CHARACTERIZATIONS OF FINITE ABELIAN AND MINIMAL NONABELIAN GROUPS

Yakov Berkovich
University of Haifa, Israel

In memory of Semen L. Gramm (1916-2009)

Abstract. In this note we present the following characterizations of finite abelian and minimal nonabelian groups: (i) A group $G$ is abelian if and only if $G' = \Phi(G)'$. (ii) A group $G$ is either abelian or minimal nonabelian if and only if $\Phi(G)' = H'$ for all maximal subgroups $H$ of $G$.

We also prove a number of related results.

In this note $G$ is a nonidentity finite group and $p, q$ are distinct primes. Our notation is standard for finite group theory (see [3] and [6]).

In what follows we use freely some known properties of $\Phi$-subgroups ([7], see also [3, §1]). For example, if $M \triangleleft G$, then $\Phi(M) \leq \Phi(G)$, and $G$ is nilpotent if and only if $G' \leq \Phi(G)$ which is equivalent to normality of all maximal subgroups in $G$ (Wielandt; see Lemma J(d)). It follows from Schur-Zassenhaus’ theorem that orders $|G|$ and $|G/\Phi(G)|$ have the same prime divisors (see Lemma J(g)). If $P \in \text{Syl}_p(G)$ is $G$-invariant, then $P \cap \Phi(G) = \Phi(P)$ ([1]; see Lemma J(a) and its proof following Lemma J (for more general result where $P$ is a normal Hall subgroup of $G$, see [2, Theorem 3.1]). We also use the Miller-Moreno-Redei description of minimal nonabelian groups (see Lemma J(b,c); for proofs, see [3, Exercise 1.8a] and [6, Lemma 11.2]). For example, if $G$ is a minimal nonabelian group, then $\Phi(G)$ is primary cyclic if $G$ is nonnilpotent, and $G$ is prime-power with $|G'| = p$ if $G$ is nilpotent.

2010 Mathematics Subject Classification. 20D15.

Key words and phrases. Maximal subgroup, abelian, minimal nonabelian, minimal nonnilpotent and Frobenius groups, Frattini subgroup, derived subgroup.
If $G$ is abelian, then $G' = \Phi(G)'$ and it appears that this property characterizes abelian groups (Theorem 1). If $G$ is either abelian or minimal nonabelian, then $H' = \Phi(G)'$, and Theorem 3 shows that this property is characteristic for groups all of whose maximal subgroups are abelian.

Let $\Gamma_1$ (or $\Gamma_1'$) be the set of maximal (nonnormal maximal) subgroups of a group $G$. A group $G$ is not nilpotent if and only if it the set $\Gamma_1'$ is not empty (Lemma J(d)).

In Lemma J we collected most known results cited in what follows so that our note is self contained modulo Lemma J.

**Lemma J.** Let $G$ be a finite group.

(a) R. Baer ([1]). If $P \in \text{Syl}(G)$ is $G$-invariant, then $\Phi(P) = P \cap \Phi(G)$.

(b) (Redei; see [3, Exercise 1.8a].) If $G$ is a nilpotent minimal nonabelian group, then $G$ is a $p$-group, $|G'| = p, Z(G) = \Phi(G)$ is of index $p^2$ in $G$ and one of the following holds:

(i) $p = 2$ and $G$ is the ordinary quaternion group,

(ii) $G = \langle a, b \mid a^{p^n} = b^{p^n} = 1, a^b = a^{1+p^m-1}, m > 1 \rangle$ is metacyclic of order $p^{m+n}$.

(iii) $G = \langle a, b \mid a^{p^n} = b^{p^n} = c^3 = 1, c = [a, b], [a, c] = [b, c] = 1 \rangle$ is nonmetacyclic of order $p^{m+n+1}$.

If $\exp(G) = p$, then $p > 2$ and $|G| = p^3$.

(c) (Miller-Morone; see [3, Lemma 10.8]) If $G$ is a nonnilpotent minimal nonabelian group, then $|G| = p^6 q^5, G = P \cdot Q$, where $P \in \text{Syl}_7(G)$ is cyclic, $Q = G' \in \text{Syl}_9(G)$ is a minimal normal subgroup of $G, Z(G) = \Phi(G)$ has index $p$ in $P$.

(d) (Wielandt) A group $G$ is nilpotent if and only if $G' \leq \Phi(G)$ (or, what is equivalent, $G/\Phi(G)$ is nilpotent).

(e) If $G$ is nilpotent and noncyclic, then $G/G'$ is also noncyclic.

(f) ([7]) If a Sylow $p$-subgroup is normal in $G/\Phi(G)$, then a Sylow $p$-subgroup is normal in $G$. If $H$ is normal in $G$, then $\Phi(H) \leq \Phi(G)$.

(g) $\pi(G/\Phi(G)) = \pi(G)$, where $\pi(G)$ is the set of prime divisors of the order of $G$.

(h) ([4]) If $N_F(F \cap H) \neq F \cap H \neq N_H(F \cap H)$ for any two distinct $F, H \in \Gamma_1$, then either $G$ is nilpotent or $|G/\Phi(G)|$ is (nonnilpotent) minimal nonabelian.

(i) If $G$ is a nonabelian $p$-group such that $G' \leq Z(G)$ has exponent $p$, then $\Phi(G) \leq Z(G)$.

(j) (Fitting; see [3, corollary 6.5]) Let $Q$ be a normal abelian Sylow $q$-subgroup of a group $G$. Then $Q \cap Z(G)$ is a direct factor of $G$.

Let us prove Lemma J(a). Since $\Phi(P) \leq D = \Phi(G) \cap P$ (Lemma J(f)), it suffices to prove the reverse implication. To this end, one may assume that $\Phi(P) = \{1\}$. Let $H$ be a $p'$-Hall subgroup of $G$. Then $P = D \times L$, where $L$
is $H$-admissible (Maschke). In that case, $G = (HL) \cdot D$, a semidirect product with kernel $D$, so $D = \{1\}$ since $D \leq \Phi(G)$. Thus, $\Phi(P) = \Phi(G) \cap P$.

Let us prove Lemma J(i). Take $x, y \in G$. Since $G$ is of class 2, we have $1 = [x, y]^p = [x, y^p]$ so that $U_1(G) \leq Z(G)$, and we obtain $\Phi(G) = G^U U_1(G) \leq Z(G)$.

**Theorem 1.** The following conditions for a group $G$ are equivalent:

(a) $G' = \Phi(G)'$.
(b) $G$ is abelian.

**Proof.** Obviously, (b) $\Rightarrow$ (a) so it remains to prove the reverse implication. We have $G' = \Phi(G)' \leq \Phi(G)$ so $G$ is nilpotent (Lemma J(d)); then $G = P_1 \times \cdots \times P_k$, where $P_1, \ldots, P_k$ are Sylow subgroups of $G$. We have

1. \[ \Phi(G) = \Phi(P_1) \times \cdots \times \Phi(P_k), \]
2. \[ G' = P_1' \times \cdots \times P_k', \]
3. \[ \Phi(G)' = \Phi(P_1)' \times \cdots \times \Phi(P_k)'. \]

Equality (1) follows from Lemma J(a) and (2), (3) are known consequences of (1). Since $\Phi(G)' = G'$, it follows from (2) and (3) that $\Phi(P_i)' = P_i'$ so $P_i$ satisfies the hypothesis for $i = 1, \ldots, k$. To complete the proof, it suffices to show that $P_i$ is abelian for $i = 1, \ldots, k$, so one may assume that $k = 1$, i.e., $G$ is a $p$-group. Assume that $G$ is nonabelian of the least possible order. We get $\Phi(G)' = G' > \{1\}$ so that $\Phi(G)$ is nonabelian. Let $R < \Phi(G)'$ be $G$-invariant of index $p$. Since $R < \Phi(G)' = G' < \Phi(G)$, we get

\[ \Phi(G/R)' = (\Phi(G)/R)' = \Phi(G)' / R = G' / R = (G/R)', \]

whence the nonabelian group $G/R$ satisfies the hypothesis so we must have $R = \{1\}$, by induction. In that case, $|\Phi(G)| = |G'| = p$ so $G' \leq Z(G)$. By Lemma J(i), $\Phi(G) \leq Z(G)$ so that $\Phi(G)$ is abelian, contrary to what has been said above.

**Lemma 2.** Let $G$ be a noncyclic $p$-group and $D = \langle \Phi(H) \mid H \in \Gamma_1 \rangle$. Then $D \leq \Phi(G)$ and $|\Phi(G) : D| \leq p$ with equality only for $p > 2$.

**Proof.** Since all members of the set $\Gamma_1$ are $G$-invariant, we have $D < G$ and $D \leq \Phi(G)$ (Lemma J(i)). Since all maximal subgroups of the quotient group $G/D$ are elementary abelian, $G/D$ is either elementary abelian or nonabelian of order $p^3$ and exponent $p$ (Lemma J(d)). In the first case, $D = \Phi(G)$. In the second case, $p > 2$ and $|\Phi(G) : D| = |\Phi(G/D)| = p$. In both cases $|\Phi(G) : D| \leq p$.

If $H \in \Gamma_1$, then $\Phi(G) < H$ so $\Phi(G)' \leq H'$. Below we consider the extreme case when $\Phi(G)' = H'$ for all $H \in \Gamma_1$.

**Theorem 3.** The following assertions for a group $G$ are equivalent:
(a) \( \Phi(G)^\prime = H^\prime \) for all \( H \in \Gamma_1 \).
(b) \( G \) is either abelian or minimal nonabelian.

**Proof.** Obviously, \((b) \Rightarrow (a)\). Therefore, it remains to prove the reverse implication. In what follows one may assume that \( G \) is nonabelian.

(i) The subgroups \( \Phi(G) \), \( \Phi(G)^\prime = H^\prime \) are \( G \)-invariant for all \( H \in \Gamma_1 \). If \( \Phi(G) \) is abelian, then all maximal subgroups of \( G \) are also abelian, by hypothesis, and \( G \) is minimal nonabelian so \((b)\) holds. Next we assume that \( \Phi(G)^\prime > \{1\} \); then all members of the set \( \Gamma_1 \) are nonabelian. Assume, in addition that \( G \) is a \( p \)-group. Let \( R < \Phi(G)^\prime \) be \( G \)-invariant of index \( p \) and set \( \bar{G} = G/R \); then \( \Phi(\bar{G}) \) is nonabelian and \( |\Phi(\bar{G})^\prime| = p \). Take \( H \in \Gamma_1 \). We have

\[
\Phi(\bar{G})^\prime = \Phi(G/R)^\prime = (\Phi(G)/R)^\prime = \Phi(G)^\prime / R = H^\prime / R = \bar{H}^\prime
\]

is of order \( p \) so, by Lemma J(i), \( \Phi(\bar{H}) \leq Z(\bar{H}) \) and hence \( \Phi(\bar{H}) \leq Z(\Phi(G)) \) since \( \Phi(\bar{H}) \leq \Phi(\bar{G}) \leq \bar{H} \). Setting \( D = \langle \Phi(\bar{H}) \mid H \in \Gamma_1 \rangle \), we conclude that \( D \leq Z(\Phi(G)) \) so \( \Phi(G) \) is abelian since \( |\Phi(G) : D| \leq p \), by Lemma 2, contrary to what has been said already. Thus, if \( G \) is a \( p \)-group, then \((a) \Rightarrow (b)\).

In what follows we assume that \( G \) is not a prime-power group. Write \( T = \Phi(G)^\prime \); then \( T = H^\prime \) for all \( H \in \Gamma_1 \), by hypothesis. Therefore, all maximal subgroups of \( G/T \) are abelian so \( G/T \) is either abelian or minimal nonabelian. As in (i), we may assume that \( \Phi(G) \) is nonabelian.

(ii) Suppose that \( G \) is nilpotent; then \( G = P_1 \times \cdots \times P_k \), where \( P_i \in \text{Syl}_p(G) \), \( i = 1, \ldots, k, k > 1 \). It follows from (2) and (3) which are true for any nilpotent group that \( \Phi(P_i)^\prime = H_i^\prime \) for all maximal subgroups \( H_i \) of \( P_i \) so \( P_i \) is either abelian or minimal nonabelian, by (i). Then \( \Phi(P_1) \) is abelian for all \( i \) so that \( \Phi(G) \) is abelian, in view of (1), contrary to the assumption. Thus, the theorem is true provided \( G \) is nilpotent. Next we assume that \( G \) is not nilpotent. In that case, by Lemma J(d), \( G/T \) is nonnilpotent.

(iii) It remains to consider the case where \( G/T \) is minimal nonabelian and nonnilpotent. It follows from the structure of \( G/T \) (Lemma J(c)) that \( \Phi(G/T) = \Phi(G)/T \) is cyclic. Remembering that \( \Phi(G) \) is nilpotent and \( T = \Phi(G)^\prime \), we conclude that \( \Phi(G) \) is cyclic (Lemma J(e)) so abelian, a final contradiction.

**Corollary 4.** Suppose that a group \( G \) is neither abelian nor minimal nonabelian. Then there exists a nonabelian \( H \in \Gamma_1 \) such that \( \Phi(G)^\prime < H^\prime \).

**Proof.** We have \( \Phi(G)^\prime \leq H^\prime \) for all \( H \in \Gamma_1 \). Assume that \( \Phi(G)^\prime = H^\prime \) for all nonabelian \( H \in \Gamma_1 \); then \( \Phi(G) \) is nonabelian (take a nonabelian \( H \in \Gamma_1 \)) so the set \( \Gamma_1 \) has no abelian members. In that case, \( \Phi(G)^\prime = H^\prime \) for all \( H \in \Gamma_1 \), by hypothesis, so that \( G \) is minimal nonabelian (Theorem 3), contrary to the hypothesis.
We claim that the following conditions for a group $G$ are equivalent: (a) $H' \leq \Phi(G)$ for all $H \in \Gamma_1$, (b) either $G$ is nilpotent or $G/\Phi(G)$ is nonnilpotent minimal nonabelian, (c) $N_F(F \cap H) \neq F \cap H \neq N_H(H \cap F)$ for all distinct $F,H \in \Gamma_1$. Obviously, (a) $\Leftrightarrow$ (b) and (b) $\Rightarrow$ (c). Next, (b) follows from (c), by Lemma J(h).

Below we use some known results on Frobenius groups (see [6, §10.2]). Recall that $G$ is said to be a Frobenius group if there is a non-identity $H < G$ such that $H \cap H^x = \{1\}$ for all $x \in G - H$. In that case, $G = H \cdot N$ is a semidirect product with kernel $N$, Sylow subgroups of $H$ are either cyclic or generalized quaternion.

**Lemma 5.** The following conditions for a nonnilpotent group $G$ are equivalent:

(a) All members of the set $\Gamma_1^n$ are abelian.

(b) $G/\Z(G) = (U/\Z(G)) \cdot (Q_1/\Z(G))$ is a Frobenius group with elementary abelian kernel $Q_1/\Z(G) = (G/\Z(G))'$ and a cyclic complement $U/\Z(G)$, $U \in \Gamma_1$.

**Proof.** If $U, V$ are two distinct abelian maximal subgroups of a non-abelian group $G$, then $U \cap V = \Z(G)$.

Suppose that $G$ satisfies condition (a). Given $H \in \Gamma_1^n$, set $H_G = \bigcap_{x \in G} H^x$; then $H_G = \Z(G)$, by the previous paragraph. Set $G = G/\Z(G)$; then $\tilde{G} = \tilde{H} \cdot \tilde{Q}_1$ is a Frobenius group with kernel $\tilde{Q}_1$ and complement $\tilde{H}$. Since $\tilde{H}$ is abelian, it is cyclic. There is in $Q_1$ an $\tilde{H}$-invariant Sylow subgroup, by Sylow’s theorem. Therefore, since $\tilde{H}$ is maximal in $\tilde{G}$, the subgroup $\tilde{Q}_1$ is a $p$-group; moreover, $\tilde{Q}_1$ is a minimal normal subgroup of $\tilde{G}$ so it is elementary abelian. All nonnormal maximal subgroups of $\tilde{G}$ are conjugate with $\tilde{H}$ (Schur-Zassenhaus) so all members of the set $\Gamma_1^n$ are conjugate in $G$ (indeed, all members of the set $\Gamma_1^n$, being abelian, contain $\Z(G)$). Thus, (a) $\Rightarrow$ (b).

Conversely, every group such as in (b), satisfies condition (a). In fact, if $H \in \Gamma_1^n$, then $\Z(G) < H$. By (b), $H/\Z(G)$ is cyclic so $H$ is abelian.

The proof of Lemma 5 shows that if the set $\Gamma_1^n$ has at least one abelian member, then the group $G$ has the same structure as in Lemma 5(b).

**Corollary 6.** A nonnilpotent group $G$ satisfies $\Phi(G)' = H'$ for all $H \in \Gamma_1^n$ if and only if $G = G/\Phi(G)'$ is as in Lemma 5(b).

**Proof.** Suppose that $\tilde{G} = G/\Phi(G)'$ is as in Lemma 5(b) and $\tilde{H}$ is a nonnormal maximal subgroup of $\tilde{G}$; then $\tilde{H}$ is cyclic, by hypothesis, so $H' \leq \Phi(G)'$. Since $\Phi(G) < H$, we get $\Phi(G)' \leq H'$ so $H' = \Phi(G)'$ and whence $G$ satisfies the hypothesis.

Now suppose that $\Phi(G)' = H'$ for all $H \in \Gamma_1^n$. Then all nonnormal maximal subgroups of $G/\Phi(G)'$ are abelian so it is as in Lemma 5(b).
Proposition 7. Suppose that a nonnilpotent group $G$ is not minimal nonabelian and such that $M' = \Phi(G)$ for all nonabelian $M \in \Gamma_1$. Then

(a) $G = P \cdot Q$ is a semidirect product with kernel $Q = G' \in \text{Syl}_p(G)$, $P \in \text{Syl}_q(G)$ is of order $p$, $\Phi(G) = \Phi(Q) > \{1\}$. Set $H = P\Phi(G)$. The set $\Gamma_1$ consists of two conjugacy classes of subgroups with representatives $H$ and $Q$.

(b) $P^G = G$, where $P^G$ is the normal closure of $P$ in $G$.

(c) $G/\Phi(Q)$ is minimal nonabelian so that $Q/\Phi(Q)$ is a minimal normal subgroup of $G/\Phi(G)$.

(d) $H' < \Phi(Q)$ if and only if $G$ is minimal nonnilpotent.

(e) If $\Phi(Q)$ is abelian, then either $G$ is minimal nonnilpotent or a Frobenius group.

Proof. By hypothesis, all maximal subgroups of $G/\Phi(G)$ are abelian so it is nonnilpotent and minimal nonabelian, by Lemma J(d,e), and (c) is proven. It follows that the set $\Gamma_1$ is the union of two conjugacy classes of subgroups. We have $\Phi(G/\Phi(G)) = \{1\}$ so that $|G/\Phi(G)| = pq^b$, where a subgroup of order $p$ is not normal in $G/\Phi(G)$ (Lemma J(b,c)). By hypothesis, the set $\Gamma_1$ has a nonabelian member so $\Phi(G') > \{1\}$. By Lemma J(g), $\pi(G) = \pi(G/\Phi(G)) = \{p, q\}$. By Lemma J(f), $Q \in \text{Syl}_p(G)$ is normal in $G$. We have $G = P \cdot Q$, where $P \in \text{Syl}_q(G)$. By Lemma J(a), $\Phi(Q) = Q \cap \Phi(G)$. By the above, $\Phi(G) = P_1 \times \Phi(Q)$, where $P_1$ is a maximal subgroup of $P$. The subgroup $H = P\Phi(Q) \in \Gamma_1$ and all nonnormal maximal subgroups of $G$ are conjugate in $G$ (see the second sentence of this paragraph). Since $P_1Q/Q$ is the unique normal maximal subgroup of $G/Q$, it follows that $G/Q \cong P$ is cyclic.

As above, $P_1 \leq \Phi(G)$ since $|G/\Phi(G)| = pq^b$ (here $P_1$ is maximal in cyclic subgroup $P$). Next, $P_1Q = P_1 \times Q = M \in \Gamma_1$. Assume that $P_1 > \{1\}$ and $M$ is nonabelian. We have $P_1 \not\leq M' = \Phi(G)$, a contradiction. Thus, if $P_1 > \{1\}$, then $M$ is abelian. Similarly, $P_1 \not\leq (P\Phi(G))' = H'$ so that $H$ is also abelian. It follows that $G$ is minimal nonabelian, contrary to the hypothesis. Thus, $P_1 = \{1\}$ so that $|P| = p$ and $Q \in \Gamma_1$. This completes the proof of (a). By Lemma J(a), $\Phi(G) = \Phi(Q)$.

Since $Q$, the unique normal maximal subgroup of $G$, has index $p$ in $G$, it follows that $P^G = G$, and the proof of (b) is complete.

Suppose that $H' < \Phi(G)$. Then $H$ is abelian, by hypothesis. In that case, $C_G(\Phi(G)) \geq H^G = G$ so $\Phi(G) = Z(G)$ since $Z(G/\Phi(G)) = \{1\}$, and we conclude that $G$ is minimal nonnilpotent since $G/\Phi(G)$ is minimal nonabelian. The proof of (d) is complete.

Assume that $\Phi(Q)(= \Phi(G))$ is abelian and $G$ is not minimal nonnilpotent. Then, by (d), $H = P\Phi(Q)$ is nonabelian so $H' = \Phi(Q)$, by hypothesis. In that case, by Lemma J(j), $N_H(P) = P$ so that $H$ is a Frobenius group since $|P| = p$. Since $G/\Phi(Q)$ is a Frobenius group, it follows that $G$ is also a
Let a nonabelian group $G$ be such that $H' = \Phi(H)$ for all nonabelian $H \leq G$. Then $G$ has no minimal nonnilpotent subgroups, by Lemma J(d). It follows that $G$ is nilpotent. Let $A \leq G$ be minimal nonabelian. Then $A$ is a $p$-subgroup for some prime $p$ and $A' = \Phi(A)$ has order $p$ and index $p^2$ in $A$ (Lemma J(b)) so that $|A| = |A'|p^2 = p^3$. Suppose that $P \leq \text{Syl}_p(G)$ is nonabelian. Then $P$ is among the 2-groups described in [8] (see also [5, §90]). Now suppose that $G = Q \times H$, where $Q \in \text{Syl}_2(G)$ is nonabelian and $H > \{1\}$. Assume that $Z \leq H$ is cyclic of order $p^2$. Then the subgroup $K = Q \times Z$ does not satisfy the hypothesis since $\Phi(Z) \not\leq K'$ and $\Phi(Z) \leq \Phi(K)$. Thus, either $G$ is a prime power or $G = Q \times H$, where $Q \in \text{Syl}_q(G)$ is nonabelian and $\exp(H) > 1$ is square free. It follows that if $H$ is also nonabelian, then $\exp(Q) = q$ and we conclude that $q > 2$.

**Problems**

1. Study the nonabelian $p$-groups $G$ of exponent $> p$ satisfying $\Omega_1(\Phi(H)) = \Omega_1(H)$ for all $H \leq G$ (obviously, we must have $p > 2$ since for any 2-group $G$ we have $\Phi(G) = \Omega_1(G)$).

2. Classify the $p$-groups $G$ satisfying $H' = \Phi(G)'$ for all those $H \leq G$ that are neither abelian nor minimal nonabelian.

3. Suppose that $G$ is a $p$-group with $|G/\Phi(G)| = p^d$. Given $i \leq d$, let $\Gamma_i$ be the set of all normal subgroups $N$ of $G$ such that $G/N$ is elementary abelian of order $p^i$. Study the $p$-groups $G$ such that $H' = \Phi(G)'$ for all $H \in \Gamma_i$ (for $i = 1$, see Theorem 3).

**Acknowledgements.**

I am indebted to the referee for a number of constructive remarks.

**References**


Y. Berkovich
Department of Mathematics,
University of Haifa,
Mount Carmel, Haifa 31905
Israel

Received: 8.12.2008.
Revised: 5.5.2009. & 27.6.2009.