ADDITIVE SELECTIONS OF \((\alpha, \beta)\)-SUBADDITIVE SET VALUED MAPS

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Abstract. It is proved that an \((\alpha, \beta)\)-subadditive set valued map with close, convex and equibounded values in a Banach space has exactly one additive selection if \(\alpha, \beta\) are positive numbers and \(\alpha + \beta \neq 1\).

The study of subadditive set valued maps (s.v. maps) is related to the classical Hyers-Ulam stability problem for the Cauchy functional equation. This problem is cf.\cite{1} and \cite{4} the next one:

If \(f : \mathbb{R} \rightarrow \mathbb{R}\) is a measurable solution of the inequality

\[
|f(x + y) - f(x) - f(y)| < \varepsilon
\]

where \(\varepsilon > 0\) then there exists a linear function \(a(x) = mx, m \in \mathbb{R}\), such that \(|f(x) - a(x)| < \varepsilon\) for every \(x \in \mathbb{R}\).

The inequality (1) can be written on the form

\[
f(x + y) - f(x) - f(y) \in B(0, \varepsilon)
\]

where \(B(0, \varepsilon)\) is the ball with centre 0 and radius \(\varepsilon\).

Hence we have

\[
f(x + y) + B(0, \varepsilon) \subseteq f(x) + B(0, \varepsilon) + f(y) + B(0, \varepsilon)
\]

and denoting by \(F(x) = f(x) + B(0, \varepsilon), x \in \mathbb{R}\), we get

\[
F(x + y) \subseteq F(x) + F(y), \quad x, y \in \mathbb{R}
\]

and

\[
a(x) \in F(x), \quad x \in \mathbb{R},
\]

which means that the s.v. map \(F\) has the additive selection \(a\).


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Now one may ask in what conditions a subadditive s.v. map (i.e. a s.v. map that satisfies relation (2) admits an additive selection. An answer at this question is given in [1]. It is also proved that there exists subadditive s.v. maps that have not additive selection (see [1]).

In the sequel we shall generalize the notion of subadditive s.v. map and we shall give for such s.v. maps an existence and uniqueness theorem of an additive selection.

Let $X$ be a real vector space. We shall denote by $\mathcal{P}(X)$ and $\mathcal{P}_0(X)$ the family of all subsets, respectively nonempty subsets of $X$. If $Y$ is a real normed space the family of all close and convex subsets of $Y$ will be denoted by $\text{ccl}(Y)$.

If $A, B$ are nonempty subsets of the real vector space $X$ and $\lambda, \mu$ are real numbers we define
\begin{align*}
A + B &= \{x \mid x = a + b, \ a \in A, \ b \in B\} \\
\lambda A &= \{x \mid x = \lambda a, \ a \in A\}.
\end{align*}

The next properties will be often used in what follows
\begin{align*}
\lambda (A + B) &= \lambda A + \lambda B \\
(\lambda + \mu) A &\subseteq \lambda A + \mu A.
\end{align*}

If $A$ is a convex set and $\lambda \mu \geq 0$ we have (cf. [3])
\begin{align*}
(\lambda + \mu) A &= \lambda A + \mu A.
\end{align*}

A set $A \subseteq X$ is said to be a cone if $K + K \subseteq K$ and $\lambda K \subseteq K$ for all $\lambda > 0$. If the zero vector from $X$ belongs to $K$ we say that $K$ is a zero cone.

**Definition 1.** Let $X, Y$ be real vector spaces, $K \subseteq X$ a cone and $\alpha > 0$, $\beta > 0$. A s.v. map $F : K \to \mathcal{P}_0(Y)$ is called $(\alpha, \beta)$-subadditive if
\begin{align*}
F(\alpha x + \beta y) &\subseteq \alpha F(x) + \beta F(y)
\end{align*}
for all $x, y \in X$.

For $\alpha = \beta = 1$, $F$ is called subadditive s.v. map.

Let us remark that there exists $(\alpha, \beta)$-subadditive s.v. map that are not subadditive.

The s.v. map $F : [0, +\infty) \to \mathcal{P}_0(\mathbb{R})$ given by
\begin{align*}
F(x) &= \left[\sqrt{x}, +\infty\right)
\end{align*}
is $\left(\frac{1}{2}, \frac{1}{2}\right)$-subadditive but is not subadditive.

For all $x, y \geq 0$ we have
\begin{align*}
F\left(\frac{x + y}{2}\right) &= \left[\sqrt{\frac{x + y}{2}}, +\infty\right)
\end{align*}
and
\[ \frac{1}{2}F(x) + \frac{1}{2}F(y) = \left[ \frac{\sqrt{x} + \sqrt{y}}{2}, +\infty \right] . \]

Taking account of
\[ \sqrt{\frac{x + y}{2}} \geq \frac{\sqrt{x} + \sqrt{y}}{2} \]
it results that
\[ F\left( \frac{x + y}{2} \right) \subseteq \frac{F(x) + F(y)}{2} \]
that means \( F \) is \( \left( \frac{1}{2}, \frac{1}{2} \right) \)-subadditive, but it is not subadditive because
\[ F(x + y) = \left[ \sqrt{x + y}, +\infty \right] \nsubseteq \left[ \sqrt{x} + \sqrt{y}, \infty \right] = F(x) + F(y) \]
for \( x > 0, y > 0 \).

In the following theorem we shall give a sufficient condition for an \( (\alpha, \beta) \)-subadditive multifunction to admit an additive selection. Let us recall that if \( F : X \to P(Y) \) is a s.v. map a function \( f : X \to Y \) with the property \( f(x) \in F(x) \) for all \( x \in X \) is said to be a selection of \( F \).

**Theorem 2.** Let \( X \) be a real vector space, \( K \subseteq X \) a zero cone, \( Y \) a Banach space and \( \alpha > 0, \beta > 0 \).
If \( F : K \to \text{ccl}(Y) \) is \( (\alpha, \beta) \)-subadditive s.v. map which satisfies the conditions:
1) \( \alpha + \beta \neq 1 \)
2) \( \sup \{ \text{diam} F(x) : x \in X \} < +\infty \)
then \( F \) admits exactly one additive selection.

**Proof.** Using \( (\alpha, \beta) \)-subadditivity of \( F \) and convexity of its values we have for all \( x \in K \)
\[(0.7) \quad F((\alpha + \beta)x) \subseteq \alpha F(x) + \beta F(y) = (\alpha + \beta)F(x) \]
and replacing \( x \) by \( (\alpha + \beta)^n x, n \in \mathbb{N} \), in (7) we obtain
\[ F((\alpha + \beta)^{n+1}x) \subseteq (\alpha + \beta)F((\alpha + \beta)^n x) \]
and
\[ \frac{F((\alpha + \beta)^{n+1}x)}{(\alpha + \beta)^{n+1}} \subseteq \frac{F((\alpha + \beta)^n x)}{(\alpha + \beta)^n} . \]

I. Let \( \alpha + \beta > 1 \).

Denoting by \( F_n(x) = \frac{F((\alpha + \beta)^n x)}{(\alpha + \beta)^n}, x \in K, n \in \mathbb{N} \), it results that \( (F_n(x))_{n \geq 0} \) is a decreasing sequence of closed subsets of the Banach space \( Y \).
We have also
\[ \text{diam} F_n(x) = \frac{1}{(\alpha + \beta)^n} \text{diam} F((\alpha + \beta)^n x) \]
and taking account of the condition 2) we obtain

\[
\lim_{n \to \infty} diam F_n(x) = 0.
\]

Using the Cantor theorem for the sequence \((F_n(x))_{n \geq 0}\) the intersection \(\bigcap_{n \geq 0} F_n(x)\) is a singleton and we denote this intersection by \(a(x)\) for all \(x \in K\).

Thus we obtain a function \(a : K \to Y\) which is a selection of \(F\) because \(a(x) \in F_0(x) = F(x)\) for all \(x \in K\).

We shall show that \(a\) is additive. We have

\[
F_n(\alpha x + \beta y) = \frac{F((\alpha + \beta)^n(\alpha x + \beta y))}{(\alpha + \beta)^n} = \frac{F(\alpha(\alpha + \beta)^n x + \beta(\alpha + \beta)^n y)}{(\alpha + \beta)^n} \subseteq
\]

\[
\alpha F((\alpha + \beta)^n x) + \beta F((\alpha + \beta)^n y) = \frac{\alpha F((\alpha + \beta)^n x)}{(\alpha + \beta)^n} + \beta \frac{F((\alpha + \beta)^n y)}{(\alpha + \beta)^n} = aF_n(x) + \beta F_n(y)
\]

for all \(x, y \in K\) and \(n \in \mathbb{N}\).

From the last relation it results that

\[
a(\alpha x + \beta y) = \bigcap_{n \geq 0} F_n(\alpha x + \beta y) \subseteq \bigcap_{n \geq 0} (\alpha F_n(x) + \beta F_n(y))
\]

and

\[
\|a(\alpha x + \beta y) - (\alpha a(x) + \beta a(y))\| \leq diam(\alpha F_n(x) + \beta F_n(y)) \leq \alpha diam F_n(x) + \beta diam F_n(y) \to 0,
\]

when \(n \to \infty\), cf.(8).

So

\[
a(\alpha x + \beta y) = \alpha a(x) + \beta a(y),
\]

for all \(x, y \in K\).

Putting \(x = y = 0\) in (9) it results \(a(0) = 0\) and for \(y = 0\) and \(x = 0\) in (9) we obtain

\[
a(\alpha x) = \alpha a(x)
a(\beta y) = \beta a(y)
\]

for all \(x, y \in K\). Using (10) we have

\[
a(x + y) = a \left( \frac{x}{\alpha} + \frac{y}{\beta} \right) = \alpha a \left( \frac{x}{\alpha} \right) + \beta a \left( \frac{y}{\beta} \right) = a(x) + a(y)
\]

for all \(x, y \in K\), thus \(a\) is additive.

II. For \(\alpha + \beta < 1\), we replace \(x\) in (7) by \(\frac{x}{(\alpha + \beta)^n+1}\), \(n \in \mathbb{N}\), and we obtain multiplying this relation by \((\alpha + \beta)^n\)

\[
(\alpha + \beta)^n F \left( \frac{x}{(\alpha + \beta)^n} \right) \subseteq (\alpha + \beta)^{n+1} F \left( \frac{x}{(\alpha + \beta)^{n+1}} \right).
\]
The sequence \((F_n'(x))_{n \geq 0}\),

\[ F'_n(x) = (\alpha + \beta)^n F\left(\frac{x}{(\alpha + \beta)^n}\right), \]

is increasing and it follows that the sequence of positive numbers \((\text{diam } F'_n(x))_{n \geq 0}\) is increasing too. We have

\[ \text{diam } F'_n(x) = (\alpha + \beta)^n \text{diam } F\left(\frac{x}{(\alpha + \beta)^n}\right) \]

and

\[ \lim_{n \to \infty} \text{diam } F'_n(x) = 0, \]

hence \(F'_n(x)\) is single valued for all \(x \in K\). The s.v. map \(F\) is single valued

\[ F(\alpha x + \beta y) = \alpha F(x) + \beta F(y) \]

for all \(x, y \in K\), and using the same methods as in case I, we obtain the additivity of \(F\). We conclude that \(F\) is its own selection.

We have proved the existence of an additive selection of \(F\) and let us prove the uniqueness.

Suppose that \(F\) has two additive selection \(a_1, a_2 : K \to Y\). We have

\[ na_i(x) = a_i(nx) \in F(nx) \]

for all \(n \in \mathbb{N}, x \in K, 1 \leq i \leq 2\). We have

\[ n\|a_1(x) - a_2(x)\| = \|na_1(x) - na_2(x)\| = \|a_1(nx) - a_2(nx)\| \leq \text{diam } F(nx) \]

for all \(x \in K, n \in \mathbb{N}\), and taking account of the condition 2) from Theorem 1, it results \(a_1(x) = a_2(x)\) for all \(x \in K\). 

**Corollary 3.** ([1]). Let \(X\) be a real vector space, \(K \subseteq X\) a cone with zero and \(Y\) a Banach space. If \(F : K \to \text{ccl}(Y)\) is a subadditive s.v. map with

\[ \sup\{\text{diam } F(x) : x \in K\} < +\infty \]

then \(F\) admits exactly one additive selection.

**Proof.** We take \(\alpha = \beta = 1\) in Theorem 1. 

**Corollary 4.** Let \(X\) be a real vector space, \(K \subseteq X\) a cone with zero, \(Y\) a normed space. If \(F : K \to \mathcal{P}_0(Y)\) is (\(\alpha, \beta\))-subadditive with \(\alpha + \beta < 1\) and convex values and

\[ \sup\{\text{diam } F(x) : x \in K\} < +\infty, \]

then \(F\) is single valued and additive.

**Proof.** It results from the proof of Theorem 1, part II. 

Remark 5. There exists $(\alpha, \beta)$-subadditive s.v. maps with $\alpha + \beta = 1$, which satisfies the conditions of Theorem 1 and have not additive selection.

Concave s.v. maps are $(\alpha, \beta)$-subadditive s.v. maps with $\alpha + \beta = 1$. It is known that if $f, g : [0,1) \to \mathbb{R}$ are two functions such that $f$ is concave and $g$ is convex and $f(x) \leq g(x)$, for each $x \in \mathbb{R}$ then the s.v. map $F : \mathbb{R} \to P_0(\mathbb{R})$, $F(x) = [f(x), g(x)]$ is concave (see [3]). Using this result it follows that

$$F(x) = \left[ \frac{x}{x+1}, 1 \right], \quad x \in [0, \infty)$$

is concave without additive selection, because if $a : \mathbb{R} \to \mathbb{R}$ is an additive selection of $F$ we must have $a(x) = mx$ for each $x \in [0, +\infty) \cap \mathbb{Q}$ ($m \in \mathbb{R}$) and

$$\frac{x}{x+1} \leq mx \leq 1, \quad \forall x \in [0, +\infty) \cap \mathbb{Q}.$$ 

The last inequality is impossible because taking the limit it results $\lim_{x \to \infty} mx = 1$.

References


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