OSCILLATION AND MULTILINEAR STIELTJES INTEGRAL

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Abstract. In this note we consider oscillation of regulated functions. We improve and simplify the proof of the existence theorem for multilinear Stieltjes integral in the Riemann-Stieltjes and Moore-Pollard sense and introduce multilinear Henstock-Kurzweil-Stieltjes integral.

1. Introduction

Notations. Let $X,Y$ and $X_j, j = 1,\ldots,p$ be linear normed spaces. Let $L(X_1,\ldots,X_p;Y)$ denote the linear normed space of bounded multilinear transformations $A : X_1 \times \cdots \times X_p \to Y$.

The existence of the Stieltjes multilinear integral of $f_i$ relativ to $g$, in the case when the function $g$ is of bounded semivariation, $f_i$ are regulated functions, and $X_j, j = 1,\ldots,p$ are Banach spaces, was proved in [4]. In the present paper we simplify and improve the proof, assuming only $Y$ to be a Banach space. Furthermore we suggest a definition of the multilinear Stieltjes integral in Henstock-Kurzweil sense.

Definition 1.1. Let $(M,d)$ be a metric space, $(X,|\cdot|)$ a linear normed space and $A$ a subset of $M$. Let $f$ be a mapping of $A$ into $X$. The oscillation of $f$ in $A$, is defined to be

$$\omega(f,A) = \sup \{|f(t) - f(s)|, \ s,t \in A\}$$

Let $a$ be a cluster point of $A$. The oscillation of $f$ at the point $a$ with respect to $A$ is

$$\omega(f,a,A) = \inf_V \omega(A \cap V)$$

where $V$ runs over the set of neighborhoods of $a$.

Definition 1.2. A mapping $f : [a,b] \to X$ is called a regulated function if it has one-sided limits at every point of $[a,b]$.
Remark 1.3. If $f : [a, b] \mapsto X$ is a regulated function and $s_0 \in (a, b)$ then the oscillation of the function $f$ at the point $s_0$ is
\[
\omega(f, s_0, [a, b]) = 
\max\{|f(s_0 + 0) - f(s_0)|, |f(s_0) - f(s_0 - 0)|, |f(s_0 + 0) - f(s_0 - 0)|\},
\]
and similarly
\[
\omega(f, a, [a, b]) = |f(a + 0) - f(a)|, \quad \omega(f, b, [a, b]) = |f(b) - f(b - 0)|.
\]

Lemma 1.4. Let $f : [a, b] \mapsto X$ be a regulated function and $\epsilon > 0$. Then there exists a subdivision $E$ of the interval $[a, b]$,
\[
E = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 \leq t_1 \leq \cdots \leq t_n = b
\]
such that the oscillation of $f$ in each of the open intervals $I_i = (t_{i-1}, t_i)$ is
\[
\omega(f, (t_{i-1}, t_i)) < \epsilon.
\]

Proof. Given $\epsilon > 0$. For every $x \in [a, b]$ there is an open interval $V_x = (x - \delta_x, x + \delta_x)$ such that $|f(s) - f(t)| < \epsilon$ if either both $s, t$ are in $(x - \delta_x, x)$ or both in $(x, x + \delta_x)$.
The intervals $U_x = (x - \delta_x/2, x + \delta_x/2)$ cover $[a, b]$. There exists a finite subfamily of such intervals $U_{x_i}, i = 1, \ldots, n - 1$, where $x_i$ is an increasing sequence, which is a covering of $[a, b]$. We take $t_i = x_i, i = 1, \ldots, n - 1, t_0 = a$ and $t_n = b$. Then either $x_i \in V(x_{i-1})$ or $x_{i-1} \in V(x_i)$ and hence
\[
|f(s) - f(t)| < \epsilon \quad \text{i.e.} \quad \omega(f, (t_{i-1}, t_i)) < \epsilon.
\]

Corollary 1.5. Given $\epsilon > 0$. Let $Q$ denote the set of the points at which the oscillation of a regulated function $f$ is $\geq \epsilon$. Then $Q$ is a finite set.

2. Semivariation

Definition 2.1. Let $A \in L(X_1, \ldots, X_k, \ldots, X_p; Y)$, $g : [a, b] \mapsto X_k$,
\[
P = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 \leq t_1 \leq \cdots \leq t_n = b.
\]
The function $g$ is of bounded semivariation relative to $A$ if there exists a positive constant $M$ such that
\[
\sum_{i=1}^{n} A[x_i^1, \ldots, x_i^{k-1}, g(t_i) - g(t_{i-1}), x_i^{k+1}, \ldots, x_i^p]
\]
is less than
\[
M \cdot \max_i |x_i^1| \cdots \max_i |x_i^{k-1}| \cdot \max_i |x_i^{k+1}| \cdots \max_i |x_i^p|
\]
for all subdivisions $P$ of $[a, b]$ and all $x_i^j \in X_j, \quad j = 1, \ldots, p, \quad j \neq k, \quad i = 1, \ldots, n$.
\[
P = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 \leq t_1 \leq \cdots \leq t_n = b.
\]
The semivariation of \( g \) relative to \( A \), \( SV(g, A, [a, b]) \), is defined as

\[
\sup_P \left\{ \left| \sum_{i=1}^{n} A[x_i^1, \ldots, x_i^{k_i-1}, g(t_i) - g(t_{i-1}), x_i^{k_i+1}, \ldots, x_i^{p_i}] \right|, \right. \\
\left. |x_i^j| \leq 1, \quad x_i^j \in X_j \right\}
\]

The supremum is taken over all subdivisions \( P \) and all \( x_i^j \in X_j, |x_i^j| \leq 1 \).

**Remark 2.2.** It is obvious that if \( g \) is of bounded variation than \( g \) is also of bounded semivariation.

The proofs of the next two lemmas follow from Definition 2.1.

**Lemma 2.3.** If \([a_i, b_i], \quad i = 1, \ldots, n\), are non-overlapping intervals such that

\[
\bigcup^n_i [a_i, b_i] \subseteq [a, b]
\]

then

\[
\left| \sum_{i=1}^{n} A[x_i^1, \ldots, x_i^{k_i-1}, g(b_i) - g(a_i), x_i^{k_i+1}, \ldots, x_i^{p_i}] \right|
\]

is less than

\[
\max_i |x_i^1| \cdot \max_i |x_i^{k_i-1}| \cdot \max_i |x_i^{k_i+1}| \cdot \cdots \cdot \max_i |x_i^{p_i}| \cdot SV(g, A, [a, b]).
\]

**Lemma 2.4.** If \( c \leq a \leq b \leq d \), then \( SV(g, A, [a, b]) \leq SV(g, A, [c, d]) \)

**Lemma 2.5.** Let \( Y \) and \( X_j, j = 1, \ldots, p \), be linear normed spaces.
Let \( A \in L(X_1, \ldots, X_k, \ldots, X_p; Y) \), let \( g : [a, b] \mapsto X_k \) be a function of bounded semivariation and let

\[
P = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 \leq t_1 \leq \cdots \leq t_n = b.
\]

Suppose that the vectors \( v_i^j, u_i^j \in X_j \), \( j \neq k \), \( i = 1, \ldots, n \), satisfy

\[
|v_i^j - u_i^j| < \epsilon,
\]

and denote \( M_j = \sup \{|1, |v_i^j|, |u_i^j|\} \).

Then the sum

\[
|S| = \left| \sum_{i=1}^{n} \left\{ A[v_i^1, \ldots, v_i^{k_i-1}, g(t_i) - g(t_{i-1}), v_i^{k_i+1}, \ldots, v_i^{p_i}] \\
- A[u_i^1, \ldots, u_i^{k_i-1}, g(t_i) - g(t_{i-1}), u_i^{k_i+1}, \ldots, u_i^{p_i}] \right\} \right|
\]

is less than \( \epsilon \cdot M \cdot SV(g, A, [a, b]) \), where \( M = p \cdot M_1 \cdots M_{k-1} \cdot M_{k+1} \cdots M_p \).
PROOF. Since $A$ is a multilinear operator we can rewrite $S$.
\[
|S| = \left| \sum_{i=1}^{n} \left\{ A[v_i^1, v_i^2, \ldots, v_i^{k-1}, g(t_i) - g(t_{i-1}), v_i^{k+1}, \ldots, v_i^p] \right. \right.
\]
\[
- A[u_i^1, v_i^2, \ldots, v_i^{k-1}, g(t_i), v_i^{k+1}, \ldots, v_i^p] \right\}
\]
\[
+ \sum_{i=1}^{n} \left\{ A[u_i^1, v_i^2, \ldots, v_i^{k-1}, g(t_i) - g(t_{i-1}), v_i^{k+1}, \ldots, v_i^p] \right. \right.
\]
\[
- A[u_i^1, u_i^2, \ldots, v_i^{k-1}, g(t_i), v_i^{k+1}, \ldots, v_i^p] \right\}
\]
\[
+ \sum_{i=1}^{n} \left\{ A[u_i^1, u_i^2, \ldots, u_i^{k-1}, g(t_i) - g(t_{i-1}), u_i^{k+1}, \ldots, v_i^p] \right. \right.
\]
\[
- A[u_i^1, u_i^2, \ldots, u_i^{k-1}, g(t_i), u_i^{k+1}, \ldots, v_i^p] \right\} \right.
\]
So we have
\[
|S| \leq \epsilon \cdot M_2 \cdots M_{k-1} \cdot M_{k+1} \cdots M_p \cdot SV(g, A, [a, b])
\]
\[
+ M_1 \cdot \epsilon \cdot M_2 \cdots M_{k-1} \cdot M_{k+1} \cdots M_p \cdot SV(g, A, [a, b])
\]
\[
\ldots
\]
\[
+ M_1 \cdots M_{k-1} \cdot M_{k+1} \cdots M_{p-1} \cdot \epsilon \cdot SV(g, A, [a, b])
\]
\[
< \epsilon \cdot M \cdot SV(g, A, [a, b]).
\]

\[
\boxdot
\]

**Lemma 2.6.** Let $Y$ and $X_j, j = 1, \ldots, p$, be linear normed spaces. Let $A \in L(X_1, \ldots, X_k, \ldots, X_p; Y)$, let $g : [a, b] \mapsto X_k$ be a function of bounded semivariation and let \(P = \{t_0, t_1, \ldots, t_n\}\), \(a = t_0 < t_1 < \cdots < t_n = b\).

Suppose that the vectors \(v_i^j, u_i^j, y_i^j, x_i^j \in X_j, j \neq k, i = 1, \ldots, n\), satisfy \(|v_i^j - u_i^j| < \epsilon, |y_i^j - x_i^j| < \epsilon\), and that \(M_j = \sup_i \{1, |v_i^j|, |v_i^j|, |x_i^j|, |y_i^j|\}\).

Then the sum
\[
|S| = \left| \sum_{i=1}^{n} \left\{ A[v_i^1, v_i^2, \ldots, v_i^{k-1}, g(t_i) - g(t_{i-1} + 0), v_i^{k+1}, \ldots, v_i^p] \right. \right.
\]
\[
- A[u_i^1, u_i^2, \ldots, v_i^{k-1}, g(t_i - 0) - g(t_{i-1} + 0), u_i^{k+1}, \ldots, u_i^p] \right\}
\]
\[
+ \sum_{i=1}^{n-1} \left\{ A[y_i^1, y_i^2, \ldots, y_i^{k-1}, g(t_i + 0) - g(t_i - 0), y_i^{k+1}, \ldots, y_i^p] \right. \right.
\]
\[
- A[x_i^1, x_i^2, \ldots, x_i^{k-1}, g(t_i - 0) - g(t_i - 0), x_i^{k+1}, \ldots, x_i^p] \right\} \right.
\]
is less than $\epsilon \cdot M \cdot SV(g, A, [a, b])$, where $M = p \cdot M_1 \cdots M_{k-1} \cdot M_{k+1} \cdots M_p$. 

Oscillation and Multilinear Stieltjes Integral

Let $\epsilon_1 > 0$. Since $A$ is a bounded operator we can choose points $t_i'$ and $t_i''$ such that $t_i \in (t_i', t_i'')$, $t_i'' < t_{i+1}$ and that the sum

$$|S_1| = \left| \sum_{i=1}^{n} \left\{ A[v_i^1, \ldots, v_i^{k-1}, g(t_i') - g(t_{i-1})], v_i^{k+1}, \ldots, v_i^p \right\]$$

$$- A[u_i^1, \ldots, u_i^{k-1}, g(t_i') - g(t_{i-1})], u_i^{k+1}, \ldots, u_i^p \right\} \right.$$

$$+ \sum_{i=1}^{n-1} \left\{ A[y_i^1, \ldots, y_i^{k-1}, g(t_i'') - g(t_i'), y_i^{k+1}, \ldots, y_i^p]$$

$$- A[x_i^1, \ldots, x_i^{k-1}, g(t_i'') - g(t_i'), x_i^{k+1}, \ldots, x_i^p] \right\} |$$

differs from $S$ by less than $\epsilon_1$. It follows from Lemma 2.5 that $|S_1| < \epsilon \cdot M \cdot SV(g, A, [a, b])$, so we have

$$|S| \leq |S - S_1| + |S_1| < \epsilon_1 + \epsilon \cdot M \cdot SV(g, A, [a, b]).$$

Since $\epsilon_1$ is arbitrary small, we have that

$$|S| \leq |S - S_1| + |S_1| < \epsilon \cdot M \cdot SV(g, A, [a, b]).$$

$\square$

3. Multilinear Stieltjes integral

Definition 3.1. Let $Y$ and $X_j, j = 1, \ldots, p$, be linear normed spaces. Let $A \in L(X_1, \ldots, X_k, \ldots, X_p; Y)$, let $g : [a, b] \mapsto X_k$ and let $f_j : [a, b] \mapsto X_j, j = 1, \ldots, p, j \neq k$. For the partition

$$P = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 \leq t_1 \leq \cdots \leq t_n = b,$$

we denote $\max\{|t_i - t_{i-1}| \}$ by $|P|$. Let $s_i^j, j = 1, \ldots, p, j \neq k$, be $p-1$ points arbitrarily taken from the interval $[t_{i-1}, t_i]$, by $S(P)$ we denote the Stieltjes sum

$$S(P) =$$

$$\sum_{i=1}^{n} \left\{ A[f_1(s_i^1), \ldots, f_{k-1}(s_i^{k-1}), g(t_i) - g(t_{i-1}), f_k(s_i^{k+1}), \ldots, f_p(s_i^p)] \right\}.$$

We say that the Stieltjes integral on $[a, b]$ of $f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_p$ with respect to $g$ and $A$ exists in the Riemann sense and has the value $I$ if, for every $\epsilon > 0$ there exist $\delta > 0$ such that

$$|P| < \delta \Rightarrow |I - S(P)| < \epsilon.$$
for any choice of the points \( t_i \in [a, b] \) and \( s_i^j \in [t_{i-1}, t_i] \).
We denote
\[
I = (RS) \int_{[a,b]} A (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p).
\]

**Definition 3.2.** We say that the Stieltjes integral exists in the Moore-Pollard sense and has the value \( I \) if, for every \( \varepsilon > 0 \) there exist a subdivision \( P_0 \) such that for every refinement \( P \supseteq P_0 \) we have
\[
|I - S(P)| < \varepsilon.
\]
We denote
\[
I = (MP) \int_{[a,b]} A (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p).
\]

In the case when \( g \) is a regulated function we can define Stieltjes integral in the Young-Moore-Pollard sense.

**Definition 3.3.** Let \( g : [a, b] \mapsto X_k \) be a regulated function. Let
\[
P = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 < t_1 < \cdots < t_n = b.
\]
Let \( s_i^j, j \neq k \), be \( p - 1 \) points in the open interval \( (t_{i-1}, t_i) \). By \( YS(P) \) we denote the sum
\[
\sum_{i=1}^n A[f_1(s_i^1), \ldots, f_{k-1}(s_i^{k-1}), g(t_i - 0) - g(t_{i-1} + 0), f_{k+1}(s_i^{k+1}), \ldots, f_p(s_i^p)] +
\sum_{i=1}^{n-1} A[f_1(t_i), \ldots, f_{k-1}(t_i), g(t_i + 0) - g(t_i - 0), f_{k+1}(t_i), \ldots, f_p(t_i)] +
A[f_1(b), \ldots, f_{k-1}(b), g(b) - g(b - 0), f_{k+1}(b), \ldots, f_p(b)] +
A[f_1(a), \ldots, f_{k-1}(a), g(a + 0) - g(a), f_{k+1}(a), \ldots, f_p(a)].
\]
We say that the Stieltjes integral on \([a, b]\) of \( f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_p \) with respect to \( g \) and \( A \) exists in the Young-Moore-Pollard sense and has the value \( I \) if, for every \( \varepsilon > 0 \) there exist a subdivision \( P_0 \) such that for every refinement \( P \supseteq P_0 \) we have
\[
|I - YS(P)| < \varepsilon.
\]
We denote
\[
I = (Y) \int_{[a,b]} A (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p).
\]
Similarly we define integrals
\[
\int_A (d_1 f_1, d_2 f_2, \ldots, d_p f_p),
\]
where \(d_i f_i\) denotes \(f_i\) or \(df_i\), see A. Halilovic [2].

In the next theorem we assume only \(Y\) to be a Banach space.

**Theorem 3.4.** Let \(Y\) be a Banach space and let \(X_j, j = 1, \ldots, p\), be linear normed spaces over the same field. Let \(A \in L(X_1, \ldots, X_k, \ldots, X_p; Y)\), let \(g: [a,b] \mapsto X_k\) be a regulated function of bounded semivariation and let \(f_j: [a,b] \mapsto X_j, j = 1, \ldots, p, j \neq k\) be regulated functions. Then

(i) **The Stieltjes integral**

\[
I = (Y) \int_A (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p)
\]
exists in the Young-Moore-Pollard sense.

(ii) **The Stieltjes integral**

\[
I = (MP) \int_A (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p)
\]
exists in the Moore-Pollard sense if and only if the functions \(g: [a,b] \mapsto X_k\) and \(f_j: [a,b] \mapsto X_j, j = 1, \ldots, p, j \neq k\), satisfy conditions (b) and (c) below.

(iii) **The Stieltjes integral**

\[
I = (RS) \int_A (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p).
\]
exists in the ordinary Riemann-Stieltjes sense if and only if the functions \(g: [a,b] \mapsto X_k\) and \(f_j: [a,b] \mapsto X_j, j = 1, \ldots, p, j \neq k\), satisfy conditions (a), (b) and (c) below.

**The conditions (a) – (c) are**

(a) \(A[f_1(s_1), \ldots, f_{k-1}(s_{k-1}), g(t + 0) - g(t - 0), f_{k+1}(s_{k+1}), \ldots, f_p(s_p)]\)

\[= A[f_1(t), \ldots, f_{k-1}(t), g(t + 0) - g(t - 0), f_{k+1}(t), \ldots, f_p(t)]\]

for all \(3^{p-1}\) combinations obtained by taking \(f_j(s_j) \in \{f_j(t - 0), f_j(t), f_j(t + 0)\}, j = 1, \ldots, p, j \neq k\), for every \(t \in (a,b)\).

(b) \(A[f_1(s_1), \ldots, f_{k-1}(s_{k-1}), g(t + 0) - g(t), f_{k+1}(s_{k+1}), \ldots, f_p(s_p)]\)

\[= A[f_1(t), \ldots, f_{k-1}(t), g(t + 0) - g(t), f_{k+1}(t), \ldots, f_p(t)]\]
for all $2^{p-1}$ combinations obtained by taking $f_j(s_j) \in \{f_j(t), f_j(t+0)\}$, $j = 1, \ldots, p$, $j \neq k$, for every $t \in [a, b]$.

(c) $A[f_1(s_1), \ldots, f_{k-1}(s_{k-1}), g(t) - g(t-0), f_{k+1}(s_{k+1}), \ldots, f_p(s_p)]$

$= A[f_1(t), \ldots, f_{k-1}(t), g(t) - g(t-0), f_{k+1}(t), \ldots, f_p(t)]$

for all $2^{p-1}$ combinations obtained by taking $f_j(s_j) \in \{f_j(t-0), f_j(t)\}$, $j = 1, \ldots, p$, $j \neq k$, for every $t \in (a, b)$.

**Proof.** Existence in the Young-Moore-Pollard sense. Given $\epsilon > 0$. It follows from Lemma 1.4 that there exists a subdivision $E$ of the interval $[a, b]$, $E = \{t_0, t_1, \ldots, t_n\}$, $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$, such that the oscillation of $f_j$ in each of the open intervals $I_i = (t_{i-1}, t_i)$, $i = 1, \ldots, n$, is less than $\epsilon$. Let $P$ be any refinement of $E$. We compare $Y(E)$ and $Y(P)$. Let $s_i^j$, $j = 1, \ldots, p$, $j \neq k$, be $p-1$ points arbitrarily chosen from the interval $[t_{i-1}, t_i]$. We suppose that $t_{i-1} = z_{i,0} < z_{i,1} < \cdots < z_{i,r(i)} = t_i$ are new points in the interval $[t_{i-1}, t_i]$ and that $u_i^e \in [z_{i,e-1}, z_{i,e}]$, $j = 1, \ldots, p$, $j \neq k$, $e = 1, \ldots, r_i$. We consider the difference $S(P) - S(E)$ in the intervals $[t_{i-1}, t_i]$. Since the points $t_i$ are in $P$ and in $E$, the terms

$A[f_1(t_i), \ldots, f_{k-1}(t_i), g(t_i + 0) - g(t_i - 0), f_{k+1}(t_i), \ldots, f_p(t_i)]$

$= A[f_1(b), \ldots, f_{k-1}(b), g(b) - g(b - 0), f_{k+1}(b), \ldots, f_p(b)]$

and

$A[f_1(a), \ldots, f_{k-1}(a), g(a + 0) - g(a), f_{k+1}(a), \ldots, f_p(a)]$

vanish, so we have.

$\Delta_i = A[f_1(s_i^1), \ldots, f_{k-1}(s_i^{k-1}), g(t_i - 0) - g(t_i - 1), f_{k+1}(s_i^{k+1}), \ldots, f_p(s_i^p)]$

$- \sum_{e=1}^{r(i)} A[f_1(u_i^1), \ldots, f_{k-1}(u_i^{k-1}), g(z_{i,e} - 0) - g(z_{i,e} - 1), f_{k+1}(u_i^{k+1}), \ldots, f_p(u_i^p)]$

$- \sum_{e=1}^{r(i)-1} A[f_1(z_{i,e}), \ldots, f_{k-1}(z_{i,e}), g(z_{i,e} + 0) - g(z_{i,e} - 0), f_{k+1}(z_{i,e}), \ldots, f_p(z_{i,e})].$

Inserting

$g(t_i - 0) - g(t_i - 1) = g(z_{i,r(i)}) - g(z_{i,r(i)-1}) + 0 + \sum_{e=1}^{r(i)-1} [g(z_{i,e} - 0) - g(z_{i,e} - 1) + g(z_{i,e} + 0) - g(z_{i,e} - 0)]$

for all $2^{p-1}$ combinations obtained by taking $f_j(s_j) \in \{f_j(t), f_j(t+0)\}$, $j = 1, \ldots, p$, $j \neq k$, for every $t \in (a, b)$. 


we obtain

\[ \Delta_i = \sum_{e=1}^{r(i)} \left\{ A[f_1(s_i^1), \ldots, f_{k-1}(s_i^{k-1}), g(z_{i,e} - 0) - g(z_{i,e} + 0), f_{k+1}(s_i^{k+1}), \ldots, f_p(s_i^p)] 
- A[f_1(u_{i,e}^1), \ldots, f_{k-1}(u_{i,e}^{k-1}), g(z_{i,e} - 0) - g(z_{i,e} + 0), f_{k+1}(u_{i,e}^{k+1}), \ldots, f_p(u_{i,e}^p)] \right\} \]

or

\[ \Delta_i = \sum_{e=1}^{r(i)-1} \left\{ A[f_1(s_i^1), \ldots, f_{k-1}(s_i^{k-1}), g(z_{i,e} - 0) - g(z_{i,e} + 0), f_{k+1}(s_i^{k+1}), \ldots, f_p(s_i^p)] 
- A[f_1(z_{i,e}), \ldots, f_{k-1}(z_{i,e}), g(z_{i,e} - 0) - g(z_{i,e} + 0), f_{k+1}(z_{i,e}), \ldots, f_p(z_{i,e})] \right\} \]

Since

\[ S(P) - S(E) = \sum \Delta_i, \]

and the oscillation in the intervals \((t_{i-1}, t_i)\) is less than \(\epsilon\), by Lemmas 2.3-2.6 we have

\[ |S(P) - S(E)| \leq \epsilon \cdot M \cdot SV(g, A, [a, b]). \]

Since \(S(P) \in Y\), and \(Y\) is a Banach space, the integral exists in the Young-Moore-Pollard sense.

Existence in the Moore-Pollard sense. Given \(\epsilon > 0\). It follows from Lemma 1.4 that there exists a subdivision \(E\) of the interval \([a, b]\),

\[ E = \{y_0, y_1, \ldots, y_m\}, \ a = y_0 \leq y_1 \leq \cdots \leq y_m = b \]

such that the oscillation of \(f_j\) in every of the open intervals \(I_l = (y_{l-1}, y_l), l = 1, \ldots, m\), is less than \(\epsilon\). It follows from the conditions (b) and (c) that there exists \(\delta > 0\) such that for \(t = y_l\) we have

\[ |A[f_1(s_j'''), \ldots, f_{k-1}(s_j'''_{k-1}), g(u) - g(t), f_{k+1}(s_j''''_{k+1}), \ldots, f_p(s_p)] - A[f_1(s_j'''), \ldots, f_{k-1}(s_j'''_{k-1}), g(u) - g(t), f_{k+1}(s_j''''_{k+1}), \ldots, f_p(s_p)]| < \frac{\epsilon}{2m} \]

if \(s_j''''' \in [t, t + \delta], j = 1, \ldots, p, j \neq k\), \(u, v \in (t, t + \delta)\), and

\[ |A[f_1(s_j'''), \ldots, f_{k-1}(s_j'''_{k-1}), g(t) - g(v), f_{k+1}(s_j''''_{k+1}), \ldots, f_p(s_p)] - A[f_1(s_j'''), \ldots, f_{k-1}(s_j'''_{k-1}), g(t) - g(v), f_{k+1}(s_j''''_{k+1}), \ldots, f_p(s_p)]| < \frac{\epsilon}{2m} \]

if \(s_j''''' \in [t - \delta, t], j = 1, \ldots, p, j \neq k\), \(u, v \in [t - \delta, t)\).
Let now $P_0 = \{t_0, t_1, \ldots, t_n\}$, $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$, be a subdivision of $[a, b]$ such that $P_0 \supseteq E$ and $|P_0| < \delta$. Let $s_i^j$, $j = 1, \ldots, p, j \neq k$, be $p - 1$ points arbitrarily chosen from the interval $[t_{i-1}, t_i]$

Let $P$ be an arbitrary refinement of $P_0$. We suppose that $t_{i-1} = z_i; 0 \leq z_i; 1 \cdots \leq z_i; r(i) = t_i$ are new points in the interval $[t_{i-1}, t_i]$ and that $u_{i,e}^k \in [z_{i,e-1}, z_{i,e}]$, $j = 1, \ldots, p, j \neq k$, $e = 1, \ldots, r_i$. We consider the difference $S(P) - S(P_0)$ in the intervals $[t_{i-1}, t_i]$.

Let

$$\Delta_i = A[f_1(s_{i}^1), \ldots, f_{k-1}(s_{i}^{k-1}), g(t_i) - g(t_{i-1}), f_{k+1}(s_{i}^{k+1}), \ldots, f_p(s_{i}^p)]$$

$$- \sum_{e=1}^{r(i)} \left\{ A[f_1(u_{i,e}^1), \ldots, f_{k-1}(u_{i,e}^{k-1}), g(z_{i,e}) - g(z_{i,e-1}), f_{k+1}(u_{i,e}^{k+1}), \ldots, f_p(u_{i,e}^p)] \right\}.$$

Inserting

$$g(t_i) - g(t_{i-1}) = \sum_{e=1}^{r(i)} [g(z_{i,e}) - g(z_{i,e-1})]$$

we obtain

$$\Delta_i = \sum_{e=1}^{r(i)} \left\{ A[f_1(s_{i}^1), \ldots, f_{k-1}(s_{i}^{k-1}), g(z_{i,e}) - g(z_{i,e-1}), f_{k+1}(s_{i}^{k+1}), \ldots, f_p(s_{i}^p)]
$$

$$- A[f_1(u_{i,e}^1), \ldots, f_{k-1}(u_{i,e}^{k-1}), g(z_{i,e}) - g(z_{i,e-1}), f_{k+1}(u_{i,e}^{k+1}), \ldots, f_p(u_{i,e}^p)] \right\}. $$

Hence

$$S(P_0) - S(P) = \sum_{i=1}^{n} \Delta_i$$

$$= \sum_{i=1}^{n} \sum_{e=1}^{r(i)} \left\{ A[f_1(s_{i}^1), \ldots, f_{k-1}(s_{i}^{k-1}), g(z_{i,e}) - g(z_{i,e-1}), f_{k+1}(s_{i}^{k+1}), \ldots, f_p(s_{i}^p)]
$$

$$- A[f_1(u_{i,e}^1), \ldots, f_{k-1}(u_{i,e}^{k-1}), g(z_{i,e}) - g(z_{i,e-1}), f_{k+1}(u_{i,e}^{k+1}), \ldots, f_p(u_{i,e}^p)] \right\}$$

$$= \sum_{i} + \sum_{e}.$$

Some of the $z_{i,e}$ are in $E$, i.e. coincide with $y_i$, and we denote by $\sum_{i}$ the sum of terms over those intervals, where at least one of the endpoints $z_{i,e}$ is
in $E$. According to (3.1) and (3.2) we have

$$\sum_{j} |f_j| < 2m \cdot \frac{\varepsilon}{2m} = \varepsilon.$$  

The oscillation of the functions $f_j$ over every interval, which build the sum denoted by $\sum'$, is less than $\varepsilon$. Hence by Lemmas 2.3 and 2.5 we have

$$\sum' \leq \varepsilon \cdot M \cdot SV(g, A, [a, b]).$$  

It follows from (3.3), (3.4) and (3.5) that

$$|S(P_0) - S(P)| = |\sum' + \sum' - \sum'| \leq \varepsilon + \varepsilon \cdot M \cdot SV(g, A, [a, b])$$

Since $S(P) \in Y$, and $Y$ is a Banach space, the integral exists in Moore-Pollard sense. *Necessity.* Suppose that one of the conditions (b), (c) for example (c) does not hold in a point $t$. We consider a subdivision $P_n$ which includes the interval $[t - 1/n, t]$. We can choose associated points $s^j_k$ so that

$$\Delta_n = |A[f_1(s^1_k), \ldots, f_k-1(s^{k-1}_k), g(t) - g(t - 0), f_{k+1}(s^{k+1}_k), \ldots, f_p(s^p_k)] - A[f_1(t), \ldots, f_k(t), g(t) - g(t - 0), f_{k+1}(t), \ldots, f_p(t)|$$

does not converge to 0 when $n \to \infty$. We compare two sums $S_1(P_n)$ and $S_2(P_n)$ which agree excepting that in the interval $[t - 1/n, t]$ we take different associated points, $s^j_k = s^j_{k-1}$ for $S_1$ and $s^j_k = t$ for $S_2$. So we have that $|S_1 - S_2| = \Delta_n$ does not converge to 0 when $n \to \infty$. Consequently (MP) $\int f$ does not exist.

*Existence in the Riemann-Stieltjes sense.* If the conditions (a), (b) are fulfilled then the integral exists in the Moore-Pollard sense, and let $I$ denote its value. By Definition 3.2, for $\varepsilon > 0$, there exists a subdivision

$$E = \{y_0, y_1, \ldots, y_m\}, \quad a = y_0 < y_1 < \ldots, < y_m = b$$

such that

$$P' \supseteq E \Rightarrow |S(P') - I| < \varepsilon.$$  

It follows, from the conditions (a), (b) and (c), that there exists $\delta > 0$ such that for $t = y_i$ we have

$$|A[f_1(s^u_1), \ldots, f_k-1(s^{u-1}_k), g(u) - g(t), f_{k+1}(s^{u+1}_k), \ldots, f_p(s^p_k)] - A[f_1(u), \ldots, f_k(u), g(u) - g(t), f_{k+1}(u), \ldots, f_p(u)]| < \frac{\varepsilon}{2m}$$

if $s^u_k, s^p_k \in [t, t + \delta], \quad j = 1, \ldots, p, \quad j \neq k, \quad u, v \in [t, t + \delta]$,

$$|A[f_1(s^u_1), \ldots, f_k-1(s^{u-1}_k), g(t) - g(u), f_{k+1}(s^{u+1}_k), \ldots, f_p(s^p_k)] - A[f_1(u), \ldots, f_k(u), g(t) - g(u), f_{k+1}(u), \ldots, f_p(u)]| < \frac{\varepsilon}{2m}$$

if $s^u_k, s^p_k \in [t - \delta, t], \quad j = 1, \ldots, p, \quad j \neq k, \quad u, v \in [t - \delta, t]$, and
\[\begin{align*}
|A[f_1(s''_1), \ldots, f_{k-1}(s''_{k-1}), g(u'') - g(u), f_{k+1}(s''_{k+1}), \ldots, f_p(s'_p)] - \\
A[f_1(s'_1), \ldots, f_{k-1}(s'_{k-1}), g(v'') - g(v), f_{k+1}(s'_{k+1}), \ldots, f_p(s'_p)]| < \frac{\epsilon}{2m}
\end{align*}\]

if \( s''_j, s'_j \in [t - \delta, t + \delta], \ j = 1, \ldots, p, \ j \neq k, \) \( u, v \in [t - \delta, t], \ u', v' \in (t, t + \delta). \)

For \( \delta \) so determined, we consider any subdivision \( P \) with \(|P| < \delta\).

Suppose

\[P = \{t_0, t_1, \ldots, t_n\}, \ a = t_0 \leq t_1 \leq \cdots \leq t_n = b,\]

and let \( s^i_j, j = 1, \ldots, p, j \neq k, \) be \( p - 1 \) points arbitrarily taken from the interval \([t_{i-1}, t_i]\). Suppose \( P_1 = P \cup E. \) We define associated points \( s^i_{1,j} = s^i_j \) in any interval \([t_{i-1}, t_i]\) which contains no points of \( E. \) In the intervals which contains \( y_i \) as an end point we chose associated points to be equal \( y_i. \) We can assume that \( \delta \) is less than \( \min |y_i - y_{i-1}| \) so that there is maximum one point \( y_i \) in any interval \([t_{i-1}, t_i]\). Because of (3.9) we have

\[(3.10) \quad |S(P) - S(P_1)| < 2m \cdot \frac{\epsilon}{2m}.
\]

Since \( P_1 \supset E \) we have

\[(3.11) \quad |I - S(P_1)| < \epsilon.
\]

It follows from (3.10) and (3.11) that

\[|I - S(P)| < 2\epsilon.
\]

It means that the integral exists in the Riemann-Stieltjes sense. We can prove the necessity of the conditions \((a), (b) \) and \((c)\) in the same way as in \((ii)\). For the condition \((a)\) we consider the intervals \((t - 1/n, t + 1/n). \)

**Remark.** For the necessity in Theorem 3.4 we do not need the assumption that \( Y \) is a Banach space, but only that \( Y \) is a linear normed space, so we have the following theorem.

**Theorem 3.5.** Let \( X_j, j = 1, \ldots, p, \) and \( Y \) be linear normed spaces over the same field. Let \( A \in L(X_1, \ldots, X_k, \ldots, X_p; Y) \) and let \( f_j : [a, b] \rightarrow X_j, \ j = 1, \ldots, p, \ j \neq k, \) and \( g : [a, b] \rightarrow X_k \) be regulated functions. Then the Stieltjes integral

\[I = (MP) \int_{[a,b]}^A (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p)
\]

exists in the Moore-Pollard sense only if the functions \( g : [a, b] \rightarrow X_k \) and \( f_j : [a, b] \rightarrow X_j, \ j = 1, \ldots, p, \ j \neq k, \) satisfy conditions \((b) \) and \((c)\) in
**Theorem 3.4;**
The Stieltjes integral

\[ I = (RS) \int_{[a,b]} (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p). \]

exists in the ordinary Riemann-Stieltjes sense only if the functions \( g : [a, b] \mapsto X_k \) and \( f_j : [a, b] \mapsto X_j, \ j = 1, \ldots, p, \ j \neq k, \) satisfy conditions (a), (b) and (c) in Theorem 3.4.

**Example.** Let \( M_{m,n} \) denote the linear normed space of all \( m \times n \) matrices. We define \( i : [-1, 1] \mapsto R \) to be

\[ i(t) = \begin{cases} 1, & \text{if } t \text{ is a rational number;} \\ 2, & \text{if } t \text{ is an irrational number.} \end{cases} \]

Let \( a, b \) and \( c \) be real numbers and \( b \neq 0 \). We define functions
\( f_1 : [-1, 1] \mapsto M_{2,3}, \ g : [-1, 1] \mapsto M_{3,2} \) and \( f_3 : [-1, 1] \mapsto M_{2,4} \) as follows

\[ f_1(t) = \begin{bmatrix} b \cdot i(t) & 0 & c \\ b \cdot i(t) & 0 & c \end{bmatrix}, \]
\[ g(t) = \begin{bmatrix} a \\ a & i(t) \\ a & t \end{bmatrix}, \]
\[ f_3(t) = \begin{bmatrix} i(t) & i(t) & i(t) & i(t) \\ d & d & d & d \end{bmatrix}. \]

The functions \( f_1, g \) and \( f_3 \) have common discontinuities at all points of the interval \([-1, 1]\). We define a multilinear operator \( A : M_{2,3} \times M_{3,2} \times M_{2,4} \mapsto M_{2,4} \) as ordinary matrix multiplication, \( A(X_{2,3}, X_{3,2}, X_{2,4}) = X_{2,3} \cdot X_{3,2} \cdot X_{2,4} \), where \( X_{i,j} \) is a matrix in \( M_{i,j} \). Let

\[ P = \{ t_0, t_1, \ldots, t_n \}, \ -1 = t_0 \leq t_1 \leq \cdots \leq t_n = 1. \]

In every interval \([t_{i-1}, t_i]\) we choose two arbitrary points \( s_i^1, s_i^3 \) and form the Stieltjes sum

\[ S(P) = \sum_{i=1}^{n} f_1(s_i^1) \cdot [g(t_i) - g(t_{i-1})] \cdot f_3(s_i^3) \]

\[ = \sum_{i=1}^{n} \begin{bmatrix} cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) \\ cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) \end{bmatrix}. \]
By Definition 3.1 we have that the $RS$ integral exists and have the value

$$I = (RS) \int_{[-1,1]} (f_1, dg, f_3) = \lim_{|P| \to 0} S(P) = \begin{bmatrix} 2cd & 2cd & 2cd & 2cd \\ 2cd & 2cd & 2cd & 2cd \end{bmatrix}$$

although the functions $f_1, g$ and $f_3$ have a common discontinuity in every point in the interval $[-1, 1]$.

**Multilinear Stieltjes integral in the Henstock-Kurzweil sense.**

Let $Y$ and $X_j, j = 1, \ldots, p$, be linear normed spaces. Let $A \in L(X_1, \ldots, X_p; Y)$ and let $f_j : [a, b] \mapsto X_j, j = 1, \ldots, p$. Let

$$P = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 < t_1 < \cdots < t_n = b,$

be a partition of $[a, b]$. If we consider multilinear Stieltjes integral in the general case, we need an ordered set $J$ which indicates functions and those coordinates where we consider "$df$". For example, if $J = (0, 1, 0, 1, 0)$ then we consider the multilinear Stieltjes integral

$$\int_{[a,b]} (f_1, df_2, f_3, df_4, f_5).$$

If we consider the integral in the Riemann sense then the integral is the limit of the sums

$$\sum_{i=1}^{n} A[f_1(s_i^1), f_2(t_i) - f_2(t_{i-1}), f_3(s_i^3), f_4(t_i) - f_4(t_{i-1}), f_5(s_i^5)],$$

where $s_i^1, s_i^3, s_i^5 \in [t_{i-1}, t_i]$, and in the Henstock sense (see definition below) the integral is the limit of the sums

$$\sum_{i=1}^{n} A[f_1(s_i), f_2(t_i) - f_2(t_{i-1}), f_3(s_i), f_4(t_i) - f_4(t_{i-1}), f_5(s_i)],$$

where $s_i \in [t_{i-1}, t_i]$.

Let the "indicator", set $J = \{e_1, e_2, \ldots, e_p\}$, where $e_j = 1$ or $e_j = 0$, be given. For given $J$ we denote

$$F^j_i = \begin{cases} f_j(s_i), & \text{if } e_j = 0; \\ f_j(t_i) - f_j(t_{i-1}), & \text{if } e_j = 1. \end{cases}$$

We define Henstock-Stieltjes sum to be

$$HS(P) = \sum_{i=1}^{n} A[F^1_i, \ldots, F^p_i].$$
Definition 3.6. We say that the multilinear Stieltjes integral on $[a, b]$ exists in the Henstock-Kurzweil sense and has the value $I$, if for every $\epsilon > 0$ there exists $\delta(s) > 0$, such that whenever a partition $P$ and points $s_i$ satisfy

$$s_i \in [x_i, x_{i-1}] \subset (s_i - \delta(s_i), s_i + \delta(s_i))$$

for $i = 1, \ldots, n$, we have

$$|I - \sum_{i=1}^{n} A[F^1_i, \ldots, F^p_i]| < \epsilon.$$

We write

$$I = (HS) \int_{[a,b]}^A (d_1 f_1, \ldots, d_p f_p),$$

where the symbol $d_j f_j$ is defined as follows

$$d_j f_j = \begin{cases} f_j, & \text{if } e_j = 0; \\ df_j, & \text{if } e_j = 1. \end{cases}$$

Remark The Stieltjes integral which we consider in the example obviously exist in the Henstock-Kurzweil sense. We compare Stieltjes integral in the Riemann, Moore-Pollard, Young and Henstock-Kurzweil sense in [5].

References


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