout this paper, A shall denote a C*-algebra, M(A) its multiplier algebra, \mathcal{H} a Hilbert space, V a Hilbert C*-module and M(V) its strict completion. If A is represented on \mathcal{H} (resp. if V \subseteq B(\mathcal{H}, \mathcal{H}_V)) \mathcal{A} \mathcal{H} (resp. \phi(V) \mathcal{H}) denotes the linear span of elements of the form a\xi, a \in A, \xi \in \mathcal{H} (resp. of the form \phi(x)(\xi) = x \otimes_k \xi, x \in V, \xi \in \mathcal{H}). The inner product of a Hilbert C*-module shall be denoted by < . | . > and of a Hilbert space by ( . | . ). B(\mathcal{H}) shall denote the C*-algebra of all bounded linear operators on \mathcal{H}. B_A(V) and K_A(V) shall denote the C*-algebra of adjointable resp. of generalized compact operators on V. \cong denotes isometric isomorphism.

2. THE CHARACTERISATION HILBERT SPACE FOR HILBERT C*-MODULES

OVER C*-ALGEBRAS OF COMPACT OPERATORS

Let V be a Hilbert K-module, where K is the C*-algebra of all compact operators on a fixed Hilbert space \mathcal{H}. Let V_e denote the subspace of V

V_e = \{ xe : x \in V \}

\(^3\) V is a full Hilbert A-module if the linear span of all products < x | y >, x, y \in V, is dense in A.
where \( e \) is a minimal projection in \( K \), i.e. \( e = F_{\xi,\xi} \) for some \( \xi \in \mathcal{H} \) of norm one\(^4\). \( V \) is a Hilbert space (not only submodule!) in the norm inherited from \( V \) and the Hilbert space inner product is given by

\[
(xe \mid ye) = \text{tr}(e < y \mid x >_V e).
\]

For details about \( V_e \), see [1].

**Proposition 2.1.** If \( V \) is a Hilbert \( K \)-module and if \( \mathcal{H}_V \) is the corresponding Hilbert space from Blecher’s theorem, then

\[
V_e \cong \mathcal{H}_V.
\]

**Proof.** Let \( e \) be the minimal projection \( F_{\xi,\xi} \) in \( K \). From Blecher’s theorem (ii) it is clear that the set \( \phi(V)\mathcal{H} \) is dense in \( \mathcal{H}_V \), so it is sufficient to define a linear map \( \psi : \mathcal{H}_V \to V_e \) on elements of the form \( (x) = x \otimes \mathbb{K} \). Set

\[
\psi(x \otimes \mathbb{K} \eta) = x F_{\eta,\xi}.
\]

\( \psi \) maps \( \mathcal{H}_V \) into \( V_e \) because \( F_{\eta,\xi} F_{\xi,\xi} = F_{\eta,\xi} \). \( \psi \) is obviously onto, since \( xe \in V_e \) is the image of \( x \otimes \mathbb{K} \xi \in \mathcal{H}_V \). Further (note that \( F_{\xi,\eta} T F_{\xi',\eta'} = (T \xi' \mid \eta) F_{\xi,\eta'} \) for all \( \xi, \xi', \eta, \eta' \in \mathcal{H} \) and \( T \in B(\mathcal{H}) \))

\[
(x F_{\eta,\xi} \mid x F_{\eta,\xi})_{V_e} = \text{tr}(e < x F_{\eta,\xi} \mid x F_{\eta,\xi} >_V e) = \text{tr}((< x F_{\eta,\xi} \mid x F_{\eta,\xi} >_V \xi \mid \xi \rangle_{\mathcal{H}_V}) = (\xi, \xi').
\]

By the expression for the inner product of elementary tensors in \( \mathcal{H}_V \), the last quantity is equal to

\[
(x F_{\eta,\xi} \otimes \mathbb{K} \eta \mid x F_{\eta,\xi} \otimes \mathbb{K} \xi)_{\mathcal{H}_V} = (x \otimes \mathbb{K} F_{\eta,\xi} \xi \mid x \otimes \mathbb{K} F_{\eta,\xi} \xi)_{\mathcal{H}_V} = (x \otimes \mathbb{K} \eta \mid x \otimes \mathbb{K} \eta)_{\mathcal{H}_V},
\]

so \( \psi \) extends to an isometric, surjective map \( \mathcal{H}_V \to V_e \).

**Corollary 2.2.** a) If \( V \) is a Hilbert \( K \)-module, then the \( C^* \)-algebra \( \mathcal{B}_K(V) \) is isomorphic to \( B(\mathcal{H}_V) \) and \( \mathbb{K}_K(V) \) is isomorphic to \( K(\mathcal{H}_V) \).

b) If \( V \) is a Hilbert \( C^* \)-module over a general \( C^* \)-algebra of compact operators \( A = \oplus_i \mathbb{K}(\mathcal{H}_i) \), let \( V_i = \mathbb{V} \mathbb{K}(\mathcal{H}_i) \). Then \( \mathcal{B}_A(V) \) is isomorphic to \( \prod_i B(\mathcal{H}_i) \) and \( \mathbb{K}_A(V) \) is isomorphic to \( \oplus_i \mathbb{K}(\mathcal{H}_i) \).

**Proof.** This is just the restatement of theorems 5 and 7 (resp. 6 and 9) from [1], using the result from the preceding proposition.

\(^4F_{\xi,\eta}\) denotes the operator \( \theta \mapsto (\theta \mid \eta)\xi \).
3. ON THE STRICT COMPLETION OF A HILBERT C*-MODULE

By Blecher’s theorem, a Hilbert \( \mathcal{A} \)-module \( V \) (with \( \mathcal{A} \subseteq \mathcal{B}(\mathcal{H}) \) nondegenerate) can be (completely) isometrically embedded in \( B(\mathcal{H}, \mathcal{H}_V) \), so it is natural to ask if the \( V \)-strict completion of \( V \) can also be realized inside \( B(\mathcal{H}, \mathcal{H}_V) \), i.e. if \( \mathcal{M}(V) \subseteq B(\mathcal{H}, \mathcal{H}_V) \). Equivalently, the question is if \( B(\mathcal{H}, \mathcal{H}_V) \) is \( V \)-strictly complete. Recall that the \( V \)-strict topology \([2]\) on \( B(\mathcal{H}, \mathcal{H}_V) \) (which is a Hilbert \( \mathcal{B}(\mathcal{H}) \)-module containing \( \phi(V) \) as a Hilbert \( \mathcal{A} \)-submodule) is defined by the family of seminorms \( t \mapsto \| t^* \phi(x) \|, \ x \in V \) and \( t \mapsto \| ta \|, \ a \in \mathcal{A} \).

We shall further identify \( V \), with its \( \phi \)-image in \( B(\mathcal{H}, \mathcal{H}_V) \).

**Proposition 3.1.** Let \( V \) be a Hilbert \( \mathcal{A} \)-module (with \( \mathcal{A} \subseteq \mathcal{B}(\mathcal{H}) \) nondegenerate) and \( \mathcal{H}_V \) the corresponding Hilbert space from Blecher’s theorem. Then \( B(\mathcal{H}, \mathcal{H}_V) \) is \( V \)-strictly complete.

**Proof.** Let \( (t_\lambda)_{\lambda \in \Lambda} \) be a \( V \)-strictly convergent net in \( B(\mathcal{H}, \mathcal{H}_V) \), i.e. the nets \( (t_\lambda a)_\lambda \) and \( (t_\lambda^* x)_\lambda \) converge in norm in \( B(\mathcal{H}, \mathcal{H}_V) \) resp. \( B(\mathcal{H}) \), for all \( a \in \mathcal{A} \) resp. \( x \in V \). Denote the (norm-)limits
\[
L(a) = \lim_\lambda t_\lambda a, \ a \in \mathcal{A},
\]
\[
R(x) = \lim_\lambda t_\lambda^* x, \ x \in V.
\]

If \( \mathcal{A} \) is an unital C*-algebra the net \( (t_\lambda)_{\lambda} \) trivially converges in norm to an operator \( t \in B(\mathcal{H}, \mathcal{H}_V) \), for if 1 is the unit element in \( \mathcal{A} \) then the net \( (t_\lambda)_{\lambda} = (t_\lambda 1)_\lambda \) converges by presumption to a \( t \in B(\mathcal{H}, \mathcal{H}_V) \) and this is the required \( V \)-strict limit of \( (t_\lambda)_{\lambda} \).

In the nonunital case, we proceed as follows:

It is easy to check that the above defined maps \( L : \mathcal{A} \to B(\mathcal{H}, \mathcal{H}_V) \) and \( R : V \to B(\mathcal{H}) \) are linear. Further, for all \( a \in \mathcal{A} \), \( x \in V \)
\[
L(a)^* x = a^* R(x) : \\
L(a)^* x = (\lim_\lambda t_\lambda a)^* x = (\lim_\lambda (t_\lambda a)^*) x = (\lim_\lambda a^* t_\lambda^*) x = \lim_\lambda a^* t_\lambda^* x = a^* R(x) \] (since the adjoint map \( * : B(\mathcal{H}, \mathcal{H}_V) \to B(\mathcal{H}_V, \mathcal{H}) \) is norm-continuous, as are right multiplication \( R_a : B(\mathcal{H}_V, \mathcal{H}) \to B(\mathcal{H}) \) by a fixed element \( x \in V \) and left multiplication \( L_a : B(\mathcal{H}) \to B(\mathcal{H}) \) by a fixed element \( a \in \mathcal{A} \)). Set
\[
t(a \xi) = L(a)(\xi)
\]
for \( a \in \mathcal{A}, \xi \in \mathcal{H} \) and
\[
t^*(x \xi) = R(x)(\xi)
\]
for \( x \in V, \xi \in \mathcal{H} \). Extending the above defined maps \( t \) and \( t^* \) linearly we get a map \( t \) defined on the dense subset \( \mathcal{A} \mathcal{H} \) of \( \mathcal{H} \) (with range in \( \mathcal{H}_V \)) and a map \( t^* \) defined on the dense subset \( V \mathcal{H} \) of \( \mathcal{H}_V \) (with range in \( \mathcal{H} \)). Both of these
maps are continuous on their domains, check this e.g. for $t$:

\[
\| t(\sum_{i=1}^{n} a_i \xi_i) \| = \| \lim_{\lambda} t_\lambda a_i \xi_i \| = \\
\| \sum_{i=1}^{n} \lim_{\lambda} t_\lambda a_i \xi_i \| = \| \lim_{\lambda} \sum_{i=1}^{n} a_i \xi_i \| \leq \\
\leq (\sup_{\lambda} \| t_\lambda \|) \| \sum_{i=1}^{n} a_i \xi_i \|.
\]

(If the net $(t_\lambda)$ is $V$-strictly convergent, then it is uniformly bounded by the uniform boundedness principle, because $A$ is nondegenerately represented on $\mathcal{H}$.)

Accordingly, $t$ and $t^*$ can be extended to bounded linear maps $t : \mathcal{H} \to \mathcal{H}_V$ and $t^* : \mathcal{H}_V \to \mathcal{H}$. These maps are mutually adjoint on the dense subsets $\mathcal{A}$ resp. $V\mathcal{H}$:

\[
(t(a\xi) | x\eta)_{\mathcal{H}_V} = (L(a)\xi | x\eta) = (\xi | L(a)^* x\eta) = (\xi | a^* R(x)\eta) = (a\xi | t^*(x\eta))_{\mathcal{H}},
\]

so $t$ and $t^*$ are mutually adjoint maps.

It is now easy to check that $t$ is the required $V$-strict limit of $(t_\lambda)_\lambda$, i.e. that $t_\lambda a \to t a$ for all $a \in A$ and $t_\lambda^* x \to t^* x$ for all $x \in V$. \hfill \Box

**Corollary 3.2.** For any Hilbert $C^*$-module $V \subseteq B(\mathcal{H}, \mathcal{H}_V)$

\[ V \subseteq \mathcal{M}(V) \subseteq B(\mathcal{H}, \mathcal{H}_V) \]

(isometrically).

Finally, note the following fact:

**Proposition 3.3.** If $V$ is a Hilbert $C^*$-module, then $\mathcal{H}_V \mathcal{H}_{\mathcal{M}(V)}$.

**Proof.** If $\mathcal{A} \subseteq B(\mathcal{H})$ then $\mathcal{M}(\mathcal{A}) \subseteq B(\mathcal{H})$. Let $\phi^\mathcal{M}$ be the map associated to the Hilbert $\mathcal{M}(\mathcal{A})$-module $\mathcal{M}(V)$ by Blecher’s theorem. $\phi(V)\mathcal{H}$ is dense in $\mathcal{H}_V$ and $\phi^\mathcal{M}\mathcal{M}(\{V\})\mathcal{H}$ is dense in $\mathcal{H}_{\mathcal{M}(V)}$. The map $\phi(x)\xi \mapsto \phi^\mathcal{M}(x)\xi$ ($x \in V \subseteq \mathcal{M}(V)$), $\xi \in \mathcal{H}$ is easily seen to be an isometry from $\phi(V)\mathcal{H}$ into $\phi^\mathcal{M}\mathcal{M}(\{V\})\mathcal{H}$, so it extends to an isometry from $\mathcal{H}_V$ into $\mathcal{H}_{\mathcal{M}(V)}$. \hfill \Box

**Remark 3.4.** Another use of the Hilbert space $\mathcal{H}_V$ is the possibility of representing $B_\mathcal{A}(V)$ (faithfully and nondegenerately) in $B(\mathcal{H}_V)$, if $V$ and $\mathcal{A}$ are as in Blecher’s theorem. This is a consequence of results from [8] and [3]. Namely, there is a more general construction - the inner tensor product of two Hilbert $C^*$-modules (details in [8]) and in [3] it is shown that the inner tensor product coincides with the Haagerup (module) tensor product. The results on embedding the $C^*$-algebra $B_\mathcal{A}(V)$ in the $C^*$-algebra of the inner
tensor product of $V$ with another module, in the special case of tensoring $V$ with $\mathcal{H}$ as in Blecher’s theorem, yield the embedding $B_A(V) \subseteq B(\mathcal{H}_V)$.

Further, it is known ([2]) that $B_A(V)$ is isomorphic to $B_{M(A)}(M(V))$, so Blecher’s theorem and its corollaries from this paper show that the following isometric embeddings are valid for any Hilbert $A$-module $V$ with $A$ faithfully and nondegenerately represented on $\mathcal{H}$ (so $A \subseteq M(A) \subseteq B(\mathcal{H})$):

$$V \subseteq M(V) \subseteq B(\mathcal{H}, \mathcal{H})(\mathcal{H}, M(V))$$

$$K_A(V) \subseteq B_A(V) \subseteq B(\mathcal{H}_V)$$

$$K_A(V) \subseteq K_{M(A)}(M(V)) \subseteq B_A(V) \cong B_{M(A)}(M(V)) \subseteq B(\mathcal{H}_V)$$

Also, the linking algebra of a Hilbert $C^*$-module (defined formally as $L = \left[ \begin{array}{cc} K_A(V) & V \\ V^* & A \end{array} \right]$ and suitably equipped with a $C^*$-algebra structure) can be represented (faithfully and nondegenerately) as a $C^*$-subalgebra of $B(\mathcal{H}_V \oplus \mathcal{H})$. In short, all important structures related to a Hilbert $C^*$-module $V$ (the algebras of adjointable and of generalized compact operators, the linking algebra, the strict completion) can be concretely represented using the Hilbert space the underlying $C^*$-algebra $A$ is represented on and Blecher’s space $\mathcal{H}_V$.

REFERENCES