AN IMPROVED INEQUALITY FOR $k$-TH DERIVATIVE OF A POLYNOMIAL

V. K. Jain
I. I. T., Khavagpur, India

ABSTRACT. For a polynomial $p(z)$ of degree $n$, we have obtained

$$|p^{(k)}(\beta)| \leq \frac{n(n-1)(n-2)\ldots(n-k+1)}{R^k} \left( \frac{1}{2\pi} \left( \max_{1 \leq t \leq n} |p(\beta t)| \right) + \max_{1 \leq t \leq 2n} |p(\beta t)| \right), \beta \neq 0 \& k \geq 1,$$

a refinement of the well known Bernstein’s inequity

$$\max_{|z|=1} |p^{(k)}(z)| \leq n(n-1)(n-2)\ldots(n-k+1) \max_{|z|=1} |p(z)|,$$

$z_1, z_2, \ldots, z_n$ being the zeros of $z^n + 1$ and $b_1, b_2, \ldots, b_{2n}$ the zeros of $z^{2n} - 1$.

The inequality is sharp.

1. Introduction and statement of results

Let $p(z)$ be a polynomial of degree $n$. It easily follows from the well known Bernstein’s theorem [3] that

$$\max_{|z|=R} |p'(z)| \leq \frac{n}{R} \max_{|z|=R} |p(z)|, R > 0,$$

with equality in (1.1) for $p(z) = \alpha z^n$. On applying inequality (1.1) again and again, we get

$$\max_{|z|=R} |p^{(k)}(z)| \leq \frac{n(n-1)\ldots(n-k+1)}{R^k} \max_{|z|=R} |p(z)|, \quad R > 0 \& k \geq 1,$$

with equality in (1.2) for $p(z) = \alpha z^n$.

We have been able to improve the inequality (1.2) and obtain a new inequality, which is sharp. More precisely, we prove

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Theorem 1.1. Let \( p(z) \) be a polynomial of degree \( n \). Then
\[
|p^{(k)}(\beta)| \leq \frac{n(n-1)\ldots(n-k+1)}{|\beta|^k} \left[ \frac{1}{2k} \left| \frac{1}{|p(\beta)|} \right| + \max_{1 \leq i \leq n} |p(\beta z_i)| \right]
\]
\[
(1.3) + \left( 1 - \frac{1}{2k-1} \right) \max_{1 \leq i \leq 2n} |p(\beta b_i)|, \beta \neq 0, \quad \& \quad k \geq 1,
\]
where \( z_1, z_2, \ldots, z_n \) are the zeros of \( z^n + 1 \) and \( b_1, b_2, \ldots, b_{2n} \) are the zeros of \( z^{2n} - 1 \). The inequality is sharp and the extremal polynomial is \( p(z) = \alpha z^n \).

2. Lemmas

For the proof of Theorem 1.1, we require the following lemmas.

Lemma 2.1. Let \( p(z) \) be a polynomial of degree \( n \) and \( z'_1, z'_2, \ldots, z'_n \) be the zeros of \( z^n + a, a \neq 0 \). Then for any complex number \( \beta \) such that \( \beta^n + a \neq 0 \), we have
\[
p'(\beta) = \frac{n \beta^{n-1}}{a + \beta^n} p(\beta) + \frac{\beta^n}{a} \sum_{i=1}^{n} \frac{p(z'_i)}{z'_i - \beta} \frac{z''_i}{(z'_i - \beta)^2}
\]
\[
(2.4) + \frac{1}{na} \sum_{i=1}^{n} \frac{z''_i \beta}{(z'_i - \beta)^2} = -\frac{n \beta^n}{(\beta^n + a)^2}
\]
This lemma is due to Aziz [1].

Lemma 2.2. Let \( p(z) \) be a polynomial of degree \( n \). Then
\[
(2.6) \max_{|z|=1} |p'(z)| \leq n \max_{1 \leq i \leq 2n} |p(b_i)|,
\]
where \( b_1, b_2, \ldots, b_{2n} \) are as in Theorem 1.1.

This lemma is due to Frappier, Rahman and Ruscheweyh [2].

Lemma 2.3. Let \( p(z) \) be a polynomial of degree \( n \). Then for \( s \geq 1 \) and \( |\beta| = 1 \),
\[
(2.7) |p^{(s)}(\beta)| \leq \frac{n-s+1}{2} \left| \frac{1}{p^{(s-1)}(\beta)} \right| + \max_{1 \leq m \leq n-s+1} |p^{(s-1)}(u'_m)|,
\]
where \( u'_1, u'_2, \ldots, u'_{n-s+1} \) are the roots of
\[
(2.8) z^{n-s+1} + a = 0
\]
\[
(2.9) a = e^{i\gamma(n-s+1)}
\]
and
\[
\gamma = \arg \beta
\]
Proof of Lemma 2.3. As 
\[ \beta^{n-s+1} + a \neq 0, \]
we have on applying Lemma 2.1 to the polynomial \( p^{(s-1)}(z) \),
\[
p^{(s)}(\beta) = \frac{(n-s+1)\beta^{n-s} - \beta^{n-s+1} + a}{\beta^{n-s+1} + a} p^{(s-1)}(\beta) + \frac{\beta^{n-s+1} + a}{(n-s+1) \beta^{n-s+1} + a} \sum_{m=1}^{n-s+1} p^{(s-1)}(u'_m) \frac{u'_m}{(u'_m - \beta)^2},
\]
which implies
\[
(2.10) \quad |p^{(s)}(\beta)| \leq \frac{n-s+1}{2} |p^{(s-1)}(\beta)| + \frac{2}{n-s+1} \left\{ \sum_{m=1}^{n-s+1} \left| \frac{u'_m}{(u'_m - \beta)^2} \right| \right\}.
\]

By the second part of Lemma 2.1, (2.10) can be written as
\[
(3.11) \quad T(z) = p(\beta z).
\]

Then
\[
(3.12) \quad |p^{(k)}(\beta)| = \frac{1}{|\beta|^k} |T^{(k)}(1)|
\]
and this completes the proof of Lemma 2.3.

3. Proof of Theorem 1.1
Now by Lemma 2.3, we have for $k \geq 2$.

\begin{equation}
|T^{(k)}(1)| \leq \frac{n-k+2}{2}(|T^{(k-1)}(1)| + \max_{1 \leq m \leq n-k+1}|T^{(k-1)}(u_m)|)
\end{equation}

where $u_1, u_2, \ldots, u_{n-k+1}$ are the roots of $z^{n-k+1} + 1 = 0$. Again by Lemma 2.3, we have

\begin{equation}
|T^{(k-1)}(1)| \leq \frac{n-k+2}{2}(|T^{(k-2)}(1)| + \max_{1 \leq j \leq n-k+2}|T^{(k-2)}(w_j)|),
\end{equation}

where $w_1, w_2, \ldots, w_{n-k+2}$ are the roots of $z^{n-k+2} + 1 = 0$. Combining (3.13) and (3.14), we obtain

\begin{align*}
|T^{(k)}(1)| &\leq \frac{n-k+1}{2} \frac{n-k+2}{2} |T^{(k-2)}(1)| + \\
&\quad \frac{n-k+1}{2} \frac{n-k+2}{2} \max_{1 \leq j \leq n-k+2} |T^{(k-2)}(w_j)| + \\
&\quad \frac{n-k+1}{2} \max_{1 \leq m \leq n-k+1} |T^{(k-1)}(u_m)|.
\end{align*}

Continuing similarly, we obtain for $k \geq 2$

\begin{align*}
|T^{(k)}(1)| &\leq \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n}{2} |T(1)| + \\
&\quad \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n}{2} \max_{1 \leq i \leq n} |T'(x_i)| + \\
&\quad \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n}{2} \max_{1 \leq h \leq n-1} |T''(x_h)| + \\
&\quad \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n}{2} \max_{1 \leq g \leq n-2} |T'''(d_g)| + \cdots + \\
&\quad \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n}{2} \max_{1 \leq \ell \leq n-k+2} |T^{(k-2)}(w_j)| + \\
&\quad \frac{n-k+1}{2} \max_{1 \leq m \leq n-k+1} |T^{(k-1)}(u_m)|,
\end{align*}

where $z_1, z_2, \ldots, z_n$ are the roots of $z^n + 1 = 0$, $x_1, x_2, \ldots, x_{n-1}$ are the roots of $z^{n-1} + 1 = 0$, $d_1, d_2, \ldots, d_{n-2}$ are the roots of $z^{n-2} + 1 = 0$, and so on.

Now, by Lemma 2.2 and Bernstein’s theorem [3], we have from (3.15), for $k \geq 2$

\begin{align*}
|T^{(k)}(1)| &\leq \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n}{2} |T(1)| + \\
&\quad \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n}{2} \max_{1 \leq i \leq n} |T'(z_i)|
\end{align*}
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\[ + \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n-1}{2} \max_{1 \leq t \leq 2n} |T(b_t)| + \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n-2}{2} (n-1)n \max_{1 \leq t \leq 2n} |T(b_t)| \]

\[ \cdots + \frac{n-k+1}{2} \frac{n-k+2}{2} (n-k+3) \cdots n \max_{1 \leq t \leq 2n} |T(b_t)| + \frac{n-k+1}{2} (n-k+2)(n-k+3) \cdots n \max_{1 \leq t \leq 2n} |T(b_t)| \]

\[ = (n-k+1)(n-k+2) \cdots (n-1)n \left[ \frac{1}{2k} (1) + \max_{1 \leq t \leq n} |T(z_t)| \right] \]

(3.16) \[ + (1 - \frac{1}{2k-1} ) \max_{1 \leq t \leq 2n} |T(b_t)| \]

Further, for \( k = 1 \), we have by Lemma 2.3

\[ |T'(1)| \leq \frac{n}{2} (\max_{1 \leq z \leq n} |T(z)|), \]

On combining (3.16) and (3.17), we get for \( k \geq 1 \)

\[ |T^{(k)}(1)| \leq (n-k+1)(n-k+2) \cdots n \left[ \frac{1}{2k} (1) + \max_{1 \leq t \leq n} |T(b_t)| \right] \]

(3.18) \[ + \max_{1 \leq z \leq n} |T(z)| \]

(3.19) \[ + (1 - \frac{1}{2k-1} ) \max_{1 \leq t \leq 2n} |T(b_t)| \]

which, by (3.11) & (3.12), implies

\[ |p^{(k)}(\beta)| \leq \frac{1}{|\beta|^k} (n-k+1)(n-k+2) \cdots n \left[ \frac{1}{2k} (|p|) \right] \]

(3.20) \[ + \max_{1 \leq z \leq n} |p(\beta z)| \]

(3.21) \[ + (1 - \frac{1}{2k-1} ) \max_{1 \leq t \leq 2n} |p(b_t)| \]

thereby proving Theorem 1.1.

References


Mathematics Department
I.I.T., Kharagpur 721302
India.

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