FIBERWISE RETRACTION AND SHAPE PROPERTIES OF
THE ORBIT SPACE

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Abstract. From the point of view of retracts and shape theory, the
category $G$-$\text{TOP}_B$ of $G$-spaces over a $G$-space $B$, where $G$ is a compact
group, is investigated. In particular, we prove that if $B$ has only one orbit
type and $E$ is a metric $G$-$\text{ANR}$ over $B$, then the orbit space $E/G$ is an
$\text{ANR}$ over $B/G$. As an application we construct a fiberwise $G$-orbit functor
$\mu: G$-$\text{SH}_B \to SH_{B/G}$ on shape level.

1. Introduction

Many of the ideas of homotopy theory belong most naturally to the
category $G$-$\text{TOP}_B$ of $G$-spaces over a given $G$-space $B$, where $G$ is a topological
group. An excellent demonstration of that provide two articles of I. M. James
and G. B. Segal [16], [17] (see also [18, Ch. 8]), which have inspired this
research. Here we study fiberwise retraction and shape properties of orbit
spaces of $G$-spaces over $B$. The first main result we establish (Theorem 3.1)
has no counterpart in the ordinary theory of retracts and provides a fiberwise
version of the following result of S. A. Antonyan [4], [5]:

Theorem 1.1. Let $G$ be a compact group, $N \subseteq G$ be a closed normal
subgroup, and $X$ be a metric $G$-$\text{A}(N)R$-space. Then $X/N$ is a $G/N$-$\text{A}(N)R$-
space. In particular $X/G$ is an $A(N)R$-space.

This result is the crucial tool in what follows. It will be applied also in
the form of the following equivalent assertion:

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Theorem 1.2. Let $G$ be a compact group, $N \subseteq G$ be a closed normal subgroup and $A$ be a metric $G$-space such that $A/N$ admits a $G/N$-equivariant closed embedding into a metric $G/N$-space $X$. Then there exists a metric $G$-space $Y$, which contains $A$ as a $G$-invariant subspace and $Y/N = X$.

The equivalence of Theorem 1.1 and Theorem 1.2 is proved in [6].

Our Theorem 3.1 generalizes the above Theorem 1.1. At the same time we show that it is not true for arbitrary base $B$. Namely, the condition "$B$ is an ANE over $B/G$" is a necessary condition (Proposition 3.2), while the condition "$B$ is a $G$-ANE over $B/G$" is sufficient in Theorem 3.1. Here $B/G$ is regarded as a $G$-space with the trivial $G$-action.

We prove for $G$ a Lie group (Theorem 3.3) that every $G$-space $B$ with all orbits of the same orbit type, is a $G$-ANE over its orbit space $B/G$.

Conversely, if $B$ is a connected $G$-ANE over $B/G$, then $B$ has only one orbit type. Theorem 3.7 is just the finite-dimensional analogue of Theorem 3.1.

In §3 we make use the results of §2 to develop a fiberwise shape theory for arbitrary $G$-spaces over a given metric $G$-space $B$, where the acting group $G$ is assumed to be compact. There are several approaches to the non-equivariant fiber shape theory [8], [10], [19], [26]. For equivariant shape theory we refer to [7], [11], [23] and [25]. The description of our fiber $G$-shape category $G\text{-}\text{SH}_B$ is based on the general construction of shape categories in [22].

Finally, applying Theorem 3.1, in particular, we establish that the $G$-orbit functor $\pi: G\text{-}\text{TOP}_B \to \text{TOP}_{B/G}$ naturally induces a corresponding functor $\mu: G\text{-}\text{SH}_B \to \text{SH}_{B/G}$ on shape level (Theorem 4.9).

2. Basic notions, facts and notations

Throughout the paper it is assumed, unless the contrary is stated, that $G$ is a compact Hausdorff group which we keep fixed. All topological spaces are assumed to be Tychonov.

The basic definitions and results of the theory of $G$-spaces or topological transformation groups can be found in G. Bredon [9] and in R. Palais [24].

A survey of the equivariant theory of retracts was given in [1]. For equivariant fiberwise theory of retracts we refer to [17], [18, Ch. 8].

By $G/H$ we always denote the left coset space of $G$ by a closed subgroup $H$, endowed with the action of $G$ by left translations.

If $X$ is a $G$-space and $N \subseteq G$ is a closed normal subgroup, then the $N$-orbit space $X/N$ admits a natural $G/N$-action defined by

\begin{equation}
(gN)(N(x)) = N(gx)
\end{equation}

where $gN \in G/N$ and $N(x)$ denotes the $N$-orbit of $x \in X$.

Hereafter, we will always mean the action (1) on the $G/N$-space $X/N$. 
If \( H \subseteq G \) is a subgroup then the class of subgroups of \( G \) which are conjugate to \( H \) is denoted by \((H)\), i.e., \((H) = \{ghg^{-1} | g \in G\}\). The class \((H)\) is often called a \( G \)-orbit type or simply an orbit type. Let \((H)\) and \((K)\) be two orbit types. One says that \((H) \leq (K)\) if \( H \) is conjugate to some subgroup of \( K \). If in addition \((H) \neq (K)\), we say that \((H) < (K)\). It is easy to see that the relation \( \leq \) is a partial ordering on the set of all \( G \)-orbit types. Now suppose that \( X \) is a \( G \)-space and \( x \in X \). The subgroup \( G_x = \{ g \in G | gx = x \} \) of \( G \) is called the stabilizer of the point \( x \); since \( G_{gx} = gG_xg^{-1} \) for any \( g \in G \) we have \((G_x) = (G_{gx})\). If \( H \subseteq G \) is a subgroup, we denote by \( X[H] \) the subset of \( H \)-fixed points of \( X \), i.e., \( X[H] = \{ x \in X | H \subseteq G_x \} \). It is well-known that \( X[H] \) is a closed \( N(H) \)-invariant subspace of \( X \), where \( N(H) \) is the normalizer of \( H \) in \( G \) (see [9, Ch. I, §5]).

Let \( X \) be a \( G \)-space and \( H \subseteq G \) be a closed subgroup. We denote by \( HS \) the subset \( \{hs | h \in H, s \in S\} \) of \( X \).

A subset \( S \subseteq X \) is called \([24, \text{p. 27}]\) an \( H \)-slice in \( X \) if

1. \( GS \) is open in \( X \) and \( S \) is closed in \( GS \),
2. \( S \) is \( H \)-invariant, i.e., \( HS = S \),
3. for each \( g \in G \) not in \( H \), \( gS \) is disjoint from \( S \).

If in addition \( GS = X \) then \( S \) is called a **global** \( H \)-slice of \( X \).

Clearly, if \( f : Z \rightarrow X \) is a \( G \)-map and \( S \) is an \( H \)-slice in \( X \), then \( f^{-1}(S) \) is an \( H \)-slice in \( Z \).

In the sequel we will need the following useful property of a slice: if \( Q \) is a global \( H \)-slice of a \( G \)-space \( X \) and \( R \) is a global \( H \)-slice of a \( G \)-space \( Y \), then for each \( H \)-equivariant map \( f : Q \rightarrow R \), the map \( F : X \rightarrow Y \) defined by \( F(qx) = gf(x) \) is a well-defined \( G \)-map (see for example [24, Proposition 1.7.10]).

The following device (due to G. Segal) called the colon construction by I. M. James [18, Ch. 4], is often useful in the study of \( G \)-spaces. For \( G \)-spaces \( X \) and \( Y \) let us denote by \((X : Y)\) the orbit space \( W/G \), where \( W \subseteq X \times Y \) is the invariant subspace consisting of pairs \((x, y)\) such that \( G_x \subseteq G_y \).

In this article we work in the category \( G\text{-TOP}_B \) of \( G \)-spaces over a given \( G \)-space \( B \), which is called the base. A \( G \)-space over \( B \) consists of a \( G \)-space \( E \) and a \( G \)-map \( p : E \rightarrow B \) called the projection.

Usually \( E \) alone is a sufficient notation. Thus \( B \) is regarded as a \( G \)-space over itself with the projection the identity map. Moreover any product \( X \times B \) of \( G \)-spaces is regarded as a \( G \)-spaces over \( B \) with the natural projection \( X \times B \rightarrow B \). Let \( X, Y \) be \( G \)-spaces over \( B \) with the projections \( p, q \), respectively. By a \( G \)-map \( f : X \rightarrow Y \) over \( B \) we mean a \( G \)-map in the ordinary sense such that \( qf = p \). With this definition of morphisms the category \( G\text{-TOP}_B \) is defined.

Let us recall the definition of \( N \)-orbit functor \( \pi : G\text{-TOP}_B \rightarrow G/N\text{-TOP}_{B/N} \) where \( N \subseteq G \) is a closed normal subgroup. For any \( G \)-space \( E \) over
with projection \( p : E \to B \) we denote \( \pi(E) = E/N, \pi(B) = B/N \) and \( \pi(p) = p/N \), where \( (p/N)(N(x)) = N(p(x)) \) for every \( N \)-orbit \( N(x) \in E/N \).

If we consider \( E/N \) and \( B/N \) as \( G/N \)-spaces with \( G/N \)-action (1), the map \( p/N : E/N \to B/N \) becomes a \( G/N \)-map. So \( E/N \) naturally is a \( G/N \)-space over \( B/N \).

Let \( F \) be another \( G \)-space over \( B \) and \( f : E \to F \) be a \( G \)-map over \( B \). Then \( f \) induces a \( G/N \)-map \( f/N : E/N \to F/N \) defined by \( (f/N)(N(x)) = N(f(x)), N(x) \in E/N \). One easily verifies that \( f/N \) is a \( G/N \)-map over \( B/N \).

Putting \( \pi(f) = f/N \) we obtain the desired functor \( \pi \).

**Remark 2.1.** Sometimes we will need to regard \( \pi(E) \) also as a \( G \)-space over \( \pi(B) \), where \( G \) acts on \( \pi(E) \) and \( \pi(B) \) via the natural homomorphism \( G \to G/N \) (see e.g., the proof of Theorem 3.1).

In this case \( \pi(f) : \pi(E) \to \pi(F) \) is a \( G \)-map over \( \pi(B) \) for any \( G \)-map \( f : E \to F \) over \( B \). So, one also can regard \( \pi \) as a functor from \( G-TOP_B \) into \( G-TOP_{B/N} \).

For future references, we will work with the diagram of \( G \)-spaces as follows:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & E \\
\downarrow{i} & & \downarrow{p} \\
X & \xrightarrow{\phi} & B
\end{array}
\]

(2.2)

where \( (X, A) \) is a metric \( G \)-pair, i.e., a pair in which \( X \) is a metric \( G \)-space, \( A \) is a closed invariant subspace of \( X \) and \( i \) is the inclusion map.

Let \( E \) be a \( G \)-space over \( B \). Then \( E \) is called a \( G \)-ANE over \( B \) (notation: \( E \in G-ANE_B \)), if the following equivariant fiberwise extension property holds for all metric \( G \)-pairs \( (X, A) \) over \( B \):

For any \( G \)-maps \( f, \phi \) which make the above diagram (2) commutative, there exist an invariant neighborhood \( U \) of \( A \) in \( X \) and a \( G \)-map \( \psi : U \to E \) such that \( \psi|A = f \) and \( p\psi = \phi|U \). If in addition we can always take \( U = X \), then we say that \( E \) is a \( G \)-AE over \( B \) (notation: \( E \in G-AE_B \)). The map \( \psi \) is called a \( G \)-extension of \( f \) over \( \phi \).

Let \( E \) be a metric \( G \)-space over \( B \). Then \( E \) is called a \( G \)-ANR over \( B \) (notation: \( E \in G-ANR_B \)) provided for any metric \( G \)-space \( X \) over \( B \) and any closed \( G \)-embedding \( E \hookrightarrow X \) over \( B \) there exist an invariant neighborhood \( U \) of \( E \) in \( X \) and a \( G \)-retraction \( r : U \to E \) over \( B \). If in addition we can always take \( U = X \), then we say that \( E \) is a \( G \)-AR over \( B \) (notation: \( E \in G-AR_B \)).

Let \( n \geq 0 \) be an integer and \( E \) be a \( G \)-space over \( B \). Then \( E \) is called a \( G \)-AN(\( E(n) \) over \( B \) if the above fiberwise extension property holds for all metric \( G \)-pairs \( (X, A) \) with \( \dim X/G \leq n \).

If in the above definitions instead of a metric \( G \)-pair \( (X, A) \) we take a \( G \)-pair \( (X, A) \) from a given class \( \mathcal{K} \) of \( G \)-spaces over \( B \), then we obtain in
a similar way the notions of $G\cdot A(N)E(K)$, $G\cdot A(N)E(n)(K)$ spaces over $B$. When the base $B$ has only one point then the above definitions become the usual definitions of $G\cdot A(N)E(K)$ and $G\cdot A(N)E(n)(K)$. In [17], [18] the case $K = \mathcal{P}$—the class of all paracompact $G$-spaces was considered.

Let $f_0, f_1: E \to E'$ be $G$-maps over $B$. A $G$-homotopy over $B$ of $f_0$ into $f_1$ is a homotopy in the ordinary sense which is a $G$-map over $B$ at each stage of the deformation. The $G$-space $E$ is called $G$-contractible over $B$ if there is a $G$-section $s: B \to E$ (i.e., $ps = \text{id}_B$) such that $sp$ and $\text{id}_E$ are $G$-homotopic over $B$.

In an obvious way one obtains a $G$-equivariant fiber homotopy category over $B$ denoted by $[G\cdot \text{TOP}_B]$, whose objects are $G$-spaces over $B$ and whose morphisms are classes of $G$-homotopic $G$-maps over $B$. There is a homotopy functor $G\cdot \text{TOP}_B \to [G\cdot \text{TOP}_B]$ which keeps the objects fixed and takes $G$-maps $f$ over $B$ into their $G$-homotopy classes $[f]$ over $B$.

3. Main results

Theorem 3.1. Let $N \subseteq G$ be a closed normal subgroup and $E$ be a metric $G\cdot \text{ANE}$ (resp., a $G\cdot \text{AE}$) over a (resp., metric) $G$-space $B$. Suppose also that $B$ is a $G\cdot \text{ANE}$ over $B/N$. Then $E/N$ is a $G/N\cdot \text{ANE}$ (resp., a $G/N\cdot \text{AE}$) over $B/N$. In particular $E/G$ is an $\text{ANE}$ (resp., an $\text{AE}$) over $B/G$ whenever $B$ is a $G\cdot \text{ANE}$ over $B/G$.

Before proceeding with the proof, let us show that the restriction on $B$ in Theorem 3.1 is essential.

First we observe that for every topological space $Z$ and each integer $n \geq 1$ the $n$-fold power $Z^n$ possesses a natural action of the symmetric group on $n$ letters $S_n$ defined by the formula: $g * (z_1, \ldots, z_n) = (z_{g^{-1}(1)}, \ldots, z_{g^{-1}(n)})$ where $g \in S_n$ and $(z_1, \ldots, z_n) \in Z^n$.

Denote $E = (Q \times Q)^n$ and $B = Q^n$ where $Q$ is the Hilbert cube. So, $E$ and $B$ can be regarded as $G$-spaces with $G = S_n$. Define the projection $p: E \to B$ as follows: $p(x_1, \ldots, x_n) = (h(x_1), \ldots, h(x_n))$ for any $(x_1, \ldots, x_n) \in E$, where $h: Q \times Q \to Q$ is the first projection. Clearly $p$ is an $S_n$-map. We claim that $E$ is an $S_n\cdot \text{AE}$ over $B$. Indeed, let $f$ and $\phi$ be $S_n$-maps making commutative the diagram (2). Denote by $T$ the discrete $S_n$-space $\{1, 2, \ldots, n\}$. Consider maps $f': A \times T \to Q \times Q$ and $\phi': X \times T \to Q$ defined by $f'(a, t) = f(a)_t$ and $\phi'(x, t) = \phi(x)_t$ respectively, where $a \in A$, $x \in X$ and $t \in T$. Consider $X \times T$ as an $S_n$-space with the diagonal action. One easily sees that $f'$ is a continuous map over $\phi'$ and that both of them are $S_n$-invariant maps. Therefore these maps induce canonically continuous maps $f^*: (A \times T)/S_n \to Q \times Q$ and $\phi^*: (X \times T)/S_n \to Q$ such that $f^*$ is a map over $\phi^*$.

Clearly $(X \times T)/S_n$ is metric, $(A \times T)/S_n$ is closed in $(X \times T)/S_n$ and $Q \times Q$ is an AE over $Q$ with $h$ the projection (because the Hilbert cube is an AE). Hence there is a continuous extension $\psi^*: (X \times T)/S_n \to Q \times Q$ of $f^*$ over $\phi^*$. Let
Since in this case the diagonal action of $E$ and $X$, we are in position to apply the above Theorem 1.2, according to which there exist a $\pi_2$-extension of $f$ over $\phi$ and the claim is proved.

However $E/S_n$ is not an $AE$ over $B/S_n$ as it is shown by V. V. Fedorchuk [13], [14, p. 242]. In fact $E/S_n$ is not even an $ANE$ over $B/S_n$. Indeed, since $E \in S_n-AE_B$, and $B$ is metric, $E$ is $S_n$-contractible over $B$. This easily implies the contractibility of $E/S_n$ over $B/S_n$ and since a fiberwise contractible $ANE$ is a fiberwise $AE$, we conclude that $E/S_n$ is not an $ANE$ over $B/S_n$.

**Proof of Theorem 3.1.** Let $(X,A)$ be a metric $G/N$-pair and let $f: A \to E/N$, $\phi: X \to B/N$ be $G/N$-maps such that $(p/N)f = \phi|A$, where $p/N: E/N \to B/N$ is the canonical $G/N$-map induced by the projection $p: E \to B$. We must show that $f$ admits a $G/N$-neighborhood extension $\Phi: W \to E/N$ over $\phi$. Define $A' \subseteq A \times E$ to be the pull-back of the $G$-space $E$ with respect to $f$. Then $A'$ is a $G$-invariant subspace of $A \times E$ endowed with the diagonal action of $G$ (by Remark 2.1 we can consider $A$ as a $G$-space).

Since in this case $N$ acts trivially on $A$ we have $A'/N = A$ (see [15, §4.1]). Let $\lambda: A' \to A$, $f': A' \to E$ be the corresponding projections. Since $A'$ is metric, we are in position to apply the above Theorem 1.2, according to which there is a metric $G$-space $X'$ which contains $A'$ as a $G$-invariant closed subspace and $A'/N = X$. Let denote by $\mu: X' \to X$ the orbit map and by $j: A' \to X'$ the inclusion map. Consider the commutative diagram

$$
\begin{array}{ccc}
A' & \xrightarrow{p f'} & B \\
\downarrow{j} & & \downarrow{\rho} \\
X' & \xrightarrow{\phi \mu} & B/N \\
\end{array}
$$

where $\rho$ is the $N$-orbit map. As $B$ is a $G$-$ANE$ over $B/N$ there exists a $G$-extension $F': U \to B$ of $p f'$ over $\phi \mu$ defined on some $G$-neighborhood $U$ of $A'$ in $X'$.

So, we have $(\phi \mu)|U = \rho F'$ and $F'|A' = p f'$. Since $E \in G$-$ANE_B$ it follows that there exist a $G$-neighborhood $V$ of $A'$ in $U$ and a $G$-extension $F: V \to E$ of $f$ over $F'$. Denote shortly by $\Phi$ the $G/N$-map $F/N$. We claim that $\Phi: V/N \to E/N$ is the desired $G/N$-extension of $f$ over $\phi$. Indeed, first we note that $W = V/N$ is a $G/N$-neighborhood of $A$ in $X$. Now let $a \in A$ and let $\nu: E \to E/N$ be the $N$-orbit map. Then $a = \lambda(a') = \mu(a')$ for some $a' \in A'$, and hence, $\Phi(a) = \nu F(a') = \nu f'(a') = f(a') = f(a)$, i.e., $\Phi i = f$. For every $x \in V/N$ there is $x' \in V$ such that $x = \mu(x')$ and then $\Phi(x) = \nu F(x')$. Consequently, $(p/N)\Phi(x) = (p/N)\nu F(x') = \rho F(x') = \rho F'(x') = \phi \mu(x') = \phi(x)$, i.e., $\Phi$ is a map over $\phi$ and the proof in the “$G$-$ANE$” case is completed.

If in addition $E$ is $G$-$AE$ over $B$ and $B$ is metric, then $E$ is $G$-contractible over $B$ by [16]. This implies that $E/N$ is $G/N$-contractible over $B/N$. Since
Concerning the restriction on base $B$ in Theorem 3.1 we have the following result (for simplicity we consider only the case $N = G$):

**Proposition 3.2.** Let $G$ be a compact Lie group and $B$ be a metric $G$-space. Suppose that for any $G$-space $E$ over $B$ which is a $G$-ANE$_B$, the orbit space $E/G$ is an ANE over $B/G$. Then $B$ is an ANE over $B/G$.

**Proof.** Take $E = G \times B$ with the $G$-action $g \cdot (h, b) = (hg^{-1}, gb)$. Since $G$ is a $G$-ANE [24, Corollary 1.6.7] we conclude that $E$ is a $G$-ANE over $B$. By the hypothesis it then follows that $E/G$ is an ANE over $B/G$. Now observe that the map $\phi: (G \times B)/G \to B$ defined by $\phi([g, b]) = gb$, $[g, b] \in (G \times B)/G$ is a homeomorphism over $B/G$ (see, for example [9, p. 113]). Thus $B$ is an ANE over $B/G$.

**Theorem 3.3.** Let $G$ be a compact Lie group and suppose $B$ is an arbitrary Tychonov $G$-space with all orbits of type $G/H$. Then $B$ is a $G$-ANE over $B/N$ for every closed normal subgroup $N \subseteq G$. Conversely, if $B$ is connected and $B$ is a $G$-ANE over $B/G$, then $B$ has only one orbit type (even in the case of $G$ an arbitrary compact group).

For the proof of Theorem 3.3 we need the following three lemmas:

**Lemma 3.4.** Let $G$ be a compact Lie group, $H \subseteq G$ a closed subgroup and $N \subseteq G$ be a closed normal subgroup. Then $G/H$ is a $G$-ANE over $G/HN$.

**Proof.** Let $(X, A)$ be a metric $G$-pair and let $f, \phi$ be $G$-maps such that the diagram (2) with $E = G/H$, $B = G/HN$ and $p: G/H \to G/HN$ the natural projection, is commutative. Put $Q = \phi^{-1}(eHN)$ where $e$ is the unity of $G$ and denote $S = Q \cap A$. Then $Q$ is a global $HN$-slice of $X$, $S$ is a closed $HN$-invariant subspace of $Q$ and $f$ maps $S$ into $(HN)/H = p^{-1}(eHN) \subseteq G/H$. Since $(HN)/H$ is an $HN$-ANE [24, Corollary 1.6.7], $f|S$ admits an $HN$-equivariant extension $F$ defined on some $HN$-invariant neighborhood $V$ of $S$ in $Q$. Then setting $F(gv) = gF(v)$ for each $g \in G$, $v \in V$, we obtain a $G$-extension $F$ of $f$ over $\phi$ defined on the $G$-neighborhood $GV$ of $A$ in $X$.

**Lemma 3.5.** Let $E$ be a $G$-space over $B$ and suppose that there exists an open invariant covering $\{V_j\}_{j \in J}$ of $B$ such that $E_j = p^{-1}(V_j)$ is a $G$-ANE$_{V_j}$ (resp., a $G$-ANE$_{V_j}$) for each index $j$. Then $E$ is a $G$-ANE$_B$ (resp., a $G$-ANE$_B$).

**Proof.** Because of [18, Proposition 8.48] it is sufficient to show that $(Z: E)$ is an $AE$ (resp., an ANE) over $(Z: B)$ for all metric $G$-spaces $Z$. Clearly $\{(Z: V_j)\}_{j \in J}$ is an open covering of $(Z: B)$. Since $E_j$ is a $G$-ANE
over $\mathcal{V}' [18, \text{Proposition } 8.48]$ using again $[18, \text{Proposition } 8.48]$, we have that $(Z; E_j)$ is an AE (resp., an ANE) over $(Z; \mathcal{V}_j)$ for each index $j$. So, by $[18, \text{Proposition } 8.25]$, $(Z; E)$ is an AE (resp., an ANE) over $(Z; B)$.

**Lemma 3.6.** Let $H \subset G$ be a closed subgroup and $E$ be a $G$-ANE (resp., a $G$-AE) over $B$. Then $E$ is an $H$-ANE (resp., an $H$-AE) over $B$.

**Proof.** We consider only the “$H$-ANE” case. The “$H$-AE” case is similar. Let $(X, A)$ be a metric $H$-pair, $f, \phi$ be $H$-maps such that the diagram (2) is commutative. Consider the $G$-space $Z$ for which $X$ is a global $H$-slice (for $Z$ one can take the twisted product $G \times_H X [9, \text{Ch. II, } \S3]$). According to $[24, \text{Theorem } 1.7.10]$ the maps $f, \phi$ uniquely determine $G$-equivariant maps $f' : GA \rightarrow E$ and $\phi' : Z \rightarrow B$ such that $f'|A = f, \phi'|X = \phi$. Clearly $pf' = \phi'|GA$. Since $E \in G$-ANE$_B$ there exist a $G$-neighborhood $U$ of $GA$ in $Z$ and a $G$-extension $\psi' : U \rightarrow E$ of $f'$ over $\phi'$. Putting $V = U \cap X$ and $\psi = \psi'|V$ we obtain the desired $H$-equivariant extension $\psi : V \rightarrow E$ of $f$ over $\phi$.

**Proof of Theorem 3.3.** By Theorem 5.8 of $[9, \text{Ch. II}]$ the $G$-space $B$ constitutes a fibre bundle over $B/G$. Therefore we can cover $B$ by open sets $V_\alpha$ of the form $V_\alpha = G/H \times U_\alpha$ where $G$ acts trivially on $U_\alpha$. For $B/N$ we have the open invariant covering $\{V_\alpha/N\}$ and according to Lemma 3.5, it suffices to show that $V_\alpha$ is a $G$-ANE over $V_\alpha/N$. Since $V_\alpha/N = (G/H \times U_\alpha)/N = G/HN \times U_\alpha$ and since $G/H$ is a $G$-ANE over $G/HN$ (Lemma 3.4), it then follows that $G/H \times U_\alpha$ is a $G$-ANE over $G/HN \times U_\alpha$. This proves the first part of Theorem 3.3.

To prove the second part, suppose that $B$ is a connected $G$-space, which is a $G$-ANE over $B/G$. Suppose also that $B$ has more than one orbit type. Then one can find two different orbit types in $B$, say $(H_1)$ and $(H_2)$, such that either $(H_1) < (H_2)$ or $(H_1)$ and $(H_2)$ are not comparable. Let $W$ denotes the normalizer of $H_2$ in $G$. Consider $B[H_2]$ the set of $H_2$-fixed points of $B$. Clearly $B[H_2]$ is a $W$-invariant subspace of $B$, so it can be regarded as a $W$-space. Put $X = B/H_2$ the $H_2$-orbit space of $B$. Since $H_2$ is a closed normal subgroup of $W$, the group $W$ acts naturally on $X$. Set $A = B[H_2]$. Evidently $A$ can be regarded as a closed $W$-invariant subspace of the $W$-space $X$. Consider the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow j & & \downarrow p \\
X & \xrightarrow{\phi} & B/G
\end{array}
$$

where $f$ is the natural inclusion and $\phi(H_2(x)) = G(x)$ for all $H_2(x) \in B/H_2 = X$. Clearly $f$ and $\phi$ are $W$-maps and the diagram commutes. Since $B$ is a $W$-ANE over $B/G$ (Lemma 3.6), there exist a $W$-invariant neighborhood $U$ of $A$
in $X$ and a $W$-extension $\psi : U \to B$ of $f$ over $\phi$. As $H_2$ acts trivially on $X$ it follows that the image $\psi(U)$ must lie in $B[H_2]$. Let $r : B \to X$ denote the $H_2$-orbit map. Putting $\psi' = \psi r$, $V = r^{-1}(U)$ we get the following commutative diagram

$$
\begin{array}{c}
A = B[H_2] \xrightarrow{f} B[H_2] \subset B \\
\downarrow \quad \downarrow p \\
V \xrightarrow{\psi'} V/G \subset B/G
\end{array}
$$

where $p' = p|V$.

We claim that there is a point $y \in V$ such that $G(y) \cap B[H_2] = \emptyset$. Indeed, if the contrary is true, then $Gv = GB[H_2]$. Since $G$ is compact, $GB[H_2]$ is closed (see [9, Ch. I, §1, Corollary 1.3] or [24, Proposition 1.1.1]) and since $V$ is open, $GV$ is open as well. So $GB[H_2]$ being a closed-open subset of $B$ must coincide with $B$ as $B$ is connected. But this is impossible, because in $B$ there is a point $z \in B$ with $G_z = H_1$ and we have $(G_z) = (H_1) \neq (H_2)$, while $(G_z) \geq (H_2)$ for all $x \in GB[H_2]$. This completes the proof of the claim. Now by the commutativity of the above diagram we have $p\psi'(y)=p'(y) = p(y)$ i.e., $\psi'(y)$ and $y$ have the same $G$-orbit. This is a contradiction since $\psi'(y) \in B[H_2]$ and $G(y) \cap B[H_2] = \emptyset$.

**Theorem 3.7.** Let $N \subseteq G$ be a closed normal subgroup and $E$ be a metric $G$-ANE(k) over a $G$-space $B$, where $k \geq 0$ is an integer. Suppose also that $B$ is a $G$-ANE(k) over $B/N$. Then $E/N$ is a $G/N$-ANE(k) over $B/N$.

Furthermore, if $B$ is metric, $\dim B/G \leq k$, $\dim E/G \leq k - 1$ and $E$ is a $G$-AE(k) over $B$, then $E/N$ is a $G/N$-AE(k) over $B/N$.

**Proof.** The proof of the first part of this theorem is analogous to that of Theorem 3.1, the only additional condition on the dimension of the orbit space also holds because $X'/N = X$ implies $X'/G = X/G$.

Consider the “$G$-AE” case. Since $B$ is metric and $\dim B/G \leq k$, the condition $E \in G$-AE$_B(k)$ easily implies that the projection $p : E \to B$ admits an equivariant section $s : B \to E$. Consider the following commutative diagram of $G$-maps:

$$
\begin{array}{c}
E \times \{0\} \cup E \times \{1\} \xrightarrow{f} E \\
\downarrow \quad \downarrow p \\
E \times [0,1] \xrightarrow{\phi} B
\end{array}
$$

where $f(x,0) = x$, $f(x,1) = sp(x)$ and $\phi(x,t) = p(x)$ for $x \in E$, $t \in [0,1]$. Since $\dim E/G \leq k - 1$ we have $\dim (E \times [0,1])/G \leq k$. Therefore, using the condition $E \in G$-AE$_B(k)$, we obtain a $G$-contraction of $E$ over $B$. This
implies that $E/N$ is $G/N$-contractible over $B/N$. Since $E/N$ is a $G/N$-$ANE(k)$ over $B/N$ by the previous case, it then follows that $E/N$ is a $G/N$-$AE(k)$ over $B/N$ [17, Proposition 2.3].

4. Equivariant fiberwise shape

Throughout of this section we assume that $B$ is a given metric $G$-space. Here we define a shape category for arbitrary $G$-spaces over $B$. In our development we follow the method of resolutions introduced in the case of ordinary shape by S. Mardesić [20], [21] and extended to the equivariant case in [7]. A general procedure is described in [22, Ch. I, §2], which associates a shape category $\text{SH}_T; P$ with every pair consisting of a category $T$ and of a dense subcategory $P$. The equivariant shape category over a base $B$ is the shape category associated in this way with the pair $T = [G-TOP_B], P = [G-ANR_B]$ where $[G-ANR_B]$ denotes the full subcategory of $[G-TOP_B]$ consisting of all $G$-spaces over $B$, which have the fiberwise $G$-homotopy type of some $G$-$ANR_B$ space.

In the realization of the outlined program the crucial tool is the notion of fiberwise $G$-resolution defined below.

Consider inverse systems $\mathbf{E} = (E_\lambda, p_{\lambda\lambda'}, \Lambda)$ in the category $\text{G-TOP}_B$. This means that every $E_\lambda$ is a $G$-space over $B$ and every $p_{\lambda\lambda'}: E_\lambda \to E_{\lambda'}$ is a $G$-map over $B$. If every $E_\lambda$ is a $G$-$ANR$ over $B$ we will say that $\mathbf{E}$ is a $G$-$ANR_B$-system.

We refer to [22, Ch. I, §§1,2] for definitions of basic terms (pro-category, expansion, dense subcategory, etc.).

If $E$ is a $G$-space over $B$, a morphism in pro $\text{G-TOP}_B$, $p: E \to \mathbf{E}$ consists of $G$-maps over $B$, $p_\lambda: E \to E_\lambda$ such that $p_{\lambda\lambda'} = p_{\lambda\lambda'} p_{\lambda\lambda'}$ for $\lambda \leq \lambda'$.

**Definition 4.1.** A morphism $p: E \to \mathbf{E}$ of pro $\text{G-TOP}_B$ is called a $G$-resolution of the $G$-space $E$ over $B$, provided for every $P \in G-ANR_B$ and every open covering $\omega$ of $P$ the following conditions are satisfied:

$(R_1)$ If $f: E \to P$ is a $G$-map over $B$, then there exist a $\lambda \in \Lambda$ and a $G$-map $h: E_\lambda \to P$ over $B$ such that $hp_{\lambda}$ and $f$ are $\omega$-near.

$(R_2)$ There is an open covering $\omega'$ of $P$ with the following property: whenever $\lambda \in \Lambda$ and $h_0, h_1: E_\lambda \to P$ are $G$-maps over $B$ such that $h_0 p_{\lambda}$ and $h_1 p_{\lambda}$ are $\omega'$-near, then there is a $\lambda' \geq \lambda$ such that $h_0 p_{\lambda'}$ and $h_1 p_{\lambda'}$ are $\omega$-near.

If each $E_\lambda$ is a $G$-$ANR_B$ then we say that $p$ is a $G$-$ANR_B$-resolution of $E$.

**Theorem 4.2.** Every $G$-space $E$ over $B$ admits a $G$-$ANR_B$-resolution $q: E \to \mathbf{E}$. 
PROOF. It follows from the proof of [7, Theorem 1] that $E$ admits an ordinary (not over $B$) $G$-$ANR$ resolution $r: E \to X = (X_\lambda, r_{\lambda\lambda}, \Lambda)$ which satisfies the following strongest condition instead of $(R_1)$ with $B$ the singleton:

$(R'_1)$ If $f: E \to L$ is a $G$-map in some $G$-$ANR$ space $L$ then there are an index $\lambda \in \Lambda$ and a $G$-map $h: X_\lambda \to L$ such that $hr_\lambda = f$.

(see also [20, the proof of Theorem 13, formula (8)]). For each $\lambda, \lambda' \in \Lambda$ with $\lambda \leq \lambda'$ define the $G$-maps $t_{\lambda}: E \to X_\lambda \times B$ and $s_{\lambda\lambda'}: X_{\lambda'} \times B \to X_\lambda \times B$ by putting $t_{\lambda}(x) = (r_{\lambda}(x), p(x))$ and $s_{\lambda\lambda'} = r_{\lambda\lambda'} \times id_B$, where $p: E \to B$ is the projection. Then $t_{\lambda}$ and $s_{\lambda\lambda'}$ become $G$-maps over $B$ if we consider each $X_\lambda \times B$ as a $G$-space over $B$ with the usual projection $X_\lambda \times B \to B$.

Let $M$ be the set of all pairs $\mu = (\lambda, U)$ where $\lambda \in \Lambda$ and $U$ is an invariant open neighborhood of $t_{\lambda}(E)$ in $X_\lambda \times B$. We order $M$ by putting $\mu \leq \mu' = (\lambda', U')$ whenever $\lambda \leq \lambda'$ and $s_{\lambda\lambda'}(U') \subseteq U$. For every $\mu = (\lambda, U) \in M$ let $E_\mu = U$, $q_\mu = t_{\lambda}: E \to U$ and $q_{\mu\mu'} = s_{\lambda\lambda'}|U': U' \to U$ if $\mu \leq \mu'$.

Clearly $E = (E_\mu, q_{\mu\mu'}, M)$ is an inverse system of $G$-$ANR$ spaces over $B$ and $q = (q_\mu): E \to E$ is a morphism of the category $\text{pro} - G$-$TOP_B$. We claim that $q$ satisfies both conditions $(R_1)$ and $(R_2)$.

For $(R_1)$ let $P$ be a $G$-$ANR$ over $B$ and $f: E \to P$ be a $G$-map over $B$. Then there are normed linear $G$-space $L$ such that $L \in G$-$AE$ and $L$ contains $P$ as a closed invariant subspace [2, Corollary 5 and Corollary 8]. This implies a closed equivariant embedding over $B$ of $P$ into $L \times B$. Since $P \in G$-$ANR_B$ there is an invariant neighborhood $V$ of $P$ in $L \times B$ and a $G$-retraction $\eta: V \to P$ over $B$. Let $\alpha: L \times B \to L$ denotes the first projection. As $L \in G$-$ANR$, according to the condition $(R'_1)$ there exist an index $\lambda \in \Lambda$ and a $G$-map $\phi: X_\lambda \to L$ such that

$$\phi r_\lambda = \alpha f.$$

Define the map $\Phi: X_\lambda \times B \to L \times B$ as the product $\phi \times id_B$ and put $U = \Phi^{-1}(V)$, $\mu = (\lambda, U)$. Then (3) implies $t_{\lambda}(E) \subseteq U$ and therefore $\mu \in M$. Now setting $h = \eta(\Phi|U)$ we obtain a $G$-map $h: E_\mu \to P$ such that $hq_\mu = f$ and $(R_1)$ is satisfied.

For $(R_2)$ consider an open covering $\omega$ of $P$. We claim that $\omega' = \omega$ has the desired property. Indeed, let $\mu = (\lambda, U) \in M$ and $h_0, h_1: U \to P$ be $G$-maps over $B$ such that $hq_\mu$ and $h_1q_\mu$ are $\omega$-near $G$-maps. For each $x \in E$ there is a $W_x \in \omega$ such that $h_0q_\mu(x)$, $h_1q_\mu(x) \in W_x$. By continuity of $h_0$ and $h_1$ there is an open neighborhood $O_{\mu}$ of $q_\mu(x)$ in $U$ such that $h_0(y), h_1(y) \in W_x$ for all $y \in O_{\mu}$. Then $O = \bigcup_{x \in E} O_{\mu}$ is an open neighborhood of $q_\mu(E)$ in $E_\mu = U$ and the maps $h_0|O$ and $h_1|O$ are $\omega$-near. Since $G$ is compact and $q_\mu(E)$ is an invariant subset of $U \cap O$ one can find an open invariant neighborhood $U'$ of $q_\mu(E)$ in $U \cap O$ [24, Proposition 1.1.14]. We now put $\mu' = (\lambda, U') \in M$. Note that $\mu \leq \mu'$ because $s_{\lambda\lambda'}(U') = U' \subseteq U$. The maps $h_0q_{\mu\mu'} = h_0|U'$ and $h_1q_{\mu\mu'} = h_1|U'$ are indeed $\omega$-near, because $U' \subseteq O$. This verifies condition $(R_2)$. 
The notion of a $G$-$\text{ANR}_B$-expansion is obtained by specializing the general categorical notion of expansion with respect to a category $\mathcal{T}$ and its subcategory $\mathcal{P}$ [22, Ch. I, §2]. In our case $\mathcal{T} = [G-\text{TOP}_B]$ and $\mathcal{P} = [G-\text{ANR}_B]$. So we have the following

**Definition 4.3.** A $G$-expansion over $B$ or a fiberwise $G$-expansion of a $G$-space $E$ over $B$ consists of an inverse system $[E] = (E_\lambda, [p_{\lambda\lambda'}], \Lambda)$ in $[G-\text{TOP}_B]$ and a morphism $[p] : E \to [E]$ in $\text{pro}-[G-\text{TOP}_B]$, i.e., a collection of fiberwise $G$-homotopy classes $[p_\lambda]$ of $G$-maps $p_\lambda : E \to E_\lambda$, $\lambda \in \Lambda$ over $B$ such that $p_{\lambda\lambda'}p_\lambda \simeq_G p_\lambda$ over $B$ for $\lambda \leq \lambda'$, satisfying the following two conditions:

1. If $P$ is a $G$-$\text{ANR}_B$ and $f : E \to P$ is a $G$-map over $B$ then there exist a $\lambda \in \Lambda$ and a $G$-map $h : E_\lambda \to P$ over $B$ such that $hp_\lambda \simeq_G f$ over $B$.
2. If $P$ is a $G$-$\text{ANR}_B$, $\lambda \in \Lambda$ and $h_0, h_1 : E_\lambda \to P$ are $G$-maps over $B$ satisfying $h_0 p_\lambda \simeq_G h_1 p_\lambda$ over $B$, then there is a $\lambda' \geq \lambda$ such that $h_0 p_{\lambda\lambda'} \simeq_G h_1 p_{\lambda\lambda'}$ over $B$.

A $G$-$\text{ANR}_B$-expansion $[p]$ is a $G$-expansion such that each $E_\lambda$ has the fiberwise $G$-homotopy type of some $G$-$\text{ANR}_B$.

Clearly every inverse system $E = (E_\lambda, p_{\lambda\lambda'}, \Lambda)$ in the category $G$-$\text{TOP}_B$ induces an inverse system $[E] = (E_\lambda, [p_{\lambda\lambda'}], \Lambda)$ in the category $[G-\text{TOP}_B]$. Moreover, every morphism $p = (p_\lambda, \Lambda) : E \to [E]$ in $\text{pro}-[G-\text{TOP}_B]$ induces a morphism $[p] = ([p_\lambda], \Lambda) : E \to [E]$ in $\text{pro}-[G-\text{TOP}_B]$.

In our development of equivariant fiberwise shape the next result is important.

**Theorem 4.4.** Let $E$ be a $G$-space over $B$. If $p : E \to [E]$ is a $G$-resolution over $B$, then the induced morphism $[p] : E \to [E]$ is a $G$-expansion over $B$.

The proof of this theorem proceeds in the same way as the proof of the analogous result in the case of ordinary shape (see the proof of [22, Ch. I §6.1, Theorem 2]) and of the equivariant one (see [7, Theorem 2]). However in the equivariant fiberwise case some specific difficulties arise. Below we show how to come over this difficulties. To be more rigorous and more complete, we repeat in our proof some of the arguments stated in the proof of [22, Ch. I, §6.1, Theorem 2].

We need the following three lemmas which are fiberwise analogues of [7, Proposition 3, Proposition 4 and Lemma 5].

First we recall that if $\omega$ is a covering of a space $Y$, then two maps $f, f' : X \to Y$ are said to be $\omega$-near provided every $x \in X$ admits a $V \in \omega$ such that $f(x), f'(x) \in V$. For a homotopy $F : X \times I \to Y$ we say that it is an $\omega$-homotopy provided every $x \in X$ admits a $V \in \omega$ such that $F(x \times I) \subseteq V$.

**Lemma 4.5.** Let $Y$ be a $G$-$\text{ANR}_B$. Then every open covering $\mathcal{U}$ of $Y$ admits an open covering $\mathcal{U}'$ of $Y$ such that whenever $f_0, f_1 : X \to Y$ are $\mathcal{U}'$-near $G$-maps over $B$ from an arbitrary $X \in G$-$\text{TOP}_B$, then there exists
an equivariant $U$-homotopy $F$ over $B$ from $f_0$ to $f_1$. Moreover, if for a given $x \in X$, $f_0(x) = f_1(x)$ then $F|x \times I$ is constant.

PROOF. By [2, Corollary 5] one can assume that $Y$ is a closed invariant subspace of a normed linear $G$-space $L$. If $p : Y \to B$ is the projection then the map $h : Y \to L \times B$ defined by $h(y) = (y, p(y))$ is a closed $G$-embedding over $B$. As $Y \in G$-$ANR_B$ and $L \times B$ is metric, there exists an open invariant neighborhood $D$ of $h(Y)$ in $L \times B$ and an equivariant retraction $r : D \to h(Y)$ over $B$. Let $\mathcal{V}$ be an open covering of $D$ which refines $r^{-1}(h(\mathcal{U}))$ and consists of sets of the form $V_i \times W_j$ where $V_i$ is an open ball from $L$ and $W_j$ is an open set from $B$. Put $\mathcal{V}' = \{(V_i \times W_j) \cap h(Y) : V_i \times W_j \in \mathcal{V}\}$. We claim that $\mathcal{U}' = h^{-1}(\mathcal{V}')$ has the desired property.

Let $f_0, f_1 : X \to Y$ be $\mathcal{U}'$-near $G$-maps over $B$ from an arbitrary $X \in G$-$TOP_B$ with projection $l : X \to B$. We define a $G$-homotopy $\Phi : X \times I \to L$ from $f_0$ to $f_1$ by putting

$$\Phi(x, t) = (1 - t)f_0(x) + tf_1(x), \quad (x, t) \in X \times I.$$  

As for every $x \in X$ there is an element $h^{-1}((V_i \times W_j) \times h(Y)) \in \mathcal{U}'$ which contains both $f_0(x)$ and $f_1(x)$, we see that $V_i$ contains both $f_0(x)$ and $f_1(x)$.

Since $V_i$ is convex, we conclude that $\Phi(x \times I) \subseteq V_i$. As $l(x) = p(f_0(x))$ is the second coordinate of $h(f_0(x))$, it then follows that $W_j$ contains $l(x)$, which implies that $(\Phi(x, t), l(x)) \in V_i \times W_j \subseteq D$ for all $t \in I$. However, $V_i \times W_j$ is contained in a set $r^{-1}(h(U))$ where $U \in \mathcal{U}$. Therefore the map $F : X \times I \to Y$ defined by $F(x, t) = h^{-1}r(\Phi(x, t), l(x))$ is a well-defined equivariant $U$-homotopy over $B$ from $f_0$ to $f_1$. Moreover, if $f_0(x) = f_1(x)$ then $\Phi|x \times I$ is constant and so is $F|x \times I$. $F$ is equivariant because $f_0, f_1$ and $r$ are equivariant and $G$ acts linearly on $L$.

LEMMA 4.6. Let $X$ be a metric $G$-space over $B$, let $A \subseteq X$ be a closed invariant subset and let $Y$ be a $G$-$ANR_B$. Moreover, let $f_0, f_1 : A \to Y$ be equivariant maps over $B$ and let $F : A \times I \to Y$ be an equivariant homotopy over $B$ from $f_0|A$ to $f_1|A$. Then there exist an invariant neighborhood $V$ of $A$ in $X$ and an equivariant homotopy $F : V \times I \to Y$ over $B$ from $f_0|V$ to $f_1|V$, which extends $F$.

PROOF. Straightforward.

LEMMA 4.7. Let $X \in G$-$TOP_B$, let $P, P' \in G$-$ANR_B$ and let $f : X \to P'$, $h_0, h_1 : P' \to P$ be $G$-maps over $B$ such that

$$h_0f \simeq_G hf \quad \text{over} \quad B.$$  

Then there exist a $P'' \in G$-$ANR_B$ and $G$-maps $f' : X \to P''$, $h : P'' \to P'$ over $B$ such that

$$hf' = f,$$  

(4.6)
Proof. By (4.5) there exists an equivariant homotopy \( Q : X \times I \to P \) over \( B \) from \( h_0 f \) to \( h_1 f \). Let \( C(I, P) \) be the \( G \)-space of all continuous maps \( \varphi : I \to P \) endowed with the compact-open topology and the \( G \)-action defined by \( (g \varphi)(t) = g(\varphi(t)) \), \( g \in G \), \( \varphi \in C(I, P) \), \( t \in I \). Consider the invariant subspace \( F(I, P) \) of \( C(I, P) \) consisting of all those maps \( \varphi : I \to P \) for which the composition \( p \varphi \) is a constant map, where \( p : P \to B \) is the projection. In what follows we will consider \( F(I, P) \) as a \( G \)-space over \( B \) equipped with the projection \( F_p : F(I, P) \to B \) defined by the formula \( F_p(\varphi) = p \varphi(0) \) (observe that \( p \varphi(0) = p \varphi(t) \) for all \( t \in I \)).

Let \( q : X \to F(I, P) \) be the \( G \)-map over \( B \) defined by \( q(x)(t) = Q(x, t) \), \( x \in X \), \( t \in I \). We now define \( P'' \subseteq P' \times F(I, P) \) by

\[
P'' = \{(y, \varphi) \in P' \times F(I, P) \mid \varphi(0) = h_0(y), \ \varphi(1) = h_1(y)\}.
\]

As in the proof of [7, Lemma 5], it can be checked that \( P'' \) is an invariant subset of \( P' \times F(I, P) \) and that \( f'(x) = (f(x), q(x)) \) defines a \( G \)-map \( f' : X \to P'' \). Moreover, \( f' \) also becomes a \( G \)-map over \( B \) if we define the projection \( l : P'' \to B \) by \( l(y, \varphi) = p'(y) \), where \( p' : P' \to B \) is the projection of \( P' \) (this follows easily from the fact that \( f \) is a \( G \)-map over \( B \)). Let \( h : P'' \to P' \) be the first cartesian projection. Then \( h \) is a \( G \)-map over \( B \) and (6) holds.

In order to verify (7) consider the \( G \)-homotopy \( H : P'' \times I \to P \) given by

\[
H((y, \varphi), t) = \varphi(t), \quad (y, \varphi) \in P'' \times I, \ t \in I.
\]

In the proof of [7, Lemma 5] it was shown that \( H \) is a \( G \)-homotopy from \( h_0 h \) to \( h_1 h \). Let us show that \( H \) is also a homotopy over \( B \).

Indeed, \( ph_H((y, \varphi), t) = p \varphi(t) \). By definition of \( F(I, P) \) we have \( p \varphi(t) = p \varphi(0) \) for all \( t \in I \) and \( \varphi(0) = h_0(y) \), implying \( ph_H((y, \varphi), t) = ph_0(y) \). As \( h_0 \) is a map over \( B \), we conclude that \( ph_0(y) = l(y) \), and hence \( ph_H((y, \varphi), t) = l(y) \). This verifies (7).

The proof of Lemma 4.7 will be completed if we show that \( P'' \) is a \( G \)-ANR\(_B \) or equivalently a \( G \)-ANE\(_B \).

Let \( Z \) be a metric \( G \)-space over \( B \), let \( A \subseteq Z \) be a closed invariant subset and let \( k : A \to P'' \) be an equivariant map over \( B \). We shall find an invariant neighborhood \( V \) of \( A \) in \( Z \) and an equivariant extension \( \tilde{k} : V \to P'' \) of \( k \) over \( B \). Denote by \( h' : P'' \to F(I, P) \) the second cartesian projection, which clearly is equivariant. Verify that \( h' \) is also a map over \( B \). Indeed, \( F_p h'(y, \varphi) = F_p(\varphi) = p \varphi(0) = ph_0(y) = l(y) \). Therefore \( h' k : A \to F(I, P) \) is a \( G \)-map over \( B \) and it induces an equivariant homotopy \( K : A \times I \to P \) over \( B \) defined by

\[
K(a, t) = (h' k(a))(t), \quad (a, t) \in A \times I.
\]

In [7, p. 221] it is shown that \( K \) is a \( G \)-homotopy from \( h_0 k \) to \( h_1 k \).
Since $P'$ is a $G$-$ANR_{B}$ and $hk : A \to P'$ is a $G$-map over $B$, there exist an invariant neighborhood $U$ of $A$ in $Z$ and an equivariant map $\tilde{k}' : U \to P'$ over $B$, which extends $hk$. One can now apply Lemma 4.6 to $h_{0}\tilde{k}'$, $h_{1}\tilde{k}'$ and $K$ and conclude that there exist an equivariant neighborhood $V$ of $A$ in $U$ and a $G$-homotopy $\tilde{K} : V \times I \to P$ over $B$ from $h_{0}\tilde{k}'|V$ to $h_{1}\tilde{k}'|V$.

Consider the $G$-map $\tilde{k}'' : V \to F(I, P)$ given by $\tilde{k}''(z)(t) = \tilde{K}(z, t)$. As $\tilde{K}$ is a homotopy over $B$, we see that $\tilde{k}''(z)$ really belongs to $F(I, P)$ and $\tilde{k}''$ is a map over $B$. It is also continuous, equivariant and extends $h'k$ [7, the proof of Lemma 5].

Now we define

$$\tilde{k} : V \to P' \times F(I, P) \quad \text{by} \quad \tilde{k}(z) = (\tilde{k}'(z), \tilde{k}''(z)), \quad z \in V,$$

which clearly is equivariant and extends $k$. As $\tilde{k}'$ and $\tilde{k}''$ both are maps over $B$, we conclude that $k$ is a map over $B$ too. Finally $k(z) \in P''$ for every $z \in V$ because

$$\tilde{k}''(z)(0) = \tilde{K}(z, 0) = h_{0}\tilde{k}'(z)$$

and

$$\tilde{k}''(z)(1) = \tilde{K}(z, 1) = h_{1}\tilde{k}'(z).$$

This completes the proof of Lemma 4.7.

**Proof of Theorem 4.4.** We must verify conditions $(E_{1})$ and $(E_{2})$ of Definition 4.3

$(E_{1})$. Let $P \in G$-$ANR_{B}$ and let $f : E \to P$ be a $G$-map over $B$. By Lemma 4.5 we can choose an open covering $\omega$ of $P$ such that any two $\omega$-near $G$-maps over $B$ into $P$ are $G$-homotopic over $B$. By property $(R_{1})$ there are a $\lambda \in \Lambda$ and a $G$-map $h : E_{\lambda} \to P$ over $B$, such that $hp_{\lambda}$ and $f$ are $\omega$-near maps and therefore $hp_{\lambda} \simeq_{G} f$ over $B$.

$(E_{2})$. Let $P \in G$-$ANR_{B}$, $\lambda \in \Lambda$ and let $h_{0}$, $h_{1} : E_{\lambda} \to P$ be $G$-maps over $B$ satisfying

$$h_{0}p_{\lambda} \simeq_{G} h_{1}p_{\lambda} \quad \text{over} \quad B. \tag{4.8}$$

We must find a $\lambda' \geq \lambda$ such that

$$h_{0}p_{\lambda'} \simeq_{G} h_{1}p_{\lambda'} \quad \text{over} \quad B. \tag{4.9}$$

Again by Lemma 4.5 we can choose an open covering $\omega$ of $P$ such that any two $\omega$-near $G$-maps over $B$ into $P$ are $G$-homotopic over $B$. Choose $\omega'$ according to $(R_{2})$. Consider the fiber product (pull-back) $P \times_{B} P \subseteq P \times P$ which naturally becomes a $G$-space over $B$. The maps $h_{0}p_{\lambda}$ and $h_{1}p_{\lambda}$ determine a $G$-map $f : E \to P' = P \times_{B} P$ over $B$ such that

$$g_{0}f = h_{0}p_{\lambda} \quad \text{and} \quad g_{1}f = h_{1}p_{\lambda}. \tag{4.10}$$
where \(g_0, g_1 : P \times_B P \to P\) are the projections (which are \(G\)-maps over \(B\)) on the first and second factor respectively. By (8) and (10) we have

\[
\text{(4.11)} \quad g_0f \simeq_G g_1f \quad \text{over} \quad B.
\]

Using the property \(P \in G\text{-ANR}_B\), it can be easily proved (see [18, p. 240]) that then \(P' = P \times_B P\) is a \(G\text{-ANR}_B\). By Lemma 4.7 there is a \(P'' \in G\text{-ANR}_B\) and there are \(G\)-maps over \(B\), \(f' : E \to P''\) and \(h : P'' \to P'\) such that

\[
\text{(4.12)} \quad hf' = f \quad \text{and} \quad g_0h \simeq_G g_1h \quad \text{over} \quad B.
\]

Let \(\omega''\) be an open covering of \(P''\) which refines the coverings \((g_0h)^{-1}(\omega')\) and \((g_1h)^{-1}(\omega')\). Applying the property \((R_1)\) we find a \(\lambda'' \in \Lambda\) and a \(G\)-map \(\psi : E_{\lambda''} \to P''\) over \(B\) such that \(\psi p_{\lambda''}\) and \(f'\) are \(\omega''\)-near. Clearly, one can assume that \(\lambda'' \geq \lambda\).

Consequently, \(g_0h\psi p_{\lambda''}\) and \(g_0hf'\) are \(\omega''\)-near maps. However, by (10) and (12) we have

\[
\text{(4.13)} \quad g_0hf' = g_0f = h_0p_\lambda.
\]

so that \(g_0h\psi p_{\lambda''}\) and \(h_0p_{\lambda''} p_{\lambda''}\) are \(\omega''\)-near maps. Therefore, by \((R_1)\) there is an index \(\lambda_0 \geq \lambda''\) such that the maps \(g_0h\psi p_{\lambda'\lambda_0}\) and \(h_0p_{\lambda\lambda_0}\) are \(\omega\)-near and thus

\[
\text{(4.14)} \quad g_0h\psi p_{\lambda'\lambda_0} \simeq_G h_0p_{\lambda\lambda_0} \quad \text{over} \quad B.
\]

Similarly there is an index \(\lambda_1 \geq \lambda''\) such that

\[
\text{(4.15)} \quad g_1h\psi p_{\lambda'\lambda_1} \simeq_G h_1p_{\lambda\lambda_1} \quad \text{over} \quad B.
\]

Now let \(\lambda' \geq \lambda_0, \lambda_1\). Then we have by (14), (12) and (15)

\[
\text{which is the desired homotopy (9).}
\]

Theorem 4.2 and Theorem 4.4 immediately imply the following

**Corollary 4.8.** Every \(G\)-space \(E\) over \(B\) admits a \(G\text{-ANR}_B\)-expansion, i.e., \([G\text{-ANR}_B]\) is a dense subcategory of \([G\text{-Top}_B]\).

We will now define the \(G\)-shape category over \(B\), denoted by \(G\text{-Sh}_B\), as a shape category \(SH(T, P)\) with \(T = [G\text{-Top}_B], P = [G\text{-ANR}_B]\). One also has a \(G\)-shape functor over \(B\), \(G\text{-Sh}_B : [G\text{-Top}_B] \to G\text{-Sh}_B\). When \(B\) is a one-point \(G\)-space, clearly \(G\text{-Sh}_B\) is naturally isomorphic to \(G\text{-SH}\) the \(G\)-shape category constructed in [7] (see also [3]). In the case of \(G\) the trivial group and \(E\) a metric spaces over a given metric base \(B\), the fiberwise shape category \(SH_B\) was considered by T. Yagasaki [26]. Using the method of resolutions, V.H. Baladze [8] later constructed a fiberwise shape category for arbitrary spaces over a given base \(B\). Fiberwise shape of compact metric
spaces over a compact metric base has been previously considered by H. Kato [19] and by M. Clapp and L. Montejano [10].

**Theorem 4.9.** Let $G\text{-}Sh_B: G\text{-}TOP_B \to G\text{-}SH_B$ denotes the $G$-shape functor over the base $B$ and let $\pi: G\text{-}TOP_B \to G/N\text{-}TOP_B / N$ denotes the $N$-orbit functor for any closed normal subgroup $N \subseteq G$. Suppose that $B$ is a $G$-ANE over $B/N$. Then there is a unique functor $\mu: G\text{-}SH_B \to G/N\text{-}SH_B / N$ such that $\mu \circ G\text{-}Sh_B = G/N\text{-}Sh_B \circ \pi$.

For the proof we need the following propositions:

**Proposition 4.10.** Let $N \subseteq G$ be a closed normal subgroup and let $[p] = ([p_x]): E \to E = \{\xi, [p_x]\}, \Lambda$ be a fiberwise $G$-expansion of $E$ over $B$. Then using the notations of \S 2, $[p/N] = ([p_x/N]): E/N \to E/N = (E_x/N, [p_x/N], \Lambda)$ is a fiberwise $G/N$-expansion of $E/N$ over $B/N$.

**Proof.** In this proof we will shortly denote by $\varphi'$ the map $\pi(\varphi) = \varphi/N$ induced by a $G$-map $\varphi$ (see \S 2).

We must check the conditions (E1) and (E2).

\begin{enumerate}
  \item [(E1):] For let $\pi: E/N \to P$ be a $G/N$-map over $B/N$ where $P$ is a $G/N$-ANR$_B/N$. Let $r': E \to B$, $r': E/N \to B/N$, $\alpha: P \to B/N$, $\beta_B: E \to E/N$ and $\beta_B': B \to B/N$ be the involving maps and projections. We have $r'\beta_E = \beta_B r$ and $\alpha \pi = r'$. Now consider $Q \subseteq P \times B$ the pull-back of maps $\alpha: P \to B/N$ and $\beta_B: B \to B/N$. If $\kappa_P: Q \to P$ and $\kappa_B: Q \to B$ are the natural projections, then $\alpha \kappa_P = \beta_B \kappa_B$. If we consider the $G/N$-spaces $P$ and $B/N$ as $G$-spaces then the $G/N$-map $\alpha$ can be regarded as a $G$-map (see Remark 2.1), and therefore $Q$ naturally becomes a $G$-space over $B$ with the projection $\kappa_B$. It is well-known [18, p. 240] and easy to prove that $Q$ is a $G$-ANR$_B$. Define the map $h: E \to Q$ by putting $h(x) = (\kappa \beta_E(x), r(x))$. One easily verifies that $h$ is a well-defined $G$-map over $B$. Since $[p]$ is a $G$-expansion over $B$, there exist $\lambda \in \Lambda$ and a $G$-map $f_1: E_{\lambda} \to Q$ over $B$ such that $fp_{\lambda} \simeq G h$ over $B$. Then $f_1$ induces a $G/N$-map $f': E_{\lambda}/N \to Q/N$ over $B/N$. Denote by $\xi_{\lambda}: E_{\lambda} \to E_{\lambda}/N$ the $N$-orbit projection, then $\kappa_P f = f' \xi_{\lambda}$. Clearly, the property $fp_{\lambda} \simeq G h$ over $B$ implies $f'p_{\lambda}' \simeq G/N h$ over $B/N$. As $Q/N = P$ (see [15, \S 4.1]) we conclude that (E1) holds.

\item [(E2):] Let $\overline{\pi}_{\lambda}, \overline{\pi}'_{\lambda}: E_{\lambda}/N \to P$ be $G/N$-maps over $B/N$ such that

$$\overline{\pi}_{\lambda} = \overline{\pi}'_{\lambda} \simeq G/N \overline{\pi}_{\lambda}'$$

over $B/N$, where $P$ is an arbitrary $G/N$-ANR$_{B/N}$ with the projection $\alpha: P \to B/N$.

Now let $Q$ be the pull-back of maps $\alpha: P \to B/N$ and $\beta_B: B \to B/N$, as before $Q$ is a $G$-ANR$_B$. For $i = 0, 1$ define the $G$-map $f_i: E\lambda \to Q$ by $f_i(x) = (\overline{\pi}_{\lambda}(x), s_{\lambda}(x))$, $x \in E_{\lambda}$, where $s_{\lambda}: E_{\lambda} \to B$ is the projection. One easily verifies that $f_i$ is a $G$-map over $B$. We claim that $f_0p_{\lambda} \simeq G f_1p_{\lambda}$ over $B$. Indeed, let $F_t: E\lambda \to P$, $0 \leq t \leq 1$ be a $G/N$-homotopy over $B/N$ from $\overline{\pi}_{\lambda}$ to $\overline{\pi}'_{\lambda}$ (see [16]). This $G$-homotopy can be lifted to a $G$-homotopy $\Phi_t :
$E \to Q$ over $B$ by putting $\Phi_i(y) = (F_i\beta_E(y), r(y))$ for all $y \in E$. Let us check that $\Phi_i = f_ip_{\lambda i}$, $i = 0, 1$. In fact $\Phi_i(y) = (F_i\beta_E(y), r(y)) = (f_ip_{\lambda i}\beta_E(y), r(y))$. Since $p_{\lambda i}\beta_E = \xi_ip_{\lambda i}$ and $r = s_{\lambda i}p_{\lambda i}$, we have $\Phi_i(y) = (f_i\xi_ip_{\lambda i}(y), s_{\lambda i}p_{\lambda i}(y)) = f_ip_{\lambda i}(y)$, i.e., $\Phi_i = f_ip_{\lambda i}$.

Hence one can apply the property $(E_2)$ to $p: E \to E$. Since $Q$ is a $G$-ANR$_B$ there exists an index $\mu \geq \lambda$ such that

$$f_0p_{\lambda \mu} \simeq_G f_1p_{\lambda \mu} \text{ over } B.$$  

This $G$-homotopy induces a $G/N$-homotopy $\overline{f_0p_{\lambda \mu}} \simeq_{G/N} \overline{f_1p_{\lambda \mu}} \text{ over } B/N$ (note that $Q/N = P$). This verifies $(E_2)$ and completes the proof.

In a similar way one can prove the following

**Proposition 4.11.** Let $N \subseteq G$ be a closed normal subgroup and let $p = (p_{\lambda})$ be a fiberwise $G$-resolution of $E$ over $B$. Then with the notations of $(2), (p/N): E/N \to E/N = (E/N, N/N, \Lambda)$ is a fiberwise $G/N$-resolution of $E/N$ over $B/N$.

Theorem 3.1 and Proposition 4.10 immediately imply the following

**Proposition 4.12.** Let $N \subseteq G$ be a closed normal subgroup and let $p: E \to E = (E_{\lambda}, [p_{\lambda}], \Lambda)$ be a fiberwise $G$-ANR$_B$-expansion of $E$ over $B$. Suppose that $B$ is a $G$-ANE over $B/N$. Then $p/N = ([p/N]): E/N \to E/N = (E/N, N/N, \Lambda)$ is a $G/N$-ANR$_B/N$-expansion of $E/N$ over $B/N$.

**Proof of Theorem 4.9.** According to [22, Ch. I, §2.3] the objects of $G$-SH$_B$ are $G$-spaces over $B$ and the morphisms of $G$-SH$_B$ between $G$-spaces $E$ and $F$ over $B$ are given by triples $([p], [q], [f])$ where $[p]: E \to [E]$ and $[q]: F \to [F]$ are $G$-ANR$_B$-expansions of $E$ and $F$ respectively, and $[f]: [E] \to [F]$ is a morphism of $pro-[G$-TOP$_B]$ (see [22, Ch. I, §1.1]). In particular, one can take for $[p]$ and $[q]$ morphisms induced by $G$-ANR$_B$-resolutions $p$ and $q$ (Theorems 4.2 and 4.4). According to Proposition 4.12, $[p/N]: E/N \to E/N$ and $[q/N]: F/N \to E/N$ are $G$-ANR$_B/N$-expansions.

Let $[f] = (f, [f_{\alpha}])$ where $f: A \to \Lambda$ is a function and $f_{\alpha}: E_{\phi(\alpha)} \to F_{\alpha}$ is a $G$-map. Clearly $[f/N] = ([\phi, [f_{\alpha}/N]])$ is a morphism of $pro-[G/N$-TOP$_B/N]$.

Now we define $\mu: G$-SH$_B \to G/N$-SH$_B/N$ by putting $\mu(E) = E/N$ for objects of $G$-SH$_B$ and $\mu([p], [q], [f]) = ([p/N], [q/N], [f/N])$ for morphisms of $G$-SH$_B$. One easily verifies (by virtue of Proposition 4.12) that $\mu$ is the desired functor. The uniqueness of $\mu$ is also easy to check.

**Theorem 4.9** has the following immediate corollaries:

**Corollary 4.13.** Let $N \subseteq G$ be a closed normal subgroup and $E$, $F$ be $G$-spaces over $B$ with $G$-SH$_B(E) = G$-SH$_B(F)$. Suppose that $B$ is a $G$-ANE over $B/N$. Then we have $G/N$-SH$_B/N(E/N) = G/N$-SH$_B/N(F/N)$. 
In particular $\text{Sh}_{B/G}(E/G) = \text{Sh}_{B/G}(F/G)$ whenever $B$ is a $G$-ANE over $B/G$.

**Corollary 4.14.** Let $N \subseteq G$ be a closed normal subgroup, $E$ and $F$ be any $G$-spaces. If $G$-$\text{Sh}(E) = G$-$\text{Sh}(F)$ then $G/N$-$\text{Sh}(E/N) = G/N$-$\text{Sh}(F/N)$. In particular $\text{Sh}(E/G) = \text{Sh}(F/G)$.

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