A QUASISTATIC FRICTIONAL CONTACT PROBLEM WITH NORMAL COMPLIANCE AND FINITE PENETRATION FOR ELASTIC MATERIALS

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Abstract. We consider a quasistatic unilateral contact problem with finite penetration between an elastic body and an obstacle, say a foundation. The constitutive law is assumed to be nonlinear and the contact is modelled with normal compliance associated to a version of Coulomb’s law of dry friction. Under a smallness assumption on the contact functions, we establish the existence of a weak solution to the problem. The proofs are based on arguments of time-discretization, compactness and lower semicontinuity.

1. Introduction

Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve a complicated surface phenomena, and are modeled with highly nonlinear initial boundary value problems. Taking into account various frictional contact conditions associated with behavior laws becoming more and more complex leads to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. A first attempt to study frictional contact problems within the framework of variational inequalities was made in [8]. The mathematical, mechanical and numerical state of the art can be found in [13]. In [11] we find a detailed analysis of the contact problem in elasticity with the mathematical and numerical studies. In this work we consider a quasistatic contact problem between a nonlinear elastic body and an obstacle say a foundation.

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We assume that the contact is modelled with normal compliance similar to the one in [10] where a dynamic frictionless contact problem was studied for elastic-visco-plastic materials. Under this condition the interpenetration of the body's surface into the foundation is allowed and justified by considering the interpenetration and deformation of surface asperities. The friction is modelled with a version of Coulomb's law of dry friction. We assume that the forces applied to the body vary slowly in time so that the acceleration in the system is negligible. In this case we can study a quasistatic approach of the process. For linear elastic materials the quasistatic contact problem using a normal compliance law has been studied in [1] by considering incremental problems and in [12] by another method using a time-regularization. The quasistatic unilateral contact problem with local or nonlocal friction has been solved respectively in [14] and in [5] by using a time-discretization. The same method was also used in [16] to solve the quasistatic unilateral contact problem with a modified version of Coulomb's law of dry friction for nonlinear elastic materials. In [2] the quasistatic contact problem with Coulomb friction was solved by an established shifting technique used to obtain increased regularity at the contact surface and by the aid of auxiliary problems involving regularized friction terms and a so-called normal compliance penalization technique. Signorini's problem with friction for nonlinear elastic materials has been solved in [6] by using a fixed point method. Also the quasistatic contact problem with normal compliance and friction has been solved in [15] for nonlinear viscoelastic materials by the same fixed point arguments. In the book [9] the authors resolve the quasistatic contact problems in viscoelasticity and viscoplasticity. Carrying out the variational analysis, the authors systematically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators, and fixed points of operators. These two latest arguments were used in [3] to solve recently two dynamic frictionless contact problems for elastic-visco-plastic materials.

In this paper we propose a variational formulation written in the form of two variational inequalities. By means of Euler’s implicit scheme as in [5,16], the quasistatic contact problem leads us to solve a well-posed variational inequality at each time step. Finally under a smallness assumption on the contact functions we prove by using lower semicontinuity and compactness arguments that the limit of the discrete solution is a solution to the continuous problem.

2. Variational formulation

We consider a nonlinear elastic body that initially occupies a domain \( \Omega \) in \( \mathbb{R}^d \), \( d = 2, 3 \). \( \Omega \) is supposed to be open, bounded, with a Lipschitz boundary \( \Gamma \). \( \Gamma \) is decomposed into three measurable parts \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \) where \( \Gamma_1, \Gamma_2, \Gamma_3 \) are disjoint open sets and \( \text{meas} \, \Gamma_1 > 0 \). The body is subjected to volume forces of density \( \phi_1 \), prescribed zero displacements and tractions \( \phi_2 \).
on the part $\Gamma_1$ and $\Gamma_2$, respectively. On $\Gamma_3$ the body is in unilateral contact with friction with a foundation.

Under these conditions, the classical formulation of the mechanical problem of frictional contact of the nonlinear elastic body is the following.

**Problem $P_1$.** Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$ such that

\[
\begin{align*}
\text{div } \sigma + \phi_1 &= 0 \text{ in } \Omega \times (0,T), \\
\sigma &= F(x,\varepsilon) \quad \text{in } \Omega \times (0,T), \\
u &= 0 \quad \text{on } \Gamma_1 \times (0,T), \\
\sigma \nu &= \phi_2 \quad \text{on } \Gamma_2 \times (0,T), \\
u \leq g, \sigma \nu + p_\nu(u_\nu) &= 0 \quad \text{on } \Gamma_3 \times (0,T), \\
\left| \sigma \right| &\leq p_r(u_\nu) \quad \text{on } \Gamma_3 \times (0,T), \\
\left| \sigma \right| &< p_r(u_\nu) \Rightarrow \tilde{u}_\tau = 0 \\
\left| \sigma \right| &= p_r(u_\nu) \Rightarrow \exists \lambda \geq 0 \text{ s.t. } \sigma \tau = -\lambda \tilde{u}_\tau
\end{align*}
\]

In the study of the mechanical problem $P_1$ we adopt the following notations and hypotheses: we use the function spaces

\[
H = \left( L^2(\Omega) \right)^d, \quad Q = \{ \tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} \in L^2(\Omega) \},
\]

\[
H_1 = \left( H^1(\Omega) \right)^d, \quad Q_1 = \{ \tau \in Q; \text{div } \tau \in H \}.
\]

$H, Q$ are Hilbert spaces equipped with the respective inner products:

\[
\langle u, v \rangle_H = \int \Omega u_i v_i dx, \quad \langle \sigma, \tau \rangle_Q = \int \Omega \sigma_{ij} \tau_{ij} dx.
\]

We denote by $S_d$ the space of second order symmetric tensors in $\mathbb{R}^d$ ($d = 2, 3$); $u = (u_i)$, the displacement field; $\sigma = (\sigma_{ij})$, the stress tensor;

\[
\varepsilon = \varepsilon(u(x,t)) = (\varepsilon_{ij}(u(x,t))) = \frac{1}{2}( u_{i,j}(x,t) + u_{j,i}(x,t)), \quad i, j \in \{1, \ldots, d\},
\]

where $u_{i,j}(x,t) = \frac{\partial u_i(x,t)}{\partial x_j}$, $(x,t) \in \Omega \times (0,T)$, the strain tensor; $\text{div } \sigma = (\sigma_{ij,j})$, the divergence of $\sigma$. We denote by $u_\nu$ and $u_\tau$ the normal and the tangential components of $u$ on $\Gamma$ given by

\[
u \cdot u, \quad u_\tau = u \cdot \tau
\]

where $\nu$ is the outward unit normal vector to $\Gamma$. We also denote by $\sigma_\nu$ and $\sigma_\tau$ the normal and the tangential components of $\sigma$, and we note that when $\sigma \in Q_1$ is a regular function then

\[
\sigma_\nu = (\sigma \nu) \cdot \nu, \quad \sigma_\tau = \sigma \nu - \sigma_\nu \nu,
\]
and Green’s formula holds:

\[ \langle \sigma, \varepsilon (v) \rangle_Q + \langle \text{div} \, \sigma, v \rangle_H = \int_{\Gamma} \sigma v \cdot v \, da \quad \forall v \in H_1. \]

Now, let \( V \) be the closed subspace of \( H_1 \) defined by

\[ V = \{ v \in H_1; v = 0 \text{ on } \Gamma_1 \}, \]

and the set of admissible displacements fields given by

\[ K = \{ v \in V; v_\nu \leq g \text{ on } \Gamma_3 \}, \]

where \( g \geq 0 \). Since \( \text{meas } \Gamma_1 > 0 \), the following Korn’s inequality holds ([8]),

\[ (2.8) \quad \| \varepsilon (v) \|_Q \geq c_\Omega \| v \|_{H_1} \quad \forall v \in V, \]

where \( c_\Omega > 0 \) is a constant which depends only on \( \Omega \) and \( \Gamma_1 \). We equip \( V \) with the inner product given by

\[ (u, v)_V = \langle \varepsilon (u), \varepsilon (v) \rangle_Q \]

and let \( \| \cdot \|_V \) be the associated norm. It follows from (2.8) that the norms \( \| \cdot \|_{H_1} \) and \( \| \cdot \|_V \) are equivalent and \( (V, \| \cdot \|_V) \) is a real Hilbert space. Moreover, by Sobolev’s trace theorem, there exists a constant \( d_\Omega > 0 \) depending only on the domain \( \Omega \), \( \Gamma_1 \) and \( \Gamma_3 \) such that

\[ (2.9) \quad \| v \|_{(L^2(\Gamma_3))^d} \leq d_\Omega \| v \|_V \quad \forall v \in V. \]

Let us equally define \( H^\frac{1}{2} (\Gamma_3) \) by

\[ H^\frac{1}{2} (\Gamma_3) = \left\{ w |_{\Gamma_3}; w \in H^\frac{1}{2} (\Gamma), \ w = 0 \text{ on } \Gamma_1 \right\}. \]

\( \langle \cdot, \cdot \rangle_{\Gamma_3} \) shall denote the duality pairing on \( H^\frac{1}{2} (\Gamma_3), H^{-\frac{1}{2}} (\Gamma_3) \). Before we start with the variational formulation of Problem \( P_1 \) let us state in which sense the duality pairing \( \langle \cdot, \cdot \rangle_{\Gamma_3} \) is taken. Indeed, we define a subset \( V_0 \) of \( H_1 \) by

\[ V_0 = \{ v \in H_1; \text{div} \, \sigma (v) \in H \}, \]

and let \( \varphi \in (L^2 (\Gamma_2))^d \) and \( u \in V_0 \) such that \( \sigma (u) \nu = \varphi \) on \( \Gamma_2 \). Then as in [16] we define the normal stress \( \sigma_\nu (u) \) on \( \Gamma_3 \) as follows

\[ (2.10) \quad \langle \sigma_\nu (u), v \rangle_{\Gamma_3} = \langle \sigma (u), \varepsilon (v) \rangle_Q + \langle \text{div} \, \sigma (u), v \rangle_H - \int_{\Gamma_2} \varphi \cdot v \, da \quad \forall v \in V; \ v_\tau = 0 \text{ on } \Gamma_3. \]
In the study of the mechanical problem $P_1$, we assume that the elasticity operator $F : \Omega \times S_d \to S_d$ satisfies the following conditions

$$
\begin{align*}
(a) & \text{ there exists } M > 0 \text{ such that } \\
& |F(x, \varepsilon_1) - F(x, \varepsilon_2)| \leq M |\varepsilon_1 - \varepsilon_2|, \\
& \text{ for all } \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \in \Omega; \\
(b) & \text{ there exists } m > 0 \text{ such that } \\
& (F(x, \varepsilon_1) - F(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m |\varepsilon_1 - \varepsilon_2|^2, \\
& \text{ for all } \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \in \Omega; \\
(c) & \text{ the mapping } x \mapsto F(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\
& \text{ for any } \varepsilon \in S_d; \\
(d) & F(x, 0) = 0 \text{ for a.e. } x \in \Omega.
\end{align*}
$$

(2.11)

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0, T] ; X)$ for the space of continuous functions from $[0, T]$ to $X$; recall that $C([0, T] ; X)$ is a real Banach space with the norm

$$
\|x\|_{C([0, T] ; X)} = \max_{t \in [0, T]} \|x(t)\|_X.
$$

For $p \in [1, \infty]$ we use the standard notation of $L^p(0, T; V)$. We also use the Sobolev space $W^{1, \infty}(0, T; V)$ equipped with the norm

$$
\|v\|_{W^{1, \infty}(0, T; V)} = \|v\|_{L^\infty(0, T; V)} + \|\dot{v}\|_{L^\infty(0, T; V)},
$$

where a dot now represents the weak derivative with respect to the time variable.

The forces are assumed to satisfy

$$
\phi_1 \in W^{1, \infty}(0, T; H), \phi_2 \in W^{1, \infty}
\left(0, T; (L^2(\Gamma_2))^d\right).
$$

Let $f : [0, T] \to V$ given by

$$
(f(t), v)_V = \int_\Omega \phi_1 \cdot vdx + \int_{\Gamma_2} \phi_2 \cdot vda \quad \forall \ v \in V, \ t \in [0, T].
$$

The assumption (2.12) implies that

$$
f \in W^{1, \infty}(0, T; V).
$$

We assume that the contact functions $p_r (r = \nu, \tau)$ satisfy

$$
\begin{align*}
(a) & \text{ } \ p_r : [-\infty, \infty] \to \mathbb{R}^+; \\
(b) & \text{ there exists } L_r > 0 \text{ such that } \\
& |p_r(u) - p_r(v)| \leq L_r |u - v|, \text{ for all } u, v \leq g; \\
(c) & p_r(v) = 0 \text{ for all } v < 0.
\end{align*}
$$

(2.13)
When \( u_ν < 0 \), i.e., when there is separation between the body and the obstacle then the condition (2.5) combined with hypothesis (2.13) shows that the reaction of the foundation vanishes (since \( σ_ν = 0 \)). When \( 0 \leq u_ν < g \) then \(-σ_ν = p_ν (u_ν)\) which means that the reaction of the foundation is uniquely determined by the normal displacement. When \( u_ν = g \) then \(-σ_ν ≥ p_ν (g)\) and \( σ_ν \) is not uniquely determined. We note then when \( g = 0 \) and \( p_ν = 0 \) then the condition (2.5) becomes the classical Signorini contact condition without a gap
\[
\begin{align*}
    u_ν &\leq 0, \quad σ_ν \leq 0, \quad σ_ν u_ν = 0, \\
    u_ν &\geq g, \quad σ_ν \leq 0, \quad σ_ν (u_ν - g) = 0.
\end{align*}
\]

The last two conditions are used to model the unilateral conditions with a rigid foundation.

Conditions (2.6) represent a version of Coulomb’s law of dry friction in which \( p_τ \) is a prescribed nonnegative function, the so-called friction bound and \( ˙u_τ \) the tangential velocity on the boundary. The tangential shear cannot exceed the maximal frictional resistance \( p_τ (u_ν)\). Then, if the strict inequality is satisfied, the surface adheres to the foundation and is in the so-called stick state, and when equality is satisfied there is relative sliding, the so-called slip state. Examples of normal compliance functions can be found in [1, 3, 10, 15].

Next, we define the friction functional \( j : V \times V \to \mathbb{R} \) by
\[
    j (v, w) = \int_{Γ_3} p_ν (v_ν) w_ν da + \int_{Γ_3} p_τ (v_τ) |w_τ| da, \quad ∀v, w \in V,
\]
and we assume that the initial data satisfies
\[
    (2.14) \quad u_0 ∈ K, \quad \langle F ε (u_0), ε (v - u_0) \rangle + j (u_0, v - u_0) ≥ (f (0), v - u_0)_V \quad ∀v ∈ K.
\]
Now we turn to the weak formulation of Problem P₁. As in [5], assume that \( u \) is a smooth function satisfying (2.1)–(2.7). Let \( v ∈ V \) and multiply the equilibrium of forces (2.1) by \( v - ˙u (t) \), integrate the result over \( Ω \) and use Green’s formula to obtain
\[
    \int_Ω (v) (v - ˙u (t)) dx = \int_Ω φ_1 (t) \cdot (v - ˙u (t)) dx + \int_Γ (v - ˙u (t))\ da.
\]
Taking into account the boundary condition (2.4) and \( v = 0 \) on \( Γ_1 \), we see that
\[
    \int_Γ (v - ˙u (t)) da = \int_Γ φ_2 (t) \cdot (v - ˙u (t)) da + \int_Γ φ_3 (t) \cdot (v - ˙u (t)) da.
\]
Moreover we have
\[
    \int_Γ (v - ˙u (t)) da = \int_Γ (v_ν - ˙u_ν (t)) da + \int_Γ (v_τ - ˙u_τ (t)) da,
\]
and
\[ \int_{\Gamma_3} \sigma_\nu (t) \left( v_\nu - \dot{u}_\nu (t) \right) da = \int_{\Gamma_3} (\sigma_\nu (t) + p_\nu (u_\nu (t))) \left( v_\nu - \dot{u}_\nu (t) \right) da \]
\[ - \int_{\Gamma_3} p_\nu (u_\nu (t)) \left( v_\nu - \dot{u}_\nu (t) \right) da. \]

Next, we need to the following formulation (2.15) obtained from the contact condition (2.5) and the law of friction (2.6).

(2.15)
\[ \begin{cases} 
(a) \ u_\nu \leq g, \ (\sigma_\nu + p_\nu (u_\nu)) \left( v_\nu - u_\nu \right) \geq 0 \ \forall v_\nu \leq g, \\
(b) \ \sigma_\tau (v_\tau - \dot{u}_\tau) + p_\tau (u_\nu) \left( |v_\tau| - |\dot{u}_\tau| \right) \geq 0 \ \forall v_\tau. 
\end{cases} \]

Indeed, it follows from (2.15)(b) and (2.15)(a) that the function \( u \) satisfies respectively the inequalities
\[ \int_{\Omega} (F \varepsilon(u(t))) \left( \varepsilon(v) - \varepsilon(\dot{u}(t)) \right) dx + j(u(t), v) - j(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t)) \]
(2.16)
\[ + \int_{\Gamma_3} (\sigma_\nu (u(t)) + p_\nu (u_\nu (t))) \left( v_\nu - \dot{u}_\nu (t) \right) da \quad \forall v \in V, \]
and
\[ \int_{\Gamma_3} (\sigma_\nu (u(t)) + p_\nu (u_\nu (t))) \left( z_\nu - u_\nu (t) \right) \geq 0 \quad \forall z \in K. \]

Finally, we combine (2.7), (2.16) and (2.17) to derive the variational formulation of Problem P1.

**Problem P2.** Find a displacement field \( u \in W^{1, \infty} (0, T; V) \) such that \( u(0) = u_0 \) in \( \Omega \) and such that for all \( t \in [0, T] \), \( u(t) \in K \cap V_0 \), and for almost all \( t \in [0, T] \),
\[ \langle F \varepsilon(u(t)), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q + j(u(t), v) - j(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t)) \]
(2.18)
\[ + \langle \sigma_\nu (u(t)) + p_\nu (u_\nu (t)), v_\nu - \dot{u}_\nu (t) \rangle_{\Gamma_3} \quad \forall v \in V, \]
\[ \langle \sigma_\nu (u(t)) + p_\nu (u_\nu (t)), z_\nu - u_\nu (t) \rangle_{\Gamma_3} \geq 0 \quad \forall z \in K. \]

One has the following theorem:

**Theorem 2.1.** Let \( T > 0 \) and assume that (2.11), (2.12), (2.13), and (2.14) hold. Then Problem P2 has at least one solution if
\[ L_\nu + L_\tau < \frac{m}{d_\Omega}. \]
3. A time-discretization

For the proof of Theorem 2.1, we carry a time-discretization of Problem P. We need a partition of the time interval \([0, T]\), with \(0 = t_0 < t_1 < \ldots < t_n = T\), where \(t_i = i\Delta t\), \(i = 0, \ldots, n\), with step size \(\Delta t = \frac{T}{n}\). We denote by \(u^i\) the approximation of \(u\) at time \(t_i\) and \(\Delta u^i = u^{i+1} - u^i\). For a continuous function \(w(t)\) we use the notation \(w^i = w(t_i)\). By using an implicit scheme, we obtain a sequence of incremental Problems \(P_n\) defined for \(u^0 = u_0\) by

**Problem \(P_n^i\).** Find \(u^{i+1} \in K \cap V_0\) such that

\[
\begin{align*}
\langle F(u^{i+1}) \cdot \varepsilon(w) - \varepsilon(u^{i+1}) \rangle_Q + j(u^{i+1}, w - u^i) - j(u^{i+1}, \Delta u^i) \\
\geq (f^{i+1}, w - u^{i+1})_V + \langle \sigma_{\nu} (u^{i+1}) + p_{\nu} (u^{i+1}_\nu), w - u^{i+1}_\nu \rangle_{\Gamma_3} \forall w \in V,
\end{align*}
\]

\[
\langle \sigma_{\nu} (u^{i+1}) + p_{\nu} (u^{i+1}_\nu), w - u^{i+1}_\nu \rangle_{\Gamma_3} \geq 0 \forall w \in K.
\]

As in [5] Problem \(P_n^i\) is equivalent to Problem \(Q_n^i\) defined as follows:

**Problem \(Q_n^i\).** Find \(u^{i+1} \in K \cap V_0\) such that

\[
\begin{align*}
\langle F(u^{i+1}) \cdot \varepsilon(w) - \varepsilon(u^{i+1}) \rangle_Q + j(u^{i+1}, w - u^i) - j(u^{i+1}, \Delta u^i) \\
\geq (f^{i+1}, w - u^{i+1})_V \forall w \in K.
\end{align*}
\]

We have the following result.

**Proposition 3.1.** Problem \(Q_n^i\) has a unique solution if

\[
L_{\nu} + L_{\tau} < \frac{m}{d_Q^2}
\]

To prove this proposition, let us introduce the following intermediate problem.

**Problem \(Q_{n, \eta}\).** For \(\eta \in K\), find \(w^{i+1}_\eta \in K\) such that

\[
\begin{align*}
\langle F(u^{i+1}_\eta) \cdot \varepsilon(w) - \varepsilon(u^{i+1}_\eta) \rangle_Q + j(\eta, w - u^i) - j(\eta, \Delta u^i) \\
\geq (f^{i+1}, w - u^{i+1}_\eta)_V \forall w \in K.
\end{align*}
\]

We have the following lemma.

**Lemma 3.2.** Problem \(Q_{n, \eta}\) has a unique solution.

**Proof.** Using Riesz’s representation theorem, we define the nonlinear operator \(A: V \to V\) by

\[
(Av, w)_V = \langle F(v) \cdot \varepsilon(w) \rangle_Q
\]

The hypotheses (2.11)(a) and (2.11)(b) on \(F\) imply that the operator \(A\) is strongly monotone and Lipschitz continuous; on the other hand the functional
defined on $K$ by $j_\eta(w) = j(\eta, w - u')$ is proper convex and lower semicontinuous. From the theory of elliptic variational inequalities ([4]), it follows that the inequality (3.3) has a unique solution. 

Now to prove Proposition 3.1, we define the following mapping $\Phi : K \rightarrow K$, by

$$\eta \rightarrow \Phi(\eta) = u_{\eta}. $$

The following lemma holds.

**Lemma 3.3.** $\Phi$ has a unique fixed point $\eta^*$ if

$$L_\nu + L_\tau < \frac{m}{d_\Omega^2},$$

and $u_{\eta^*}$ is a unique solution of Problem $Q_n$.

**Proof.** Let’s set $v = u_{\eta_2}$ in the inequality of Problem $Q_{n\eta_2}$ and $v = u_{\eta_1}$ in the inequality of Problem $Q'_{n\eta_2}$. Adding the resulting inequalities, we obtain the following inequality

$$\langle F_\varepsilon(u_{\eta_1}) - F_\varepsilon(u_{\eta_2}), \varepsilon(u_{\eta_1} - u_{\eta_2}) \rangle_{Q} \leq j(\eta_1, u_{\eta_2} - u') - j(\eta_1, u_{\eta_1} - u') + j(\eta_2, u_{\eta_1} - u') - j(\eta_2, u_{\eta_2} - u').$$

Using (2.11)(b), (2.9) and (2.13)(b), we obtain

$$\|\Phi(u_{\eta_1}) - \Phi(u_{\eta_2})\|_V \leq \frac{d_\Omega^2}{m} (L_\nu + L_\tau) \|\eta_1 - \eta_2\|_V.$$ 

Then when $L_\nu + L_\tau < \frac{m}{d_\Omega^2}$, $\Phi$ is contractive; thus it admits a unique fixed point $\eta^*$ and $u_{\eta^*}$ is a unique solution of Problem $Q_n$.

4. Existence of a solution

The main result of this section is to show the existence of a solution obtained as a limit of the interpolate function of the discrete solution. For this it is necessary at first to establish the following lemma.

**Lemma 4.1.** For $L_\nu + L_\tau < \frac{m}{d_\Omega^2}$, there exists positive constants $c_1$, $c_2$ such that

$$(4.1) \quad \|u^{i+1}\|_V \leq c_1 \|f^{i+1}\|_V, \quad \|\Delta u^i\|_V \leq c_2 \|\Delta f^i\|_V.$$ 

**Proof.** Set $w = 0$ in inequality (3.2); then, using the assumption (2.11)(b) on $F$, (2.13) and the relation (2.9), we obtain by a standard reasoning that for $L_\nu + L_\tau < \frac{m}{d_\Omega^2}$, there exists a constant $c_1 > 0$ such that the first inequality holds.
To prove the second inequality, set \( v = u^i \) in inequality (3.2) and set \( v = u^{i+1} \) in the same inequality satisfied by \( u^i \) and adding them up, we obtain
\[
- \langle F \varepsilon(u^{i+1}) - F \varepsilon(u^i), \varepsilon(\Delta u^i) \rangle_Q + j(u^i, u^{i+1} - u^{i-1})
- j(u^i, u^i - u^{i-1}) - j(u^{i+1}, u^i) \geq (-\Delta f^i, \Delta u^i)_V.
\]

On the other hand
\[
\begin{align*}
&j(u^i, u^{i+1} - u^{i-1}) - j(u^i, u^i - u^{i-1}) - j(u^{i+1}, u^i - u^i) \\
&= \int_{\Gamma_3} (p_r(u^i_{\nu})) (u^{i+1}_\nu - u^i_\nu) \, da + \int_{\Gamma_3} P_r (u^i_\nu) (|u^{i+1}_\nu - u^{i-1}_\nu| - |u^{i-1}_\nu - u^i_\nu|) \, da \\
&\quad - \int_{\Gamma_3} (p_r(u^{i+1}_\nu)) (u^{i+1}_\nu - u^i_\nu) \, da - \int_{\Gamma_3} P_r (u^{i+1}_\nu) |u^{i+1}_\nu - u^i_\nu| \, da.
\end{align*}
\]
Then using the relation
\[
|u^{i+1}_\nu - u^{i-1}_\nu| - |u^{i-1}_\nu - u^i_\nu| \leq |\Delta u^i|,
\]
we find
\[
\begin{align*}
&\langle F \varepsilon(u^{i+1}) - F \varepsilon(u^i), \varepsilon(\Delta u^i) \rangle_Q \\
&\quad \leq \int_{\Gamma_3} |p_r(u^{i+1}_\nu) - p_r(u^i_\nu)| |\Delta u^i| \, da \\
&\quad + \int_{\Gamma_3} |p_r(u^{i+1}_\nu) - p_r(u^i_\nu)| |\Delta u^i| \, da + (\Delta f^i, \Delta u^i)_V.
\end{align*}
\]
Therefore the assumption (2.13)(b) on the contact functions \( p_r \) (\( r = \tau, \nu \)) leads to
\[
\begin{align*}
&\langle F \varepsilon(u^{i+1}) - F \varepsilon(u^i), \varepsilon(\Delta u^i) \rangle_Q \\
&\quad \leq L_\nu \int_{\Gamma_3} |\Delta u^i|^2 \, da + L_\tau \int_{\Gamma_3} |\Delta u^i| |\Delta u^i| \, da + (\Delta f^i, \Delta u^i)_V.
\end{align*}
\]
Moreover we use (2.9) and (2.11)(b) and obtain
\[
m \|\Delta u^i\|^2_V \leq \langle F \varepsilon(u^i), \varepsilon(\Delta u^i) \rangle_Q \\
\quad \leq d^2_0 (L_\nu + L_\tau) \|\Delta u^i\|^2_V + \|\Delta f^i\|_V \|\Delta u^i\|_V.
\]
So it follows that for \( L_\nu + L_\tau < \frac{m}{2d^2_0} \), there exists a constant \( c_2 > 0 \) such that
\[
\|\Delta u^i\|_V \leq c_2 \|\Delta f^i\|_V.
\]

Next, as in [5] we define the continuous function \( u^n \) in \([0, T] \rightarrow V \) by
\[
\begin{align*}
u^n(t) &= u^i + \frac{t - t_i}{\Delta t} \Delta u^i, \quad \forall t \in [t_i, t_{i+1}], \quad i = 0, ..., n - 1.
\end{align*}
\]
Then as in [16] we have the following lemma.
Lemma 4.2. There exists a function \( u \in W^{1,\infty}(0, T; V) \), such that passing to a subsequence still denoted by \( (u^n) \) we have
\[
u^n \rightharpoonup u \quad \text{weak* in} \quad W^{1,\infty}(0, T; V).
\]

On the other hand as in [5,16] let’s introduce the following piecewise constant functions
\[
\tilde{u}^n : [0, T] \to V, \quad \tilde{f}^n : [0, T] \to V,
\]
defined by
\[
\tilde{u}^n (t) = u^{i+1}, \quad \tilde{f}^n (t) = f (t_{i+1}), \quad \forall \ t \in (t_i, t_{i+1}], \ i = 0, \ldots, n-1.
\]
As in [9] the following result holds.

Lemma 4.3. There exists a subsequence still denoted \( (\tilde{u}^n) \) such that
\[
\tilde{u}^n \rightharpoonup u \quad \text{weak* in} \quad L^\infty(0, T; V),
\]
\[
\tilde{u}^n (t) \rightharpoonup u (t) \quad \text{weakly in} \ V, \ a.e. t \in [0, T].
\]

Remark 4.4. Also as in [16] we have the following results
\[
(4.2) \quad \tilde{f}^n \rightharpoonup f \quad \text{strongly in} \ L^2(0, T; V),
\]
\[
u (t) \in K \cap V_0 \quad \text{for all} \ t \in [0, T].
\]

Now we have all the ingredients to prove the following result.

Proposition 4.5. The sequence \( (\tilde{u}^n) \) converges strongly to the function \( u \) in \( L^2(0, T; V) \) and \( u \) is a solution to Problem \( P_2 \).

Proof. From (3.2) we deduce the inequality
\[
\langle F (\epsilon (\tilde{u}^n (t)), \epsilon (w) - \epsilon (\tilde{u}^n (t))) \rangle_Q + j (\tilde{u}^n (t), w - \tilde{u}^n (t)) \\
\geq \left( \tilde{f}^n (t), w - \tilde{u}^n (t) \right)_V \quad \forall \ w \in K, \ a.e. t \in [0, T].
\]

(4.3)

To show the strong convergence, we take \( w = \tilde{u}^{n+m} \) in (4.3) and \( w = \tilde{u}^n \) in the same inequality satisfied by \( \tilde{u}^{n+m} (t) \) and adding the resulting inequalities, it follows by using (2.11)(b), the relation (2.9) and \( L_\nu + L_\tau < \frac{m}{4C_1} \), that there exists a constant \( C_1 > 0 \) such that
\[
\| \tilde{u}^{n+m} (t) - \tilde{u}^n (t) \|^2_V \\
\leq C_1 \left( \int_{\Gamma_3} \left( |p_\tau (\tilde{u}_\nu^n (t))| + |p_\tau (\tilde{u}^{n+m}_\nu (t))| \right) \| \tilde{u}^{n+m}_\tau (t) - \tilde{u}_\tau^n (t) \| \right) da \\
+ \left( \| \tilde{f}^{n+m} (t) - \tilde{f}^n (t) \|^2_V \right).
\]
Moreover using (2.13)(b) we show that there exists a constant $C_2 > 0$ such that
\[
\int_{G_3} \left( |\rho^u (t)| + |\rho^u \left( \tilde{u}^{n+m} (t) \right)| \right) \left( |\tilde{u}^{n+m} (t) - \tilde{u}^n (t)| \right) da \\
\leq C_2 \left( \|\tilde{u}^{n+m} \|_{L^2 (G_3)} + \|\tilde{u}^n \|_{L^2 (G_3)} \right) \|\tilde{u}^{n+m} (t) - \tilde{u}^n (t)\|_{L^2 (G_3)}. \]
Using the relation (2.9) and that $(\tilde{u}_n)$ is bounded in $L^\infty (0, T; V)$ we deduce
\[
\int_{G_3} \left( |\rho^u (t)| + |\rho^u \left( \tilde{u}^{n+m} (t) \right)| \right) \left( |\tilde{u}^{n+m} (t) - \tilde{u}^n (t)| \right) da \\
\leq C_3 \|f\|_{L^\infty (0, T; V)} \|\tilde{u}^{n+m} (t) - \tilde{u}^n (t)\|_{L^2 (G_3)}, \]
where $C_3 > 0$. To complete the rest of the proof we refer the reader to see [16, Lemma 7] and conclude that
\[(4.4) \quad \tilde{u}^n \to u \text{ strongly in } L^2 (0, T; V).\]

Now to show that $u$ is a solution of Problem $P_2$, in inequality (3.1) set, for $z \in V, w = u^t + z \Delta t$ and divide by $\Delta t$, we get
\[
\left\{ \begin{aligned}
&\left< F \varepsilon (u^{t+1}) , \varepsilon (z) - \varepsilon \left( \frac{\Delta u^t}{\Delta t} \right) \right>_Q + j \left( u^{t+1} , z \right) - j \left( \frac{\Delta u^t}{\Delta t} \right) \\
&\geq \left< j^{t+1} , z - \frac{\Delta u^t}{\Delta t} \right>_V + \left< \sigma_{\nu} (u^{t+1}) + p_{\nu} (\tilde{u}^{t+1}) , z_{\nu} - \frac{\Delta u_{\nu}^t}{\Delta t} \right>_{G_3} \forall z \in V,
\end{aligned} \right.
\]
from which we deduce for any $z \in L^2 (0, T; V)$ the inequality
\[
\left\{ \begin{aligned}
&\left< F \varepsilon (\tilde{u}^n (t)), \varepsilon (z (t)) - \varepsilon (\tilde{u}^n (t)) \right>_Q + j \left( \tilde{u}^n (t) , z (t) \right) - j \left( \tilde{u}^n (t) , \tilde{u}^n (t) \right) \\
&\geq \left< \tilde{f}^n (t) , z (t) - \tilde{u}^n (t) \right>_V + \left< \sigma_{\nu} (\tilde{u}^n (t)) + p_{\nu} (\tilde{u}^n (t)) , z_{\nu} - \tilde{u}_{\nu}^n (t) \right>_{G_3},
\end{aligned} \right.
\]
for a.a. $t \in [0, T]$.

Integrating both sides of the previous inequality on $(0, T)$ we obtain the inequality
\[
\int_0^T \left< F \varepsilon (\tilde{u}^n (t)), \varepsilon (z (t)) - \varepsilon (\tilde{u}^n (t)) \right>_Q \, dt + \int_0^T j \left( \tilde{u}^n (t) , z (t) \right) \, dt \\
- \int_0^T j \left( \tilde{u}^n (t) , \tilde{u}^n (t) \right) \, dt \\
\geq \int_0^T \left< \tilde{f}^n (t) , z (t) - \tilde{u}^n (t) \right>_V \, dt \\
+ \int_0^T \left< \sigma_{\nu} (\tilde{u}^n (t)) + p_{\nu} (\tilde{u}^n (t)) , z_{\nu} (t) - \tilde{u}_{\nu}^n (t) \right>_{G_3} \, dt
\]
(4.5)
Now before passing to the limit in the previous inequality we start with the proof of the following lemma.
Lemma 4.6. For all \( z \in L^2(0, T; V) \) we have:

\[
\lim_{n \to \infty} \int_0^T \langle F(\varepsilon(\tilde{\nu}^n)) + \varepsilon(z(t)) - \varepsilon(u(t)) \rangle_Q \, dt \\
= \int_0^T \langle F\varepsilon(u(t)), \varepsilon(z(t)) - \varepsilon(u(t)) \rangle_Q \, dt,
\]

(4.6)

\[
\lim_{n \to \infty} \int_0^T j(\tilde{\nu}^n(t), z(t)) \, dt = \int_0^T j(u(t), z(t)) \, dt,
\]

(4.7)

\[
\lim_{n \to \infty} \int_0^T \left( \tilde{T}^n(t), z(t) - \tilde{u}^n(t) \right)_V \, dt = \int_0^T \left( f(t), z(t) - u(t) \right)_V \, dt,
\]

(4.8)

\[
\lim_{n \to \infty} \int_0^T \langle \sigma_n(\tilde{\nu}^n(t)) + p_n(\tilde{\nu}^n(t)), z_n(t) - \tilde{u}_n(t) \rangle_{\Gamma_3} \, dt \\
= \int_0^T \langle \sigma_n(u(t)) + p_n(u(t)), z_n(t) - \tilde{u}_n(t) \rangle_{\Gamma_3} \, dt.
\]

(4.9)

Proof. For the proof of (4.6) we refer the reader to [16, Lemma 10]. To prove (4.7) we write the term \( j(\tilde{\nu}^n(t), z(t)) \) as

\[
j(\tilde{\nu}^n(t), z(t)) = j(\tilde{\nu}^n(t), z(t)) - j(u(t), z(t)) + j(u(t), z(t)).
\]

We have

\[
j(\tilde{\nu}^n(t), z(t)) - j(u(t), z(t)) \\
= \int_{\Gamma_3} (p_n(\tilde{\nu}^n(t)) - p_n(u(t))) \, z_n(t) \, da + \int_{\Gamma_3} (p_r(\tilde{\nu}^n(t)) - p_r(u(t))) \, |z_r(t)| \, da.
\]

So using (2.13)(b) and the relation (2.9), we get

\[
\left| \int_0^T j(\tilde{\nu}^n(t), z(t)) - j(u(t), z(t)) \, dt \right| \leq C_4 \|\tilde{\nu}^n - u\|_{L^2(0,T;V)} \|z\|_{L^2(0,T;V)}
\]

where \( C_4 > 0 \). Then we deduce from (4.4) that

\[
\lim_{n \to \infty} \int_0^T j(\tilde{\nu}^n(t), z(t)) - j(u(t), z(t)) \, dt = 0.
\]

Finally to prove (4.8) it suffices to use (4.2) and to prove (4.9) it suffices also to use (2.10), (2.13), (4.2) and (4.4).

Lemma 4.7. We have:

\[
\lim_{n \to \infty} \int_0^T j(\tilde{\nu}^n(t), \tilde{\nu}^n(t)) \, dt \geq \int_0^T j(u(t), u(t)) \, dt.
\]

(4.10)
Proof. First, we have
\[
\int_0^T j(\tilde{u}^n(t), \dot{u}^n(t)) \, dt = \int_0^T (j(\tilde{u}^n(t), \dot{u}^n(t)) - j(u(t), \dot{u}^n(t))) \, dt \\
+ \int_0^T j(u(t), \dot{u}^n(t)) \, dt.
\]
There exists a constant \(C_5 > 0\) such that the first term of the second side of the equality can be estimated as
\[
\left| \int_0^T (j(\tilde{u}^n(t), \dot{u}^n(t)) - j(u(t), \dot{u}^n(t))) \, dt \right| \\
\leq C_5 \|\tilde{u}^n - u\|_{L^2(0,T;V)} \|\dot{u}^n\|_{L^2(0,T;V)}.
\]
Since \(\dot{u}^n\) is bounded in \(L^2(0,T;V)\) it follows from (4.4) that
\[
\lim_{n \to \infty} \int_0^T (j(\tilde{u}^n(t), \dot{u}^n(t)) - j(u(t), \dot{u}^n(t))) \, dt = 0.
\]
For the convergence of the other term, set
\[
p_r(u^\nu(t)) = k_r(t), r = \nu, \tau, k_r(t) \geq 0 \text{ for all } t \in [0,T].
\]
Setting \(k = (k_\nu, k_\tau)\), and keeping in mind the assumptions on the functions \(p_r\), it follows that
\[
k \in C([0,T]; (L^2(\Gamma_3))^2).
\]
Moreover, if we define the function \(\phi_k\) by
\[
\phi_k(z) = \int_{\Gamma_3} k_\nu z_\nu da + \int_{\Gamma_3} k_\tau |z_\tau| da,
\]
then \(\phi_k\) is lower semicontinuous and we have
\[
\liminf_{n \to \infty} \int_0^T \phi_k(\tilde{u}^n(t)) \, dt \geq \int_0^T \phi_k(\tilde{u}(t)) \, dt,
\]
whence we deduce (4.10).

Now using Lemma 4.6 and passing to the limit in (4.5) we deduce the inequality
\[
\left\{
\begin{array}{l}
\int_0^T \left( \langle \mathcal{F}_\varepsilon(u(t)), \varepsilon(z(t)) - \varepsilon(\tilde{u}(t)) \rangle_Q + j(u(t), z(t)) - j(u(t), \tilde{u}(t)) \right) \, dt \\
\quad \geq \int_0^T \langle f(t), z(t) - \tilde{u}(t) \rangle_\nu \, dt \\
\quad + \int_0^T \langle \sigma_\nu(u(t)) + p_\nu(u_\nu(t)), z_\nu(t) - \tilde{u}_\nu(t) \rangle_{\Gamma_3} \, dt, \quad \forall z \in L^2(0,T;V).
\end{array}
\right.
\]
As in [5] from inequality (4.11) we deduce the inequality (2.18). Also integrating the inequality (4.3) with respect to time and passing to the limit we obtain the following inequality from which we deduce the inequality (2.19).

\[
\int_0^T \left( (F \varepsilon (u(t)), \varepsilon (z(t)) - \varepsilon (u(t))) \right) dt \\
\geq \int_0^T (f(t), z(t) - u(t)) \, dt, \quad \forall z \in L^2(0, T; V) \text{ such that } z(t) \in K, \\
a.e. \ t \in [0, T].
\]

Therefore we conclude that \( u \) is a solution of Problem P₂.

5. Conclusion

Our main result in this paper concerns the existence of the weak solution in the study of a quasistatic frictional contact problem with finite penetration for nonlinear elastic materials. The contact is modelled with normal compliance law associated to a version of Coulomb’s law of dry friction. Under a smallness assumption on the contact functions we show the existence of a weak solution. Finally, we note that the important question of uniqueness of the solution is not resolved here, and remains still open.

References


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