Math. Commun., Vol. **15**, No. 1, pp. 255-259 (2010)

A note on generalized absolute Cesàro summability

Hüseyin Bor[1,*]

[1] *Department of Mathematics, Erciyes University, Kayseri-38 039, Turkey*

Received December 13, 2008; accepted February 17, 2010

Abstract. In this paper, a known theorem dealing with $\mid C, 1 \mid_k$ summability methods has been generalized under weaker conditions for $\mid C, \alpha, \beta \,; \delta \mid_k$ summability methods. Some new results have also been obtained.
AMS subject classifications: 40D15, 40F05, 40G99

Key words: Cesàro summability, almost increasing sequences, summability factors

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence c_n and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = ne^{(-1)^n}$. A sequence (d_n) of positive numbers is said to be δ-quasi monotone, if $d_n > 0$ ultimately and $\Delta d_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the n-th Cesàro means of order (α, β), with $\alpha + \beta > -1$, of the sequence (s_n) and (na_n), respectively, i.e., (see [6])

$$u_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} s_v \tag{1}$$

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{2}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}),\ \alpha+\beta > -1,\ A_0^{\alpha+\beta} = 1 \quad and \quad A_{-n}^{\alpha+\beta} = 0 \text{ for } n > 0. \tag{3}$$

The series $\sum a_n$ is said to be summable $\mid C, \alpha, \beta \mid_k$, $k \geq 1$ and $\alpha + \beta > -1$, if (see [7])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta} \mid^k < \infty. \tag{4}$$

*Corresponding author. *Email address:* `bor@erciyes.edu.tr` (H. Bor)

Since $t_n^{\alpha,\beta} = n(u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta})$ (see [7]), condition (4) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{5}$$

The series $\sum a_n$ is summable $\mid C, \alpha, \beta\ ; \delta \mid_k$, $k \geq 1$, $\alpha + \beta > -1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} \mid u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta} \mid^k = \sum_{n=1}^{\infty} n^{\delta k-1} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{6}$$

If we take $\delta = 0$, then $\mid C, \alpha, \beta\ ; \delta \mid_k$ summability reduces to $\mid C, \alpha, \beta \mid_k$ summability. Also, if we take $\beta = 0$, then we get $\mid C, \alpha; \delta \mid_k$ (see [9]) summability. Furthermore, if we take $\beta = 0$ and $\delta = 0$, then $\mid C, \alpha, \beta\ ; \delta \mid_k$ summability reduces to $\mid C, \alpha \mid_k$ (see [8]) summability. It should be noted that obviously $(C, \alpha, 0)$ mean is the same as (C, α) mean.

Mazhar [10] has obtained the following theorem for $\mid C, 1 \mid_k$ summability factors of infinite series.

Theorem A. *Let (X_n) be a positive non-decreasing increasing sequence such that $\mid \Delta X_n \mid = O(X_n/n)$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent and $\mid \Delta\lambda_n \mid \leq A_n$ for all n. If*

$$\sum_{n=1}^{m} \frac{1}{n} \mid t_n \mid^k = O(X_m) \quad as \quad m \to \infty, \tag{7}$$

then the series $\sum a_n \lambda_n$ is summable $\mid C, 1 \mid_k$, $k \geq 1$.

2. The main result

The aim of this paper is to generalize Theorem A under weaker conditions for $\mid C, \alpha, \beta\ ; \delta \mid_k$ summability, by taking an almost increasing sequence instead of a positive non-decrasing sequence. We shall prove the following theorem.

Theorem 1. *Let (X_n) be an almost increasing sequence such that $\mid \Delta X_n \mid = O(X_n/n)$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent and $\mid \Delta\lambda_n \mid \leq |A_n|$ for all n. If the sequence $(\theta_n^{\alpha,\beta})$ is defined by*

$$\theta_n^{\alpha,\beta} = \mid t_n^{\alpha,\beta} \mid, \quad \alpha = 1, \beta > -1 \tag{8}$$

$$\theta_n^{\alpha,\beta} = \max_{1 \leq v \leq n} \mid t_v^{\alpha,\beta} \mid, \quad 0 < \alpha < 1, \beta > -1 \tag{9}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\delta k-1} (\theta_n^{\alpha,\beta})^k = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

then the series $\sum a_n \lambda_n$ is summable $\mid C, \alpha, \beta\ ; \delta \mid_k$ for $0 < \alpha \leq 1$, $\beta > -1$, $k \geq 1$, $\delta \geq 0$ and $\alpha + \beta - \delta > 0$.

It should be noted that if we take (X_n) as a positive non-decreasing sequence, $\alpha = 1$, $\delta = 0$ and $\beta = 0$, then we get Theorem A. In this case condition (10) reduces to condition (7). We need the following lemmas for the proof of our theorem.

Lemma 1 (see [3]). *Let (X_n) be an almost increasing sequence such that $n \mid \Delta X_n \mid = O(X_n)$. If (A_n) is a δ-quasi-monotone with $\sum n\delta_n X_n < \infty$ and $\sum A_n X_n$ is convergent, then*

$$nA_nX_n = O(1) \quad as \quad n \to \infty, \tag{11}$$

$$\sum_{n=1}^{\infty} nX_n \mid \Delta A_n \mid < \infty. \tag{12}$$

Lemma 2 (see [4]). *If $0 < \alpha \leq 1$, $\beta > -1$ and $1 \leq v \leq n$, then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \leq \max_{1 \leq m \leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{13}$$

3. Proof of Theorem 1

Let $(T_n^{\alpha,\beta})$ be the n-th (C, α, β) mean of the sequence $(na_n\lambda_n)$. Then, by means of (2) we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

First applying Abel's transformation and then using Lemma 2, we have that

$$\begin{aligned}
T_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \\
\mid T_n^{\alpha,\beta} \mid &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\
&\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \theta_n^{\alpha,\beta} \\
&= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}, \text{ say.}
\end{aligned}$$

Since

$$\mid T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} \mid^k \leq 2^k (\mid T_{n,1}^{\alpha,\beta} \mid^k + \mid T_{n,2}^{\alpha,\beta} \mid^k),$$

in order to complete the proof of Theorem 1, by using (6) it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty, \text{ for } r = 1, 2.$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1} \mid T_{n,1}^{\alpha,\beta} \mid^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1} \mid \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \Delta\lambda_v \mid^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+(\alpha+\beta-\delta)k}} \{\sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} A_v (\theta_v^{\alpha,\beta})^k\} \times \{\sum_{v=1}^{n-1} A_v\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} A_v (\theta_v^{\alpha,\beta})^k \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta-\delta)k}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} A_v (\theta_v^{\alpha,\beta})^k \int_v^{\infty} \frac{dx}{x^{1+(\alpha+\beta-\delta)k}} \\
&= O(1) \sum_{v=1}^{m} A_v v^{\delta k} (\theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{m} v A_v v^{\delta k-1} (\theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(vA_v) \sum_{p=1}^{v} p^{\delta k-1} (\theta_p^{\alpha,\beta})^k \\
&\quad + O(1) m A_v \sum_{v=1}^{m} v^{\delta k-1} (\theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\,(vA_v)|\, X_v + O(1) m A_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v\,|\Delta A_v|\, X_v + O(1) \sum_{v=1}^{m-1} A_v X_v + O(1) m A_m X_m \\
&= O(1), \text{ as } m \to \infty,
\end{aligned}$$

in view of hypotheses of Theorem 1 and Lemma 1.

Similarly, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1} \mid T_{n,2}^{\alpha,\beta} \mid^k &= O(1) \sum_{n=1}^{m} |\lambda_n|\, n^{\delta k-1} (\theta_n^{\alpha,\beta})^k \\
&= O(1) \sum_{n=1}^{m} n^{\delta k-1} (\theta_n^{\alpha,\beta})^k \sum_{v=n}^{\infty} \mid \Delta\lambda_v \mid \\
&= O(1) \sum_{v=1}^{\infty} \mid \Delta\lambda_v \mid \sum_{n=1}^{v} n^{\delta k-1} (\theta_n^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{\infty} A_v X_v < \infty,
\end{aligned}$$

by virtue of hypotheses of Theorem 1 and Lemma 1. Therefore, by (6) we get that

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty, \text{ for } r = 1, 2. \tag{14}$$

This completes the proof of Theorem 1.

If we take $X_n = logn$, $\beta = 0$, $\delta = 0$ and $\alpha = 1$, then we get a result of Mazhar [10] dealing with $\mid C, 1 \mid_k$ summability factors. Also, if we take $\beta = 0$, $\delta = 0$, then we have a new result for $\mid C, \alpha \mid_k$ summability factors. Finally, if we take $\delta = 0$, then we get another new result for $\mid C, \alpha, \beta \mid_k$ summability factors.

References

[1] S. Aljančić, D. Arandelović, *O*-regularly varying functions, Publ. Inst. Math. **22**(1977), 5–22.

[2] R. P. Boas, *Quasi positive sequences and trigonometric series*, Proc. London Math. Soc. **14A**(1965), 38–46.

[3] H. Bor, *An application of almost increasing and δ-quasi-monotone sequences*, J. Inequal. Pure Appl. Math. **3**(2002), Art. 16, (electronic).

[4] H. Bor, *On a new application of quasi power increasing sequences*, Proc. Estonian Acad. Sci. Phys. Math. **57**(2008), 205–209.

[5] H. Bor, *On the generalized absolute Cesàro summability*, Pac. J. Appl. Math. **2**(2009), 35–40 .

[6] D. Borwein, *Theorems on some methods of summability*, Quart. J. Math. **9**(1958), 310–316.

[7] G. Das, *A Tauberian theorem for absolute summability*, Proc. Camb. Phil. Soc. **67**(1970), 321–326.

[8] T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. **7**(1957), 113–141.

[9] T. M. Flett, *Some more theorems concerning the absolute summability of Fourier series and power series*, Proc. London Math. Soc. **7**(1958), 357–387.

[10] S. M. Mazhar, *On generalized quasi-convex sequence and its applications*, Indian J. Pure Appl. Math. **8**(1977), 784–790.