On an application of almost increasing sequences

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Abstract. In the present paper, a general theorem on \(|N, p_n; \delta|_k\) summability factors of infinite series has been proved under weaker conditions. Some new results have also been obtained dealing with \(|N, p_n|_k\) and \(|C, 1; \delta|_k\) summability factors.

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1. Introduction

Let \(\sum a_n\) be a given infinite series with partial sums \((s_n)\). We denote by \(u_n^\alpha\) and \(t_n^\alpha\) the n-th Cesàro means of order \(\alpha\), with \(\alpha > -1\), of the sequence \((s_n)\) and \((na_n)\), respectively, i.e.,

\[
u_n^\alpha = \frac{1}{A_\alpha n} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \\
t_n^\alpha = \frac{1}{A_\alpha n} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,
\]

where

\[
A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.
\]

A series \(\sum a_n\) is said to be summable \(|C, \alpha|_k\), \(k \geq 1\), if (see [8], [11])

\[
\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{|t_n^\alpha|^k}{n} < \infty.
\]

and it is said to be summable \(|C, \alpha; \delta|_k\), \(k \geq 1\) and \(\delta \geq 0\), if (see [9])

\[
\sum_{n=1}^{\infty} n^{\delta k-1} |t_n^\alpha|^k < \infty.
\]

Let \((p_n)\) be a sequence of positive numbers such that

\[
P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).
\]
The sequence-to-sequence transformation

\[ \sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{7} \]

defines the sequence \((\sigma_n)\) of the Riesz mean or simply the \((\bar{N}, p_n)\) mean of the sequence \((s_n)\) generated by the sequence of coefficients \((p_n)\) (see [10]). The series \(\sum a_n\) is said to be summable \(|\bar{N}, p_n|_k\), \(k \geq 1\), if (see [2], [3])

\[ \sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} | \Delta \sigma_{n-1} |^k < \infty, \tag{8} \]

and it is said to be summable \(|\bar{N}, p_n; \delta|_k\), \(k \geq 1\) and \(\delta \geq 0\), if (see [5])

\[ \sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} | \Delta \sigma_{n-1} |^k < \infty, \tag{9} \]

where

\[ \Delta \sigma_{n-1} = \sigma_n - \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{10} \]

In the special case \(p_n = 1\) for all values of \(n\) (resp. \(\delta = 0\)) \(|\bar{N}, p_n|_k\) summability is the same as \(|C,1; \delta|_k\) (resp. \(|\bar{N}, p_n|_k\)) summability. Also, if we take \(\delta = 0\) and \(k = 1\), then we get \(|\bar{N}, p_n|\) summability.

2. Known results

Bor [4] has proved the following theorem for \(|\bar{N}, p_n|_k\) summability factors.

**Theorem 1.** Let \((X_n)\) be a positive non-decreasing sequence and let there be sequences \((\beta_n)\) and \((\lambda_n)\) such that

\[ | \Delta \lambda_n | \leq \beta_n, \tag{11} \]

\[ \beta_n \to 0 \text{ as } n \to \infty, \tag{12} \]

\[ \sum_{n=1}^{\infty} n | \Delta \beta_n | X_n < \infty, \tag{13} \]

\[ | \lambda_n | X_n = O(1). \tag{14} \]

If

\[ \sum_{v=1}^{n} \frac{|t_v|^k}{v} = O(X_n) \text{ as } n \to \infty, \tag{15} \]

where \((t_n)\) is the \(n\)-th \((C,1)\) mean of the sequence \((na_n)\), and \((p_n)\) is a sequence such that

\[ P_n = O(np_n), \tag{16} \]

\[ P_n \Delta p_n = O(p_n p_{n+1}), \tag{17} \]

then the series \(\sum_{n=1}^{\infty} a_n \frac{P_{n} \lambda_{n}}{p_n n} \) is summable \(|\bar{N}, p_n|_k\), \(k \geq 1\).
Recently, Bor [7] has generalized Theorem 1 for the \( \bar{N}, p_n; \delta \mid_k \) summability factors.

**Theorem 2.** Let \((X_n)\) be a positive non-decreasing sequence and the sequences \((\beta_n)\) and \((\lambda_n)\) are such that conditions (11)-(17) of Theorem A are satisfied with condition (15) replaced by:

\[
\sum_{v=1}^{n} \left( \frac{P_v}{p_v} \right)^{\delta k} |t_v|^{k} = O(X_n) \text{ as } n \to \infty. \tag{18}
\]

If

\[
\sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} = O\left( \frac{P_v}{p_v} \right)^{\delta k} \frac{1}{P_v} \text{ as } m \to \infty, \tag{19}
\]

then the series \( \sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n} \) is summable \( \bar{N}, p_n; \delta \mid_k \), \( k \geq 1 \) and \( 0 \leq \delta < 1/k \).

It should be noted that if we take \( \delta = 0 \) in Theorem 2, then we get Theorem 1. In this case condition (19) reduces to

\[
\sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = \sum_{n=v+1}^{m+1} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) = O\left( \frac{1}{P_v} \right) \text{ as } m \to \infty,
\]

which always holds.

**3. The main result**

The aim of this paper is to prove Theorem 2 under weaker conditions. For this we need the concept of an almost increasing sequence. A positive sequence \((b_n)\) is said to be almost increasing if there exist a positive increasing sequence \((c_n)\) and two positive constants \(A\) and \(B\) such that \(Ac_n \leq b_n \leq Bc_n\) (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say \( b_n = ne^{-1}n \). Now, we shall prove the following theorem.

**Theorem 3.** Let \((X_n)\) be an almost increasing sequence. If conditions (11)-(14) and (16)-(19) are satisfied, then the series \( \sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n} \) is summable \( \bar{N}, p_n; \delta \mid_k \), \( k \geq 1 \) and \( 0 \leq \delta < 1/k \).

We need the following lemmas for the proof of Theorem 3.

**Lemma 1** (see [12]). If \((X_n)\) is an almost increasing sequence, then under conditions (12)-(13) we have that

\[
nX_n \beta_n = O(1), \tag{20}
\]

\[
\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{21}
\]
Lemma 2 (see [4]). If conditions (16) and (17) are satisfied, then we have
\[ \Delta \left( \frac{P_n}{n^2 p_n} \right) = O \left( \frac{1}{n^2} \right). \]  
(22)

Lemma 3 (see [4]). If conditions (11)-(14) are satisfied, then we have that
\[ \lambda_n = O(1) \]  
(23)
\[ \Delta \lambda_n = O \left( \frac{1}{n} \right). \]  
(24)

4. Proof of Theorem 3

Let \((T_n)\) be the sequence of an \((\tilde{N}, p_n)\) mean of the series \(\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{vp_n}\). Then, by definition, we have
\[ T_n = \frac{1}{P_n} \sum_{v=1}^{n} \sum_{r=1}^{p_v} a_v P_r \lambda_v \frac{r a_r}{r p_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_r \lambda_v}{vp_v}. \]  
(25)

Then, for \(n \geq 1\)
\[ T_n - T_{n-1} = \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} a_v \lambda_v}{vp_v}. \]

Using Abel’s transformation, we get
\[ T_n - T_{n-1} = \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n} \Delta \left( \frac{P_{v-1} P_v \lambda_v}{v^2 p_v} \right) \sum_{r=1}^{v} r a_r + \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_{v+1} \Delta \lambda_v (v+1) \frac{t_v}{v^2 p_v} = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \]

To complete the proof of Theorem 3, by Minkowski’s inequality, it is sufficient to show that
\[ \sum_{n=1}^{\infty} \left( \frac{P_n}{P_{n-1}} \right)^{\delta k + k-1} |T_{n,r}|^k < \infty, \text{ for } r = 1, 2, 3, 4. \]  
(26)
Now, applying Hölder’s inequality, we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} | T_{n,1} |^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{p_{n-1}} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} t_v \| \lambda_v \| \frac{1}{v^k} \right\}^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \right)^k p_v \| t_v \| \lambda_v \| \frac{k}{v^k}
\]

\[
\times \left\{ \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k p_v \| t_v \| \lambda_v \| \frac{k}{v^k} \sum_{n=e+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{p_{n-1}}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k \lambda_v \| \frac{k}{v^k} \left( \frac{P_v}{p_v} \right)^{\delta k}
\]

\[
= O(1) \sum_{v=1}^{m} \lambda_v \| \frac{k}{v^k} \left( \frac{P_v}{p_v} \right)^{\delta k}
\]

\[
= O(1) \sum_{v=1}^{m} \Delta \lambda_v \| \sum_{r=1}^{v} \frac{P_r}{p_r} \| \frac{k}{v^k}
\]

\[
+ O(1) \| \lambda_m \| \sum_{v=1}^{m} \frac{P_v}{p_v} \| \frac{k}{v^k}
\]

\[
= O(1) \sum_{v=1}^{m} \| \Delta \lambda_v \| X_v + O(1) \| \lambda_m \| X_m
\]

\[
= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) \| \lambda_m \| X_m
\]

\[
= O(1),
\]

as \( m \to \infty \), by (11), (14), (16), (18), (19), (22) and (24).

Now using the fact that \( \langle P_v/v \rangle = O(p_v) \) by (16), we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} | T_{n,2} |^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{p_{n-1}} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} \| \Delta \lambda_v \| p_v \| t_v \| \right\}^k
\]
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} \left( \frac{p_v}{p_v} \right)^k | \Delta \lambda_v |^k | t_v |^k p_v \\
\times \left\{ \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
= O(1) \sum_{v=1}^{m} \left( \frac{p_v}{p_v} \right)^k | \Delta \lambda_v |^k | t_v |^k p_v \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{p_{n-1}} \\
= O(1) \sum_{v=1}^{m} \left( \frac{p_v}{p_v} \right)^{\delta k} \left( \frac{P_v}{p_v} \right)^{k-1} | \Delta \lambda_v |^{k-1} | \Delta \lambda_v | | t_v |^k \\
= O(1) \sum_{v=1}^{m} \left( \frac{p_v}{p_v} \right)^{\delta k} | t_v |^k \\
= O(1) \sum_{v=1}^{m} \beta_v \left( \frac{p_v}{p_v} \right)^{\delta k} | t_v |^k = O(1) \sum_{v=1}^{m} v \beta_v \left( \frac{p_v}{p_v} \right)^{\delta k} \frac{t_v |^k}{v} \\
= O(1) \sum_{v=1}^{m} \Delta(v \beta_v) \sum_{r=1}^{v} \left( \frac{p_r}{p_r} \right)^{\delta k} \frac{t_r |^k}{r} + O(1) m \beta_m \sum_{v=1}^{m} \left( \frac{p_v}{p_v} \right)^{\delta k} \frac{t_v |^k}{v} \\
= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m \beta_m X_m \\
= O(1) \text{ as } m \to \infty,

by (11), (13), (16), (18), (19), (21), (22) and (25).

Now, since \( \Delta \left( \frac{P_v}{p_v} \right) = O \left( \frac{1}{v^2} \right) \) by Lemma 2, we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k+k-1} |T_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{p_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v | \lambda_{v+1} | | t_v | \frac{1}{v} \right\}^k \\
\times \left\{ \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{p_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v p_v | \lambda_{v+1} | \frac{1}{v} | t_v | \right\}^k \\
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \right) p_v \frac{1}{v^k} | \lambda_{v+1} |^k | t_v |^k \\
\times \left\{ \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
= O(1) \sum_{v=1}^{m} \left( \frac{p_v}{p_v} \right)^k p_v \frac{1}{v^k} | \lambda_{v+1} |^{k-1} | \lambda_{v+1} | | t_v |^k
\[
\sum_{n=v+1}^{m+1} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k-1} \frac{1}{P_n^{\gamma}} = O(1) \sum_{v=1}^{m} \left( \frac{p_v}{p_{v-1}} \right)^{k-1} \frac{1}{v^k} \mid \lambda_{v+1} \mid \mid t_v \mid^k \left( \frac{p_v^{\delta k}}{p_v} \right)
\]

\[
= O(1) \sum_{v=1}^{m} \frac{p_v^{\delta k}}{p_{v-1}^k} \left( \frac{1}{v^k} \right) \mid \lambda_{v+1} \mid \mid t_v \mid^k
\]

\[
= O(1) \sum_{v=1}^{m} \frac{p_v^{\delta k}}{p_{v-1}^k} \mid \lambda_{v+1} \mid \mid t_v \mid^k
\]

as \( m \to \infty \), by (11), (14), (16), (18), (19), (22) and (24). Finally, as in \( T_{n,3} \), we have that

\[
\sum_{n=1}^{m} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k+k-1} \mid T_{n,4} \mid^k = O(1) \sum_{n=1}^{m} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k} \left( \frac{p_n^{k-1}}{p_{n-1}^k} \right) \left( \frac{n+1}{n} \right)^k \frac{1}{n^k} \mid \lambda_n \mid \mid t_n \mid^k
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{p_n^{\delta k}}{p_{n-1}^k} \right) n^{k-1} \frac{1}{n^k} \mid \lambda_n \mid \mid t_n \mid^k
\]

\[
= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid \left( \frac{p_n^{\delta k}}{p_{n-1}^k} \right) \frac{1}{n} \mid t_n \mid^k
\]

Therefore, we get that

\[
\sum_{n=1}^{m} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k+k-1} \mid T_{n,r} \mid^k = O(1) \text{ as } m \to \infty \text{, for } r = 1, 2, 3, 4.
\]

This completes the proof of the Theorem.

If we take \( \delta = 0 \), then we get a result of Bor [6] for \( \mid \lambda_n, p_n \mid_k \) summability factors. Also, if we take \( p_n = 1 \) for all values of \( n \), then we get a new result dealing with \( \mid C, 1; \delta \mid_k \) summability factors.
References