SHORT PROOFS OF SOME BASIC CHARACTERIZATION THEOREMS OF FINITE p-GROUP THEORY

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Abstract. We offer short proofs of such basic results of finite p-group theory as theorems of Blackburn, Huppert, Ito-Ohara, Janko, Taussky. All proofs of those theorems are based on the following result: If $G$ is a nonabelian metacyclic p-group and $R$ is a proper $G$-invariant subgroup of $G'$, then $G/R$ is not metacyclic. In the second part we use Blackburn’s theory of p-groups of maximal class. Here we prove that a p-group $G$ is of maximal class if and only if $\Omega^2_2(G) = \langle x \in G \mid o(x) = p^2 \rangle$ is of maximal class. We also show that a noncyclic p-group $G$ of exponent $> p$ contains two distinct maximal cyclic subgroups $A$ and $B$ of orders $> p$ such that $|A \cap B| = p$, unless $p = 2$ and $G$ is dihedral.

This note is a continuation of the author’s previous papers [Ber1, Ber2, Ber4]. Only finite p-groups, where $p$ is a prime, are considered. The same notation as in [Ber1] is used. The $n$th member of the lower central series of $G$ is denoted by $K_n(G)$. Given a p-group $G$ and a natural number $n$, set $\Omega_n(G) = \langle x^{p^n} \mid x \in G \rangle$, $\Omega_n(G) = \langle x \in G \mid o(x) \leq p^n \rangle$, $\Omega_n^2(G) = \langle x \in G \mid o(x) = p^n \rangle$, $U^2(G) = \bar{U}_1(\bar{U}_1(G))$, $p^d(G) = |G : \Phi(G)|$, where $\Phi(G)$ is the Frattini subgroup of $G$. Next, $G'$ is the derived subgroup and $Z(G)$ is the center of $G$. A group $G$ of order $p^m$ is of maximal class if $m > 2$ and $\text{cl}(G) = m - 1$. A group $G$ is metacyclic if it contains a normal cyclic subgroup $C$ such that $G/C$ is cyclic. A group $G$ is said to be minimal nonabelian if it is nonabelian but all its proper subgroups are abelian. A p-group $G$ is regular if, for $x, y \in G$, there is $z \in \langle x, y \rangle'$ such that $(xy)^p = x^py^pz^p$. A p-group $G$ is absolutely regular if $|G/\bar{U}_1(G)| < p^p$. A p-group $G$ is powerful...
Let $G$ be a nonabelian $p$-group.

(a) (Tuan) If $G$ has an abelian subgroup of index $p$, then $|G| = p|G'||Z(G)|$.

(b) (Mann)\(^1\) If $M, N$ are two distinct maximal subgroups of $G$, then $|G'| \leq p|M'N'|$.

(c) (Blackburn) If $G/K_{p+1}(G)$ is of maximal class, then $G$ is also of maximal class.

(d) (Berkovich) Suppose that $G$ contains the unique subgroup $L$ of index $p^{p+1}$. If $G/L$ is of maximal class, then $G$ is also of maximal class (obviously, this is also true in the case $|G : L| > p^{p+1}$).

(e) (Hall) If $cl(G) < p$ or $exp(G) = p$, then $G$ is regular.

(f) (Hall) If $G$ is regular, then $exp(\Omega_n(G)) \leq p^n$ and $|\Omega_n(G)| = |G/\Omega_n(G)|$.

(g) (Blackburn) If $G$ is of maximal class and order $\leq p^{p+1}$, then $exp(G) \leq p^2$. If $|G| \leq p^2$, then $|G : \Omega_1(G)| \leq p$. If $|G| = p^{p+1}$, then $G$ is irregular and $|G/\Omega_1(G)| = p^p$.

(h) (Blackburn) A $p$-group $G$ of maximal class has an absolutely regular subgroup $G_1$ of index $p$, and $exp(G_1) = exp(G)$. In particular, if $G$ is of order $> p^{p+1}$, it has no normal subgroup of order $p^p$ and exponent $p$ since, for each $n > 1$, $G$ has at most one normal subgroup of index $p^n$. If $|G| > p^p$, then $|\Omega_1(G_1)| = p^{p-1}$. Next, all elements of the set $G - G_1$ have orders $\leq p^2$.

(i) (Berkovich) If $G$ has a nonabelian subgroup $B$ of order $p^3$ such that $G_2(B) < B$, then $G$ is of maximal class.

(j) (Lubotzky-Mann) If $G$ is powerful and $X$ is a maximal cyclic subgroup of $G$, then $X \notin \Phi(G)$.

(k) (Berkovich) If $N$ is a two-generator $G$-invariant subgroup of $\Phi(G)$, then $N$ is metacyclic.

(l) (Blackburn) If $G$ is of maximal class and order $> p^{p+1}$, then exactly $p$ maximal subgroups of $G$ are of maximal class, the $(p+1)$-th maximal subgroup $G_1$, the fundamental subgroup of $G$, is absolutely regular.

\(^1\)This result is also contained in Kazarin’s Ph.D. thesis (unpublished).
(m) (Berkovich) If $H < G$ and $N_G(H)$ is of maximal class, then $G$ is also of maximal class.

(n) (Berkovich) Let $G$ be irregular but not of maximal class. If $U < G$, $|U| < p^p$ and $\exp(U) = p$, then there is in $G$ a normal subgroup $V$ of order $p^p$ and exponent $p$ such that $U < V$.

(o) (Blackburn) If $G$ is irregular of maximal class and a normal subgroup $V$ of $G$ is of order $p^{p-1}$, then $\exp(G/V) = \frac{1}{p} \cdot \exp(G)$.

(p) (Blackburn) If an irregular group $G$ has an absolutely regular maximal subgroup $H$, then either $G$ is of maximal class or $G = H\Omega_1(G)$, where $|\Omega_1(G)| = p^p$.

(q) (Blackburn) If an irregular group $G$ has no normal subgroup of order $p^p$ and exponent $p$, it is of maximal class.

(r) (Hall’s regularity criterion) Absolutely regular $p$-groups are regular.

(s) (Berkovich) If $G$ is a $p$-group of maximal class and order $> p^{p+1}$, then $c_2(G) \equiv p^{p-2} \pmod{p^{p-1}}$.

(t) (Berkovich) If a $p$-group $G$ is neither absolutely regular nor of maximal class, then $c_2(G) \equiv 0 \pmod{p^{p-1}}$.

(u) (Berkovich) Let $G$ be not of maximal class. Then the number of subgroups of maximal class and index $p$ in $G$ is divisible by $p^2$.

(v) (Berkovich) If $G$ is neither absolutely regular nor of maximal class, then the number of subgroups of order $p^p$ and exponent $p$ is $G$ is $\equiv 1 \pmod{p}$.

(w) (Blackburn) If $\Omega_2(G)$ is metacyclic, then $G$ is also metacyclic.

2°. Blackburn [Bla1, Theorem 2.3] has proved that a $p$-group $G$ is metacyclic if and only if $G/K_3(G)\Phi(G')$ is metacyclic. This result is an important source of characterizations of metacyclic $p$-groups. Here we prove this assertion in slightly another, but equivalent form (Theorem 2). The main point of this section is to deduce from that theorem some basic results of $p$-group theory. Besides, our proof of Theorem 2 is essentially simpler than the Philip Hall’s proof presented in [Bla1].

We prove Blackburn’s result in the following form.

**Theorem 2.** The following conditions for a nonabelian $p$-group $G$ are equivalent:

(a) $G$ is metacyclic.

(b) The quotient group $G/R$ is metacyclic for some $G$-invariant subgroup $R$ of index $p$ in $G'$.  

**Remark 3.** If there is a $G$-invariant subgroup $R < G'$ such that $G/R$ is metacyclic, then $G$ is also metacyclic. Indeed, take $R \leq R_1 < G'$, where $R_1$ is

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2Since $d(G) = 2$, $R$ is characteristic in $G$, by Lemma 7(b).
we have \( 1 = \ldots \) and 
\[ (\text{see Lemma 7, below.}) \]

Blackburn’s result: 
\[ G \]

\[ G = G' \text{ is metacyclic if and only if } G = G' \]

The following lemma is a useful criterion for a p-group to be minimal nonabelian.

**Lemma 5 ([BJ1, Lemma 65.2(a)])**. *If a p-group* \( G \) *is such that* \( d(G) = 2, G' \leq Z(G) \) *and* \( \exp(G') = p, \) *then* \( G \) *is minimal nonabelian.*

**Proof.** It follows from \( \exp(G') = p \) that \( G \) is nonabelian. For \( x, y \in G, \) we have \( 1 = [x, y]^p = [x, y^p] \) so \( y^p \in Z(G) \) whence \( \Phi(G) = G' \Omega_1(G) \leq Z(G) \) and \( \Phi(G) = Z(G) \) since \( |G : Z(G)| > p. \) If \( M < G \) is maximal, then \( |M : Z(G)| = p \) so \( M \) is abelian. We are done.

**Lemma 6 ([Red]).** *If* \( G \) *is a nonmetacyclic minimal nonabelian p-group, then

\[ G = \langle a, b \mid a^{p^n} = b^{p^n} = e^p = 1, [a, b] = c, [a, c] = [b, c] = 1, m \geq n \rangle. \]

*Here* \( G' = \langle c \rangle \) *is a maximal cyclic subgroup of* \( G, Z(G) = \Phi(G) = \langle a^p, b^p, c \rangle \) *has index* \( p^2 \) *in* \( G, \) \( \Omega_1(G) = \langle a^{p^{n-1}}, b^{p^{n-1}}, c \rangle \) *is elementary abelian of order* \( p^3, \) \( \Omega_1(G) = \langle a^p, b^p \rangle \) *and* \( |G/\Omega_1(G)| = p^3 \) *if and only if* \( p > 2. \)

**Proof.** If \( a, b \in G \) are not permutable, then \( G = \langle a, b \rangle. \) If \( A, B \) are distinct maximal subgroups of \( G, \) then \( A \cap B = Z(G) \) and \( G/Z(G) \) is abelian of type \( (p, p) \) so \( \Phi(G) = Z(G). \) We have \( |G'| = \frac{1}{p} |G : Z(G)| = p \) (Lemma 1(a)).

Let \( G/G' = (U/G') \times (V/G'), \) where both factors are cyclic of orders \( p^m, p^n, \) respectively, \( m \geq n; \) then \( U \) and \( V \) are noncyclic \( (G \) is not metacyclic!) so \( \Omega_1(G) = \Omega_1(U) \Omega_1(V) \) is elementary abelian of order \( p^3 \) (indeed, \( \Omega_1(G)/G' = \Omega_1(G/G') \)) Assume that \( G' < L < G, \) where \( L \) is cyclic of order \( p^2. \) We have \( m + n > 2. \) Then \( G/G' = (C/G') \times (D/G'), \) where \( L \leq C \) and \( C/G' \) is cyclic, by \([BZ, \text{Theorem 1.16}].\) It follows from \( G' = \Phi(L) \leq \Phi(C) \) that \( 1 = d(C/G') = d(C) \) so \( C \) is cyclic. Since \( G/C \) is cyclic, \( G \) is metacyclic, contrary to the hypothesis. All remaining assertions are obvious.

It follows from Lemma 6 that a minimal nonabelian p-group \( G \) is not metacyclic if and only if \( G' \) is a maximal cyclic subgroup of \( G.\)

\(^3\)In our case, \( R_1 \) is determined uniquely since \( G'/K_3(G) \) is cyclic and \( K_3(G) \leq R_1; \) see Lemma 7, below.
Lemma 7. (a) If a $p$-group $G$ is two-generator of class 2, then $G'$ is cyclic.
(b) [Bla1, Lemma 2.2] If $G$ is a nonabelian two-generator $p$-group, then $G'/K_3(G)$ is cyclic.

Proof. (a) Since $\text{cl}(G) = 2$, then $[xy, uv] = [x, u][x, v][y, u][y, v]$ for $x, y, u, v \in G$. Let $G = \langle a, b \rangle$ and $w, z \in G$. Expressing $w, z$ in terms of $a$ and $b$ and using the above identity, we see that the commutator $[w, z]$ is a power of $[a, b]$, so $G' = \langle [a, b] \rangle$.
(b) The statement (b) follows from (a): $G/K_3(G)$ is two-generator of class 2.

Remark 8. Let $G$ be a nonmetacyclic minimal nonabelian 2-group given by $(\ast)$. We claim that if $G = AB$, where $A$ and $B$ are cyclic, then $n = 1$. Assume that this is false. Set $G = G/\langle a^4, b^4 \rangle$; then $G$ is of order $2^5$ and exponent 4 so it is not a product of two cyclic subgroups (of order $\leq 4$). This is a contradiction since $G = AB$. Let, in addition, $m > n = 1$. We claim that $G$ is indeed a product of two cyclic subgroups. Set $A = \langle a \rangle$. Then $G/\overline{U}_1(A)$ is dihedral of order 8. Let $U/\overline{U}_1(A) < G/\overline{U}_1(A)$ be cyclic of order 4. If $B_0$ is a cyclic subgroup which covers $U/\overline{U}_1(A)$, then, by the product formula, $G = AB_0$, as want to be shown.

Remark 9. Let $G$ be a nonabelian two-generator $p$-group. It follows from Lemma 6 and Theorem 2 that if $R$ is a $G$-invariant subgroup of index $p$ in $G'$, then $G$ is metacyclic if and only if $G'/R$ is not a maximal cyclic subgroup of $G/R$. In particular, we obtain the following theorem from [IO]: The derived subgroup $G'$ of a 2-group $G = AB$ ($A$ and $B$ are cyclic) is contained properly in a cyclic subgroup of $G$ if and only if $G$ is metacyclic.

Remark 10. If $G$ is a nonmetacyclic $p$-group, then it contains a characteristic subgroup $R$ such that $G/R$ is one of the following groups: (i) elementary abelian of order $> p^2$, (ii) nonabelian of order $p^3$ and exponent $p$, (iii) a 2-group, given in $(\ast)$, with $m = n = 2$, (iv) a 2-group, given in $(\ast)$, with $m = 2, n = 1$. (Obviously, groups (i)-(iii) are not products of two cyclic subgroups.) Let us prove this. If $d(G) > 2$, we have case (i) with $R = \Phi(G)$. Next assume that $d(G) = 2$. If $p > 2$, we have case (ii) with $R = K_3(G)\Phi(G')\overline{U}_1(G) = K_3(G)\overline{U}_1(G)$ (Theorem 2 and Lemmas 5-7). If $p = 2$, we have cases (iii) or (iv) with $R = K_3(G)\Phi(G')\overline{U}_2(G)$ (Corollary 4 and Lemma 6).

4The group (iv) is a product of two cyclic subgroups; see the footnote to the proof of Corollary 17.
characteristic maximal subgroups at all). In particular, a 2-group \( G \) is metacyclic if and only if \( G \) and all its maximal subgroups are two-generator. This also follows from

**Corollary 11 ([Bla1]).** Suppose that a nonabelian \( p \)-group \( G \) and all its maximal subgroups are two-generator. Then \( G \) is either metacyclic or \( p > 2 \) and \( K_3(G) = \bar{U}_1(G) \) has index \( p^3 \) in \( G \) (in the last case, \( |G: G'| = p^2 \)).

**Proof.** Suppose that \( G \) is not metacyclic. In cases (iii) and (iv) of Remark 10, \( G \) has a maximal subgroup that is not generated by two elements so \( p > 2 \). By Lemma 6, \( G \) has no nonmetacyclic epimorphic image which is minimal nonabelian of order \( > p^3 \). The group \( G \) also has no epimorphic image of order \( > p^3 \) and exponent \( p \) so \( |G: \bar{U}_1(G)| = p^3 \). Assume that \( |G : G'| > p^2 \). Let \( R \) be a \( G \)-invariant subgroup of index \( p \) in \( G' \). Then \( G/R \) is a nonmetacyclic minimal nonabelian group (Theorem 2 and Lemma 5) of order \( > p^3 \), contrary to what has just been said. Thus, \( |G : G'| = p^2 \). Then \( G/K_3(G) \) is minimal nonabelian since its center \( G'/K_3(G) \) has index \( p^2 \); moreover, that quotient group is nonmetacyclic (Remark 3). In that case, by the above, \( |G/K_3(G)| = p^3 = |G/\bar{U}_1(G)| \) so \( K_3(G) = \bar{U}_1(G) \) since \( \bar{U}_1(G) \leq K_3(G) \).

**Corollary 12 (Taussky).** Let \( G \) be a nonabelian 2-group. If \( |G : G'| = 4 \), then \( G \) is of maximal class.

**Proof.** Let \( R \) be a \( G \)-invariant subgroup of index 2 in \( G' \). Then \( G/R \) is nonabelian of order 8 so metacyclic; then \( G \) is metacyclic (Theorem 2) so \( G \) has a normal cyclic subgroup \( U < G \) such that \( G/U \) is cyclic. Since \( G' < U \), we get \( |G : U| = 2 \), and the result follows from description of 2-groups with cyclic subgroup of index 2.

**Corollary 13 (Huppert [Hup]).** Let \( G \) be a \( p \)-group, \( p > 2 \), and let \( |G/\bar{U}_1(G)| \leq p^2 \). Then \( G \) is metacyclic.

**Proof.** Assuming that \( G \) is not metacyclic, we must consider cases (i) and (ii) of Remark 10. We have there \( |G/\bar{U}_1(G)| > p^2 \), a contradiction.

**Supplement 1 to Corollary 11.** Let \( G \) be a \( p \)-group.

(a) \( G \) is metacyclic if and only if \( G/\bar{U}_2(G) \) is metacyclic.

(b) [Ber1, Theorem 3.4] \( G \) is metacyclic if and only if \( G/\bar{U}_2(G) \) is metacyclic.

**Proof.** (b) \( \Rightarrow \) (a) since \( \bar{U}_2(G) \leq \bar{U}_2(G) \) (indeed, \( \exp(G/\bar{U}_2(G)) \leq p^2 \)). If \( G \) is not metacyclic, then \( G/\bar{U}_2(G) \) is not metacyclic (Remark 10), proving (b).

**Supplement 2 to Corollary 11.** Suppose that a nonabelian \( p \)-group \( G \) and all its characteristic subgroups of index \( \frac{1}{p}|G : G'| \) are two-generator.
Then either $G$ is metacyclic or $p > 2$ and $G/K_3(G)$ is of order $p^3$ and exponent $p$. If, in addition, a nonmetacyclic $p$-group $G$ and all its characteristic subgroups are two-generator, then $K_3(G) = \tilde{U}_1(G)$.

**Proof.** By Lemma 7(b), a $G$-invariant subgroup $R$ of index $p$ in $G$ is characteristic in $G$. Suppose that $G$ is nonmetacyclic; then $G/R$ is also nonmetacyclic (Theorem 2) and minimal nonabelian (Lemma 5). Assume that $|G/R| > p^3$. Then $H/R = \Omega_1(G/R)$ is elementary abelian of order $p^3$ (Lemma 6), $d(H) > 2$, $|G/H| = p^2|G/G'|$ and $H$ is characteristic in $G$, contrary to the hypothesis. Thus, $|G/R| = p^3$ so $|G/G'| = \frac{1}{p}|G/R| = p^2$; then $p > 2$ since $G/R$ is nonmetacyclic (Corollary 12). It follows that $G/K_3(G)$ is minimal nonabelian so $|G'/K_3(G)| = p$ (Lemma 6); then $R = K_3(G)$ and $\exp(G/R) = p$ since $G/R$ is not metacyclic (Corollary 11).

Now suppose, in addition, that all characteristic subgroups of a nonmetacyclic $p$-group $G$ are two-generator. Set $\tilde{G} = G/\tilde{U}_1(G)$. Assume that $|\tilde{G}| > p^3$. Let $\tilde{G}$ be of order $p^3$; then it contains an abelian subgroup $\tilde{A}$ of index $p$ and $d(\tilde{A}) \geq d(\tilde{A}) = 3$ so, by hypothesis, $\tilde{A}$ is not characteristic in $\tilde{G}$. Then $\tilde{G}$ has another abelian maximal subgroup $\tilde{B}$. We have $\tilde{A} \cap \tilde{B} = Z(\tilde{G})$ so $\tilde{G}$ is minimal nonabelian since $d(\tilde{G}) = 2$. But a minimal nonabelian group of exponent $p$ has order $p^3$ (Lemma 6), a contradiction. Now let $|\tilde{G}| > p^2$. Then $d(\tilde{G}') = 2$, by hypothesis, so $|\tilde{G}'| = p^2$ since $\exp(\tilde{G}') = p$ (Lemma 1(k)). In that case, $|\tilde{G}| = |\tilde{G} : \tilde{G}'| |\tilde{G}'| = p^4$, contrary to the assumption. Thus, $|\tilde{G}/\tilde{U}_1(G)| = p^3$ so $K_3(G) = \tilde{U}_1(G)$ since $K_3(G) \leq \tilde{U}_1(G)$ and $|G/K_3(G)| = p^3$. $\blacksquare$

In particular, if a 2-group $G$ and all its characteristic subgroups of index $\frac{1}{3}(G : G')$ are two-generator, then $G$ is metacyclic, and this implies Corollary 12.

In the proof of Theorem 2 we use only Lemma 7(b) which is independent of all other previously proved results.

**Proof of Theorem 2.** It suffices to show that (b) $\Rightarrow$ (a). Since $G/R$ is metacyclic, it has a normal cyclic subgroup $U/R$ such that $G/U$ is cyclic. Assume that $U$ is noncyclic. Then $U$ has a $G$-invariant subgroup $T$ such that $U/T$ is abelian of type $(p, p)$. Set $G = G/T$. In that case, $R \nsubseteq T$ since $\tilde{U} = U/T$ cannot be an epimorphic image of the cyclic group $U/R$; then $G' \nsubseteq T$ so $G$ is nonabelian. Next, $G/G'$ is noncyclic so $G' < \tilde{U}$ and $|G'| = p$ since $|U| = p^2$. It follows from $G' = G'/T/T \cong G'/G' \cap T$ that $G' \cap T = R$, by Lemma 7(b). Then $R = G' \cap T < T$, a contradiction.$^5$

If a $p$-group $G$ is nonmetacyclic but all its proper epimorphic images are metacyclic, then either $G$ is of order $p^3$ and exponent $p$ or $G$ is as given in (c)

$^5$Isaacs proved the following equivalent of Theorem 2. Let $G$ be a $p$-group and let $Z < G'$ be $G$-invariant of order $p$. If $G/Z$ is metacyclic, then $G$ is metacyclic; see [Ber5, Lemma 11].
with \( m = 2 \) and \( n = 1 \). Indeed, the result is trivial for abelian \( G \). Now let \( G \) be nonabelian. Let \( R \) be a \( G \)-invariant subgroup of index \( p \) in \( G' \); then \( G/R \) is not metacyclic (Theorem 2) so \( R = \{1\} \), and we get \( |G'| = p \). By Lemma 5, \( G \) is minimal nonabelian. Now the assertion follows from Lemma 6.

**Corollary 14.** Suppose that a nonabelian and nonmetacyclic \( p \)-group \( G \) and all its maximal subgroups are two-generator, \( p > 2 \) and \( |G| = p^m \), \( m > 3 \); then \( \text{cl}(G) > 2 \). Set \( K = K_4(G) \) and \( \bar{G} = G/K \). Then one of the following holds:

(a) \( \bar{G} \) is of order \( p^4 \). In particular, if \( p = 3 \), then \( G \) is of maximal class.

(b) \( |\bar{G}| = p^5 \), all maximal subgroups of \( \bar{G} \) are minimal nonabelian (see [BJ2, Theorem 5.5] for defining relations of \( G \)).

**Proof.** By Corollary 11, \( K_3(G) = \bar{O}_3(G) \) has index \( p^3 \) in \( G \) so that \( \text{cl}(G) > 2 \) since \( m > 3 \) and \( |G : G'| = p^2 \); \( d(G) = 2 \). Then \( Z(\bar{G}) = K_3(G)/K \) has index \( p^3 \) in \( \bar{G} \) since \( \text{cl}(G) = 3 \). Let \( M < G \) be maximal; then \( |M : Z(\bar{G})| = \frac{p}{p^2} G : Z(G) = p^2 \) and, since \( d(M) = 2 \), it follows that \( M \) is either abelian or minimal nonabelian. In view of Lemma 6, \( G \) has a nonabelian maximal subgroup, say \( \bar{M} \). By Lemma 1(a), \( \bar{G} \) has at most one abelian maximal subgroup.

Suppose that \( \bar{G} \) has an abelian maximal subgroup, say \( \bar{A} \). Then \( |\bar{G}'| \leq p|\bar{M}'\bar{A}'| = p^2 \) (Lemma 1(b)) so \( |\bar{G}| = |\bar{G}'||G : G'| = p^4 \), and we get \( \text{cl}(\bar{G}) = 3 \). In particular, if \( p = 3 \), then \( G \) is of maximal class (Lemma 1(c)). Thus, \( G \) is as stated in part (a).

Now suppose that all maximal subgroups of \( \bar{G} \) are minimal nonabelian; then \( |\bar{G}| > p^4 \). If \( \bar{U}, \bar{V} \) are distinct maximal subgroups of \( \bar{G} \), then \( |\bar{G}'| \leq p|\bar{U}'\bar{V}'| = p^3 \) so \( |\bar{G}'| = p^3 \) since \( p^5 \leq |\bar{G}| = |\bar{G} : G'||G'| \leq p^5 \).

Blackburn found indices of the lower central series of groups of Corollary 14 for \( p > 3 \) (the case \( p = 3 \) is open); see [Bla2].

Our arguments in Corollary 15 and Remark 16 are based on [Jan2].

**Corollary 15 (Janko [Jan2]).** If every maximal cyclic subgroup of a noncyclic \( p \)-group \( G \) is contained in a unique maximal subgroup of \( G \), then \( G \) is metacyclic.

**Proof.** Let \( N \) be a proper normal subgroup of \( G \) and let \( U/N \leq G/N \) be maximal cyclic. Then \( U = AN \) for a cyclic \( A \). Let \( B \supseteq A \) be a maximal cyclic subgroup of \( G \); then \( B \cap N = A \cap N \) and \( U/N = BN/N \) so \( |A| = |B| \) and \( A = B \), i.e., \( A \) is a maximal cyclic subgroup of \( G \). Assume that \( K/N, M/N \) are distinct maximal subgroups of \( G/N \) containing \( U/N \). Then \( A \leq U \leq K \cap M \), contrary to the hypothesis. Thus, the hypothesis is inherited by epimorphic images.

Let \( A < G \) be maximal cyclic. Then \( A\Phi(G)/\Phi(G) \) is contained in a unique maximal subgroup of \( G/\Phi(G) \) so \( A\Phi(G) \) is maximal in \( G \), and we conclude
that \(d(G) = 2\). Assume that \(G\) is nonmetacyclic. Let \(R\) be a \(G\)-invariant subgroup of index \(p\) in \(G'\). Then \(\bar{G} = G/R\) is nonmetacyclic (Theorem 2) and minimal nonabelian (Lemma 5) so \(\bar{G}'\) is maximal cyclic in \(\bar{G}\) (Lemma 6). Since \(\bar{G}/G'\) is abelian of rank 2, \(\bar{G}'\) is contained in \(1 + p > 1\) maximal subgroups of \(\bar{G}\), contrary to the previous paragraph.

**Remark 16.** Obviously, metacyclic \(p\)-groups are powerful for \(p > 2\). Let us show (this is Janko’s result as well) that \(G\) of Corollary 15 is also powerful for \(p = 2\), unless \(G\) is of maximal class. Assume that \(G\) is not of maximal class. Then \(|G/G'| > 4\) (Corollary 12) so \(W = G/\bar{U}_2(G)\) cannot be nonabelian of order 8. It suffices to show that \(W\) is abelian. Assume that this is false. Then \(W = \langle a, b | a^4 = b^4 = 1, a^b = a^{-1}\rangle\) is the unique nonabelian metacyclic group of order 24 and exponent 4 (Corollary 15). In that case, \(W/(a^2b^2)\) is ordinary quaternion so has two distinct maximal subgroups \(U/(a^2b^2)\) and \(V/(a^2b^2)\). Since \(\langle a^2b^2\rangle\) is a maximal cyclic subgroup of \(W\), we get a contradiction. Thus, \(G\) is powerful. Then, by Lemma 1(j), if \(X < G\) is maximal cyclic, then \(X\) is not contained in \(\Phi(G)\) (Lemma 1(j)) so \(X\Phi(G)\) is the unique maximal subgroup of \(G\) containing \(X\) since \(d(G) = 2\). Thus, \(G\) satisfies the hypothesis of Corollary 15 if and only if it is powerful and metacyclic.

It follows from Corollary 13 that a \(p\)-group \(G = AB\), where \(A\) and \(B\) are cyclic, is metacyclic if \(p > 2\). This is not true for \(p = 2\), however, we have

**Corollary 17 (Ito-Ohara [10]).** If a nonmetacyclic 2-group \(G = AB\) is a product of two cyclic subgroups \(A\) and \(B\), then \(G/G'\) is of type \((2^m, 2), m > 1\).

**Proof.** Let \(R\) be a \(G\)-invariant subgroup of index 2 in \(G'\). Then \(\bar{G} = G/R\) is nonmetacyclic (Theorem 2) and minimal nonabelian (Lemma 5) as in (\(\ast\)). Since \(\bar{G} = \bar{A}\bar{B}\), we get \(n = 1\) (Remark 8). Next, \(m > 1\) (Corollary 12).

**Remark 18.** Suppose that a nonmetacyclic 2-group \(G = AB\) is a product of two cyclic subgroups \(A\) and \(B\). Since \(A \cap B = \Phi(A) \cap \Phi(B)\), we get \(\Phi(G) = \Phi(A)\Phi(B)\), by the product formula, so \(\Phi(G)\) is metacyclic (Lemma 1(k)). It follows that all subgroups of \(G\) are three-generator. By Corollary 11, \(G\) has a maximal subgroup \(M\) with \(d(M) = 3\). We claim that \(M\) is the unique maximal subgroup of \(G\) which is not generated by two elements. Indeed, let \(U, V\) be maximal subgroups of \(G\), containing \(A, B\), respectively; then \(U \neq V\). By the modular law, \(U = A(U \cap B)\) and \(V = B(V \cap A)\) so \(d(U) = 2 = d(V)\) since \(G\) is nonmetacyclic. Since the set of maximal subgroups of \(G\) is \(\{M, U, V\}\), our claim follows. In particular, \(M\) is characteristic in \(G\). Set \(\bar{G} = G/\bar{U}_2(G)\); then \(\bar{G} = \bar{A}\bar{B}\) so \(|\bar{A}| = 4 = |\bar{B}|\) since \(\bar{G}\) is of exponent 4 (in fact, \(\bar{G}\) is a group (iv) of Remark 10).

\(^6\)The author and Janko [15] have proved independently that subgroups \(U\) and \(V\) are metacyclic; see the proof of Supplement to Corollary 17 due to the author.
Suppose that $X$ is a 2-group such that $d(X) = 2$, $\exp(X) > 2$ and $\Phi(X)$ is metacyclic. We claim that $|X/\Phi(X)| \leq 2^4$. Assume that this is false. Clearly, $\Phi(X)$ is metacyclic. Then, $\Phi(X) = 1$, i.e., $\exp(X) = 2^4 \leq |\Phi(X)| \leq 2$ since $\Phi(X)$ is metacyclic of exponent $\leq 4$. By Burnside, $\Phi(X)$ cannot be nonabelian of order 8 so it is either abelian of type $(4, 2)$, or abelian of type $(4, 2)$, or $\Phi(X) = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1} \rangle$. In any case, every generating system of $\Phi(X)$ must contain an element of order 4. It follows from $\Phi(X) = 1$ that $X$ has an element of order 8, a contradiction since $\exp(X) = 4$.

**Supplement to Corollary 17.** Let $G = AB$ be a nonmetacyclic 2-group, where $A$ and $B$ are cyclic and let $G/G'$ be abelian of type $(2^m, 2)$, $m > 1$ (see Corollary 17). Then the set $\Gamma_1 = \{U, V, M\}$ is the set of maximal subgroups of $G$, where $A < U$, $B < V$, the subgroups $U, V$ are metacyclic but not of maximal class and $d(M) = 3$.

**Proof.** By Remark 18, $\Phi(G) = 1$ is metacyclic but not cyclic since $G$ has no cyclic subgroup of index 2.

Since $d(G) = 2$ and $G$ is not minimal nonabelian, we get $Z(G) < \Phi(G)$.

Assume that $U$ is of maximal class. Since $G$ is nonmetacyclic, it is not of maximal class. Then, by [Ber1, Theorem 7.4(a)], we get $d(G) = 3$, a contradiction. Similarly, $V$ is also not of maximal class.

Let us prove, for example, that $U$ is metacyclic. Assume that this is false. Then $U/\Phi(U)$ is nonmetacyclic, by Blackburn’s result [Ber1, Theorem 3.4]; in particular, $|U/\Phi(U)| \geq 2^4$ and $G/\Phi(U)$ is nonmetacyclic. Since $d(U) = 2$ and $\Phi(U)$ is metacyclic, we get $|U/\Phi(U)| = 2^4$ (see the paragraph preceding the supplement). We have $\Phi(U) = G$ and $\Phi(U) < \Phi(M)$ (otherwise, all maximal subgroups of two-generator nonmetacyclic group $G/\Phi(U)$ are two-generator, contrary to [Ber1, Theorem 3.3]). We conclude that $d(M/\Phi(U)) = 3$. Next, $G/\Phi(U) = (A\Phi(U)/\Phi(U))(B\Phi(U)/\Phi(U))$, where both factors are cyclic. Therefore, to get a contradiction, one may assume that $\Phi(U) = 1$. In that case, $|G| = 2^5$, $U = \langle x, y \mid x^4 = b^2 = z^2 = 1, z = [x, y], [x, z] = [y, z] = 1 \rangle$ is minimal nonabelian. Since $U$ is not metacyclic and two-generator, it has no normal cyclic subgroup of order 4. Since $G = AB$ is of order 25 and exponent $\leq 8$, one of the factors $A, B$ is 2-generator, namely $B$ (since $|A| \leq \exp(U) = 4$) has order 8, by the product formula. Then $\exp(V) = 8$ and $|V : B| = 2$. It follows from $\Phi(V) = \Phi_1(B)$ that $\Phi_1(B) < G$.

But $\Phi_1(B) = \Phi(B) < \Phi(G) < U$, and the cyclic subgroup $\Phi_1(B)$ of order 4 is normal in $G$. Since $G$ is not of maximal class, $\Phi_1(B) < U$.

\[ \Phi_1(B) \subseteq \Phi_1(U) \]
type $(2, 2)$. Let $x \in G - R$ be an involution. Then $D = \langle x, R \rangle \cong D_8$. By Lemma 1(i), $D G(D)$ is nonmetacyclic, a contradiction. It follows that then $G$ is either dihedral or semidihedral\(^7\). If, in addition, $G$ is nonabelian and satisfies $\Omega_1(G) = G$, then it is dihedral.

**Remark 20.** Suppose that a metacyclic 2-group $G$ of exponent $\geq 2^3$ satisfies $\Omega_2(G) = G$. Then $G$ is either generalized quaternion or $G/\Omega_1(G)$ is dihedral with $\Omega_1(G) \leq Z(G)$. Obviously, $G$ is nonabelian. If $G$ is of maximal class, it is generalized quaternion. Next assume that $G$ is not of maximal class. Then $G$ has a normal four-subgroup $R$ (Lemma 1(q)) and $R = \Omega_1(G)$ (Remark 19). If $U < G$ is cyclic of order 4, then $U \cap R = \Omega_1(U)$ so $|RU/R| = 2$. It follows that $\Omega_1(G/R) = G/R$ so $G/R$ is dihedral, by Remark 19. We claim that if $G$ is metacyclic and $G/R$ is dihedral ($R = \Omega_1(G)$ is a four-subgroup), then $R \leq Z(G)$. Indeed, let $U/\Omega_1(G) < G/\Omega_1(G)$ be of order 2; then $U$ is abelian (Remark 19). Since all such $U$ centralize $\Omega_1(G)$ and generate $G$, 1\((G/R) = Z(G)\).

**Remark 21.** Let $G$ be a 2-group. Suppose that $H = \Omega_2(G)$ is metacyclic of exponent $\geq 2^3$. Then one of the following holds: (a) $G$ is of maximal class (in that case, $H = G$), (b) $G$ is metacyclic with dihedral $G/\Omega_1(G)$ (then $H = G$ and $\Omega_1(G) \leq Z(G)$) or semidihedral (then $|G/H| = 2$). Indeed, by Lemma 1(w), $G$ is metacyclic. By Remark 20, $H$ is one of groups (a), (b). If $H$ is of maximal class, then $c_2(G) = c_2(H) \equiv 1 \pmod{4}$ so $G$ is of maximal class, by Lemma 1(p) and 1(q). Now let $H$ be not of maximal class and let $R \subset H$ be $G$-invariant of type $(2, 2)$. We have $\Omega_1(H/R) = H/R$ so $H/R$ is dihedral and $R \leq Z(H)$ (remarking 19, 20).

If $G$ is a nonmetacyclic 2-group of order $2^m$ and $m > n \geq 4$, then the number of normal subgroups $D$ of $G$ such that $G/D$ is metacyclic of order $2^n$, is even [Ber5].

3°. In this section, most proofs are based on properties of $p$-groups of maximal class and counting theorems.

Let $G$ be a $p$-group of exponent $p^e > p^2$, $p > 2$, and let $1 < k < e$. Suppose that $H < G$ is metacyclic of exponent $p^k$ such that whenever $H < L$, then $\exp(L) > p^k$. Then $G$ is also metacyclic. This is a consequence of Corollary 13 and the following

**Theorem 22.** Let $G$ be a $p$-group of exponent $p^e > p^2$ and let $1 < k < e$. Suppose that $U$ is a maximal member of the set of subgroups of $G$ having exponent $p^k$.

(a) If $U$ is absolutely regular then $G$ is also absolutely regular, $U = \Omega_k(G)$ and the subgroup $U$ is not of maximal class.
(b) If $U$ is irregular of maximal class, then $G$ is also of maximal class.

**Proof.** If $G$ is absolutely regular, then $U$ is also absolutely regular. If $G$ is a 2-group of maximal class, then $U$ is also of maximal class (and order $2^{k+1}$).

Let $G$ be of maximal class, $p > 2$ and let $U$ be absolutely regular. Then $G$ is irregular since $e > 2$ (Lemma 1(g)). Denote by $G_1$ the absolutely regular subgroup of index $p$ in $G$; then $\exp(G_1) = \exp(G) = p^k > p^k$ (Lemma 1(h)). Assume that $U < G_1$. Then $U = \Omega_k(G_1) < G_1$ since $k < e$, hence $U < G$. Since $|G : U| > p$, then all elements of the set $(G/U) - (G_1/U)$ have the same order $p$ [Ber3, Theorem 13.19], so there exists $H/U < G/U$ such that $H \not\subset G_1$ and $|H : U| = p$. Then $H$ is of maximal class [Ber3, Theorem 13.19] so $\exp(H) = \exp(U)$ (Lemma 1(h)), contrary to the choice of $U$. Now suppose that $U \not\subset G_1$. We get $k = 2$ (otherwise, $U = \Omega_k(U) \leq \Omega_k(G) \leq G_1$, by Lemma 1(h)). Assume that $\Omega_1(G_1) \not\subset U$. Let $R \leq \Omega_1(G_1)$ be a minimal $G$-invariant subgroup such that $R \not\subset U$. In that case, $|UR : R| = p$. By Lemma 1(f), (1(h) and (1(p)), $\exp(UR) = \exp(U)$, contrary to the choice of $U$. Thus, $\Omega_1(G_1) < U$ and $|U| \geq p^p$ (Lemma 1(h)); moreover, by [Ber3, Theorem 13.19], $|U| = p^p$. Let $U < H \leq G$, where $|H : U| = p$. Then $H$ is of maximal class [Ber3, Theorem 13.19] and order $p^{k+1}$ so $\exp(H) = p^2 = \exp(U)$ and $U < H$, contrary to the choice of $U$. Thus, if $G$ is irregular of maximal class, then $U$ must be also irregular of maximal class and $\Omega_k(G_1)$ has index $p$ in $U$.

In what follows we may assume that $G$ is not of maximal class.

Next we proceed by induction on $|G|$.

(i) Let $G$ be noncyclic and regular; then $U$ is absolutely regular. Then $U = \Omega_k(G)$ (Lemma 1(f)) so $\Omega_1(G) = \Omega_1(U)$ and $p^p > |U/\Omega_1(U)| = |\Omega_1(U)| = |G/\Omega_1(G)|$, whence $G$ is absolutely regular; in that case, $p > 2$.

Assume that, in addition, $U$ is of maximal class. Then $|U : \Omega_1(U)| = p$ (Lemma 1(g)) so $|\Omega_1(G/\Omega_1(G))| = p$. It follows that $G/\Omega_1(G)$ is cyclic (of order $p^p$). Let $D$ be a $G$-invariant subgroup of index $p^2$ in $\Omega_1(U) = \Omega_1(G)$, and set $C = \Gamma_G(\Omega_1(U)/D)$; then $C/D$ is abelian and $U \leq C$ so $U/D$ is abelian of order $p^k$, and we conclude that $U$ is not of maximal class, contrary to the assumption. Thus, $U$ is not of maximal class.

In what follows we assume that $G$ is irregular.

(ii) Let $U$ be absolutely regular; then $|\Omega_1(U)| = |U/\Omega_1(U)| < p^p$. We write $R = \Omega_1(U)$ and $N = N_G(R) = G$. Then $U < N$.

Assume that $N = G$. Then, by Lemma 1(a), there is in $G$ a normal subgroup $S$ of order $p|\Omega_1(U)|$ and exponent $p$ such that $R < S$. Set $H = US$. Then $H/S \cong U/R$ is of exponent $p^{k-1}$ so, since $U < H$, we get $\exp(H) = p^k$, contrary to the choice of $U$.

Now let $N < G$. Then $N$ is absolutely regular, by induction and Lemma 1(m). In that case, $U = \Omega_k(N)$ so $R = \Omega_1(N)$ is characteristic in $N$ whence $N = G$, contrary to the assumption.
(iii) In what follows we assume that $U$ is irregular of maximal class. Set $V = \Omega_1(\Phi(U))$ and $N = N_G(V)$. If $N < G$, then, by induction, $N$ is of maximal class so $G$ is also of maximal class (Lemma 1(m)), contrary to the assumption. Now let $N = G$. Then, as in (ii), $G$ has a normal subgroup $R$ of order $p^e$ and exponent $p$ such that $V < R$. Set $H = UR$; then $H/R \cong U/V$ is of exponent $p^{k-1}$. This is a contradiction since $\exp(H) = p^k = \exp(U)$ and $U < H$. \hfill \square

**Supplement 1 to Theorem 22.** Let $G$ be a $p$-group of exponent $p^c > p$, $1 < k \leq e$. Set $H = \Omega_k^p(G)$.

(a) If $H$ is absolutely regular, then $G$ is either absolutely regular or irregular of maximal class.

(b) If $H$ is of maximal class, then $G$ is also of maximal class.

**Proof.** We proceed by induction on $|G|$. One may assume that $H < G$.

(a) Suppose that $H$ is absolutely regular. Set $R = \Omega_1(H)$; then $R \triangleleft G$.

Assume that $G$ is neither absolutely regular nor of maximal class. Then $G$ contains a normal subgroup $S$ of order $p|R|$ and exponent $p$ such that $R < S$ (Lemma 1(n)). Set $U = HS$. Assume that $U$ is of maximal class. Then $|S| = |H| = \frac{1}{p}|U|$ (Lemma 1(h)), $|HS| = p^{k+1}$, $\exp(HS) = p^2$ so $k = 2$ and $H(= \Omega_2^p(G))$ is the unique maximal subgroup of $HS$ of exponent $p^2$. In that case, $c_2(G) = c_2(H) \not\equiv 0 \pmod{p^{e-1}}$ so $G$ is either absolutely regular or of maximal class (Lemma 1(s) and 1(t)), contrary to the assumption. The proof of (a) is complete.

(b) Suppose that $H$ is irregular of maximal class.

Assume that $|H| > p^{k+1}$. Then $c_2(G) = c_2(H) \equiv p^{e-2} \pmod{p^{e-1}}$, so $G$ is of maximal class (Lemma 1(s) and 1(t)), a contradiction.

It remains to consider the possibility $|H| = p^{k+1}$; then $\exp(H) = p^2$ (Lemma 1(g)) so $k = 2$. In that case, $c_2(H) = c_2(G) \equiv 0 \pmod{p^{e-1}}$ (Lemma 1(t)) so $\Omega_1(H)$ is of order $p^e$ and exponent $p$. Let $H < A \leq G$ and $|A : H| = p$. By [Ber3, Theorem 13.21], one may assume that $A$ is not of maximal class. By [Ber1, Theorem 7.4(c)], $A$ contains exactly $p + 1$ regular subgroups $T_1, \ldots, T_{p+1}$ of index $p$ which are not absolutely regular. It follows that $\exp(T_i) = p$ for all $i$ (otherwise, $T_i = \Omega_2^p(T_i) \leq \Omega_2^p(G) = H$, which is not the case). Then $T_i \cap H = \Omega_1(H)$. It follows that $\Omega_1(H)$ is contained in $p + 2$ pairwise distinct subgroups $T_1, \ldots, T_{p+1}$ of index $p$ in $A$, a contradiction since $A/\Omega_1(H)$ is of order $p^2$. \hfill \square

**Supplement 2 to Theorem 22.** Let $H$ be a metacyclic subgroup of exponent $2^k$ of a $2$-group $G$. Suppose that $H$ is maximal among subgroups of exponent $2^k$ in $G$. Then $G$ has no $H$-invariant elementary abelian subgroup of order 8 (see [Jan1]).
Proof. Assume that $G$ has an $H$-invariant elementary abelian subgroup $E$ of order 8. To get a contradiction, one may assume, without loss of generality, that $G = HE$; then $E \triangleleft G$. Set $L = H \cap E$; then $|L| \leq 4$ and $L$ is normal in $G$.

Let $L = \{1\}$. If $L_0 \leq E \cap Z(G)$ is of order 2, then $H < H \times L_0$ and $\exp(H \times L_0) = 2^k$, contrary to the choice of $H$.

Let $L$ be of order 4. Then $G = L \times E$ is of exponent $2^{k-1}$ so $\exp(G) = 2^k$, contrary to the choice of $H$.

Now let $|L| = 2$. In view of Theorem 22, one may assume that $H$ is not of maximal class. Then $H$ contains a normal abelian subgroup $R$ of type $(2, 2)$. By the product formula, $|ER| = 16$. Note that $ER$ is $H$-invariant. We also have $R < C_E(R)$ and $C_E(R)$ is $H$-invariant. Let $R < F < R \cap C_E(R)$, where $F$ is an $H$-invariant subgroup of order 8; then $F$ is elementary abelian, the quotient group $HF/R = (H/R) \times (F/R)$ has exponent $2^{k-1}$ so $\exp(HF) = 2^k$, contrary to the choice of $H$ since $H < HF$.

For related results, see [Ber4].

Let $s$ be a positive integer. A $p$-group $G$ is said to be an $L_s$-group, if $\Omega_1(G)$ is of order $p^s$ and exponent $p$ and $G = \Omega_1(G)$ is cyclic of order $> p$ ($\Omega_1(G)$ is said to be the kernel of $G$).

Below we use the following

Lemma 23 ([Ber1, Lemma 2.1]). Let $G$ be a $p$-group with $|\Omega_2(G)| = p^{p+1} < |G|$. Then one of the following holds:

(a) $G$ is absolutely regular.
(b) $G$ is an $L_p$-group.
(c) $p = 2$ and $G = \langle a, b \mid a^{2^n} = 1, a^{2^{n-1}} = b^4, a^b = a^{-1+2^{n-2}} \rangle$.

It is known that an irregular $p$-group $G$ has a maximal regular subgroup $R$ of order $p^p$ if and only if $G$ is of maximal class [Ber3, §10].

The following theorem supplements this result.

Theorem 24. Let $G$ be a $p$-group and let $H < G$ be a maximal member of the set of subgroups of $G$ of exponent $p^2$. Suppose that $|H| = p^{p+1}$. Then one of the following holds:

(a) $p = 2$ and $G$ is of maximal class.
(b) $H = \Omega_2(G)$ (see Lemma 23).

Proof. If $G$ is regular, then $H = \Omega_2(G)$ so $G$ is a group of Lemma 23. Next let $G$ be irregular. By hypothesis, $\exp(H) < \exp(G)$.

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Footnote 9: This is an easy consequence of Lemma 1(m). Indeed, write $N = N_G(R)$. If $N < G$, then $N$ is of maximal class, by induction, and we are done (Lemma 1(m)). Now let $N = G$. Take $D$, a $G$-invariant subgroup of index $p^2$ in $N$, and set $C = C_G(R/D)$. If $B/R \leq C/R$ is of order $p$, then $B$ is regular since $B/D$ is abelian of order $p^3$ (Lemma 1(e)), a contradiction since $R < B$. 
Suppose that $G$ is irregular of maximal class. It follows from [Ber3, theorems 9.5 and 9.6] that then $p = 2$, and we get case (a). Indeed, assume that $p > 2$. If $H \leq G_1$, then $H = \Omega_2(G_1)$. If $H = G_1$, then $\exp(H) = \exp(G)$, contrary to the choice of $H$. Thus, $H < G_1$. Let $U/H$ be a subgroup of $G/H$ of order $p$ not contained in $G_1/H$. Then $U$ is of maximal class and exponent $p^2$ [Ber3, Theorem 13.19], contrary to the choice of $H$. Now let $H \not\leq G_1$; then $\Omega_1(G_1) \leq H$ and $H$ is of maximal class. Let $H < F \leq G$ with $|F:H| = p$. Then $\exp(F) = \exp(H)$, contrary to the choice of $H$. The $2$-groups of maximal class satisfy the hypothesis.

In what follows we assume that $G$ is not of maximal class. Then, in view of Theorem 22, one may assume that $H$ is neither absolutely regular nor of maximal class so $\text{cl}(H) < p$. It follows that $H$ is regular (Lemma 1(e)) and $\Omega_1(H)$ is of order $p^n$ and exponent $p$. Set $N = N_G(\Omega_1(H))$; then $H < N$ since $\Omega_1(H)$ is characteristic in $H < G$. We use induction on $|G|$. Assume that $N < G$. Then, by induction, $N$ is one of groups (a,b). However, in case (b), $\Omega_1(H)$ is characteristic in $N$ (Lemma 23) so $N = G$, contrary to the assumption. On the other hand, $N$ cannot be a $2$-group of maximal class since $H$ is abelian of type $(4,2)$, by the previous paragraph.

Thus, $N = G$ so $\Omega_1(H) < G$. By hypothesis, $G/\Omega_1(H)$ has no abelian subgroup $K/\Omega_1(H)$ of type $(p,p)$ such that $H < K$, so $G/\Omega_1(H)$ is either cyclic or generalized quaternion (then $p = 2$). In that case, $\Omega_1(G) = \Omega_1(H)$ so that $\Omega_2(G) = H$. \[\]

Let a natural number $n \geq p - 1$. A $p$-group $G$ is said to be a $U^p_n$-group provided it has a normal subgroup $R$ of order $p^n$ and exponent $p$ such that $G/R$ is irregular of maximal class and, if $T/R$ is absolutely regular of index $p$ in $G/R$, then $\Omega_1(T) = R$.\footnote{It follows from Lemma 1(p) and 1(q), that $U^n_p$-groups do no exist for $n < p - 1$. The $U^2_2$-groups are classified by Janko; see [Jan3] or [BJ1, §67].} Let us prove that if a normal subgroup $R_1$ of $G$ is of exponent $p$, then $R_1 \leq R$. Assume that this is false and that every proper $G$-invariant subgroup of $R_1$ is contained in $R$; then $|RR_1 : R| = p$ so $RR_1/R < T/R$ since $G/R$ has only one minimal normal subgroup. This is a contradiction: $RR_1 \leq \Omega_1(T) = R < RR_1$. It follows that $R$ is characteristic in $G$. We call $R$ the kernel of the $U^p_n$-group $G$. It follows from Lemma 1(p) that $U^p_{p-1}$-groups are of maximal class. Note that $\exp(G) = p\exp(G/R) = \exp(T)$.

**Theorem 25.** Let $G$ be a $p$-group and let $H < G$ be a maximal member of the set of subgroups of $G$ of exponent $\exp(H)$. If $H$ is a $U^p_n$-group, then $G$ is also a $U^p_n$-group.

**Proof.** We use induction on $|G|$. In view of Theorem 22(b), one may assume that $H$ is not of maximal class so that $n > p - 1$. Let $R$ be the kernel of $H$ and set $N = N_G(R)$. If $N < G$, then $N$ is a $U^p_n$-group, by induction. In that case, $R$ is also kernel of $N$ so characteristic in $N$. It follows that $N = G$,
contrary to the assumption. Thus, \( N = G \). Then \( H/R \) is a maximal member of the set of subgroups of exponent \( \frac{1}{p} \cdot \exp(H) \) in \( G/R \) and \( H/R \) is irregular of maximal class. Then \( G/R \) is of maximal class, by Theorem 22. Let us show that \( G \) is a \( U_{n}^{p} \)-group. Let \( T/R \) be the\(^{11} \) absolutely regular subgroup of index \( p \) in \( G/R \) (Lemma 1(h)) and set \( U/R = (H/R) \cap (T/R) \). Then \( U/R \) is an absolutely regular subgroup of index \( p \) in \( H/R \) so \( \Omega_{1}(U) = R \) since \( H \) is a \( U_{n}^{p} \)-group. Let \( F/R < T/R \) be \( G \)-invariant of order \( p \). It follows from the subgroup structure of \( G/R \) (see [Ber3, §9 and Theorem 13.19]) that \( F/R \leq \Phi(G/R) < H/R \) so \( F/R \leq \Phi(H/R) < U/R \), and we get \( \exp(F) = p^{2} \) since \( F \) is not contained in \( R = \Omega_{1}(U) \). In that case, \( R = \Omega_{1}(T) \) so \( G \) is a \( U_{n}^{p} \)-group.

Remark 26. Let \( G \) be a \( p \)-group and let \( H < G \) be a maximal member of the set of subgroups of \( G \) of exponent \( \exp(H) \). If \( H \) is an \( L_{n} \)-group, then \( G \) is also an \( L_{n} \)-group. To prove this, it suffices to repeat, with small modifications, the proof of Theorem 25 and use the following easy fact: If \( C < G \) is a cyclic subgroup of order \( p^{k} > p \) which is not contained properly in a subgroup of exponent \( p^{k} \), then \( G \) is cyclic.

The following theorem is an analogue of Supplement 1 to Corollary 11 and dual, in some sense, to Theorem 22.

Theorem 27. Suppose that a \( p \)-group \( G \) is such that \( G/\tilde{U}^{2}(G) \) is of maximal class. Then \( G \) is also of maximal class.

Proof. (a) Suppose that \( G \) is regular. Then \( |G/\tilde{U}^{1}(G)| = p^{k} \), where \( k < p \), and \( |G/\tilde{U}^{2}(G)| = p^{k+1} \) (Lemma 1(g)) so \( |\tilde{U}^{1}(G) : \tilde{U}^{1}(\tilde{U}^{1}(G))| = p \), and we conclude that \( \tilde{U}^{1}(G) \) is cyclic. Let \( |\tilde{U}^{1}(G)| = p^{e} \); then \( \exp(G) = p^{e+1} \). By Lemma 1(f), \( |\Omega_{1}(1)| = p^{k} \). Since \( |G| = p^{k+1} \), it follows that \( G/\Omega_{1}(G) \) is cyclic of order \( p^{e} \). By hypothesis, \( |G : G'| = p^{2} \) so \( e = 1 \). In that case, \( \tilde{U}^{2}(G) = \{1\} \) so \( G \) is of maximal class, by hypothesis.

(b) Now let \( G \) be irregular. One may assume that \( |G| > p^{p+1} \) (otherwise, in view of Lemma 1(e), it is nothing to prove). By Lemma 1(r), \( |G/\tilde{U}^{1}(G)| \geq p^{p} \) so \( |G/\tilde{U}^{2}(G)| \geq p^{p+1} \) and we conclude that \( G/\tilde{U}^{2}(G) \) is irregular (Lemma 1(g)). By hypothesis and Lemma 1(g), we get \( |G/\tilde{U}^{1}(G)| = p^{p} \) and \( |G/G'| = p^{2} \).

(i) Let \( L \) be a normal subgroup of index \( p^{p+1} \) in \( G \). By the previous paragraph, \( \exp(G/L) > p \). Let \( R/L = \tilde{U}^{2}(G/L) \); then \( \tilde{U}^{2}(G) \leq R \). It follows from properties of irregular \( p \)-groups of maximal class\(^{12} \) that \( |G/R| \geq p^{p+1} = |G/L| \) so \( R = L \), and we conclude that \( \exp(G/L) = p^{2} \) and \( \tilde{U}^{2}(G) \leq L \).

(ii) Assume that \( L \) and \( L_{1} \) are distinct normal subgroups of the same index \( p^{p+1} \) in \( G \). Then \( \tilde{U}^{2}(G) \leq L \cap L_{1} \), by (i). In that case, \( L/\tilde{U}^{2}(G) \) and

\(^{11} \)the” since \( |G/R| > |H/R| \geq p^{p+1} \).

\(^{12} \)If \( X \) is irregular \( p \)-group of maximal class, then every its epimorphic image of order \( p^{e} \) has exponent \( p \).
\(L_1/\Omega^2(G)\) are different normal subgroups of index \(p^{r+1} > p\) in a \(p\)-group of maximal class \(G/\Omega^2(G)\), which is impossible (Lemma 1(h)). Thus, \(G\) has the unique normal subgroup, say \(L\), of index \(p^{r+1}\). By the above, \(G/L\), as a nonabelian epimorphic image of \(G/\Omega^2(G)\), is of maximal class. Then, by Lemma 1(d), \(G\) is also of maximal class.

The case \(p = 2\) of Theorem 27 follows immediately from Corollary 12.

A \(p\)-group \(G\) is of maximal class if and only if \(G/\Omega^2(G)\) is of maximal class. Indeed, \(\Omega^2(G) \leq \Omega^2(G)\) so \(G/\Omega^2(G)\) is of maximal class as a nonabelian epimorphic image of \(G/\Omega^2(G)\), and the result follows from Theorem 27.

**Remark 28.** Now we offer another argument for part (b) of the proof of Theorem 27. Let \(H/\Omega^2(G)\) be an absolutely regular subgroup of index \(p\) in \(G/\Omega^2(G)\), existing, by Lemma 1(h). Assume that \(H\) is not absolutely regular. Then, by Lemma 1(r), we have \(|H/\Omega^2(H)| \geq p^r\). Clearly, \(\Omega^2(G) \leq \Omega_1(H)\) so \(H/\Omega_1(H)\) of order \(\geq p^r\) and exponent \(p\) is an epimorphic image of the absolutely regular group \(H/\Omega^2(G)\), a contradiction.\(^\text{13}\) Thus, \(H\) is absolutely regular. Assume that \(G\) is not of maximal class. Then \(G = H\Omega_1(G)\), where \(\Omega_1(G)\) is of order \(p^r\) and exponent \(p\) (Lemma 1(p)). By hypothesis, \(|G/G'| = p^2\). We have \(G/H = \Omega_1(G)\), \(H = H\Omega_1(H) \cong (H/\Omega_1(H)) \times (\Omega_1(G)/\Omega_1(H))\) so \(|H/\Omega_1(H)| = p\), \(|H| = p|\Omega_1(H)| = p^p\) and \(|G| = p^{p+1}\). In that case, \(\Omega^2(G) = \{1\}\) so \(G\) is of maximal class, contrary to the assumption.

In Remark 29 we use the following fact. If \(G\) is neither absolutely regular nor maximal class and \(E_1, \ldots, E_r\) are all its subgroups of order \(p^r\) and exponent \(p\), then \(\bigcup_{i=1}^r E_i = \{x \in G \mid x^p = 1\}\). Indeed, if \(D\) is a normal subgroup of \(G\) of order \(p^{r-1}\) and exponent \(p\) and \(x \in G - D\) is of order \(p\), then the subgroup \(\langle x, D \rangle\) is of order \(p^r\) and exponent \(p\) so coincides with some \(E_i\).

**Remark 29.** If \(G\) is a \(p\)-group such that \(H = \Omega_1(G)\) is of maximal class, then one of the following holds: (a) \(H\) is of order \(\leq p^r\) and exponent \(p\), (b) \(G\) is of maximal class. Indeed, this is the case if \(G\) is regular, by Lemma 1(g). Now assume that \(G\) is not of maximal class and \(|H| > p^r\). Let \(E_1, \ldots, E_r\) be all subgroups of order \(p^r\) and exponent \(p\) in \(G\); then \(r > 1\) and, by Lemma 1(v), \(r \equiv 1 \pmod p\). We have \(E_i < H\) for all \(i\) so \(H\) has a \(G\)-invariant subgroup, say \(E_1\), of order \(p^r\) and exponent \(p\). It follows that \(|H| = p^{r+1}\) (Lemma 1(h)). Since \(r \geq p+1\) and \(d(H) = 2\), we get \(\exp(H) = p\) so \(H\) is regular (Lemma 1(e)), a contradiction.

**4°.** In this section we prove the following

**Theorem 30.** Let \(A\) be a maximal cyclic subgroup of order \(> p\) of a noncyclic \(p\)-group \(G\). Then there exists in \(G\) a maximal cyclic subgroup \(B\) of order \(> p\) such that \(|A \cap B| = p\), unless \(p = 2\) and \(G\) is dihedral.

\(^{13}\)This argument is similar to one from the proof of Theorem 2.
Proof. If $A$ is the unique cyclic subgroup of its order in $G$, then $p = 2$ and $G$ is of maximal class [Ber2, Remark 6.2], and the theorem is true. In what follows we assume that there is in $G$ another cyclic subgroup of order $|A|$.

Suppose that $|G : A| = p$ and $G$ is either abelian $(a) \times \langle b \rangle$ of type $(p^n, p)$ or $G = \langle a, b \mid a^{p^n} = b^p = 1, a^b = a^{1+p^{n-1}} \rangle$, $A = \langle a \rangle$, $n > 1$ and $n > 2$ if $G$ is nonabelian 2-group. In both cases $G$ has exactly $p$ cyclic subgroups of order $p^i$, $i = 2, \ldots, n$. If $n = 2$ and $B$ is a cyclic subgroup of index $p$ in $G$, $B \neq A$, then $|A \cap B| = p$. Now let $n > 2$; then $\Phi(G) = \langle a^p \rangle$. Let $B < G$ be a cyclic subgroup of order $p^2$ not contained in $\Phi(G)$. Then $B$ is a maximal cyclic subgroup of $G$ (indeed, if $B < C \leq G$ and $C$ is cyclic of order $p|B|$, then $B = \Phi(C) \leq \Phi(G)$, contrary to the choice of $B$). We have $|A \cap B| = p$ again.

If $G$ is a 2-group of maximal class and $G$ is not dihedral, it has a maximal cyclic subgroup $B$ of order 4 with $B \nsubseteq A$; then $|A \cap B| = 2$.


Suppose that $H$ is not dihedral. Then, by the above, there is in $H$ a maximal cyclic subgroup $B_1$ of order $p^2$ such that $|A \cap B_1| = p$. Let $B_1 \leq B < G$, where $B$ is a maximal cyclic subgroup of $G$. Then $A \cap B = A \cap B_1$, completing this case.

Now suppose that $H$ is dihedral. Let $H < F \leq G$, where $|F : H| = 2$. Then $A \triangleleft F$ since $A$ is characteristic in $H$. Let $A_1$ be a subgroup of order 4 in $A$; then $A_1 \triangleleft F$. In that case, $C_F(A_1)$ is maximal in $F$ and contains $A$ as a subgroup of index 2. Since $A$ is maximal cyclic subgroup of $G$, the subgroup $C_F(A_1)$ is noncyclic. Since $C_F(A_1)$ is not dihedral, it has a maximal cyclic subgroup $B_1$ of order $2$ such that $|A \cap B_1| = 2$, by induction. If $B_1 \leq B < G$, where $B$ is a maximal cyclic subgroup of $G$, then $A \cap B = A \cap B_1$, completing the proof.

Suppose that a $p$-group $G$ is neither abelian nor minimal nonabelian. We claim that then $G$ contains $p$ pairwise distinct minimal nonabelian subgroups, say $B_1, \ldots, B_p$, of the same order, say $p^n$, such that $B_1 \cap \cdots \cap B_p \geq \Phi(B_i)$ for $i = 1, \ldots, p$ (in particular, $|B_1 \cap \cdots \cap B_p| \geq p^{n-2}$). Indeed, let $B_1$ be a minimal nonabelian subgroup of $G$ of minimal order, and set $|B_1| = p^n$. Let $B_1 < U \leq G$, where $|U : B_1| = p$. It follows from the choice of $B_1$ that each maximal subgroup of $U$ is either abelian or minimal nonabelian (of order $p^n$). By [Ber6, Remark 1], $U$ contains at least $p$ distinct minimal nonabelian subgroups, say $B_1, \ldots, B_p$. If $i \neq j$, then $|B_i \cap B_j| = p^{n-1}$ so $B_i \cap B_j$ is maximal in $B_i$. It follows that $\Phi(B_i) < B_i \cap B_j$ for all $i \neq j$, and our claim follows.
Problems

1. Classify the \( p \)-groups \( G \) in which every maximal cyclic subgroup of composite order is contained in a unique maximal subgroup of \( G \).

2. Study the \( p \)-groups \( G \), all of whose maximal cyclic subgroups are not contained in \( \Phi(G) \).

3. Study the \( p \)-groups \( G, p > 2 \), such that \( K_2(G) = \Omega_1(G) \) has index \( p^p \) in \( G \).

4. Let \( H \) be a maximal member of the set of subgroups of exponent \( p > 2 \) in a \( p \)-group \( G \). Study the structure of \( G \) provided \( H \) is of maximal class.

5. Study the \( p \)-groups \( G \) such that \( G/\Omega_1(G) \) is irregular of maximal class and \( \Omega_1(G) \) is irregular.

6. Let \( H \) be a metacyclic subgroup of exponent \( 2^k > 2 \) of a 2-group \( G \). Study the structure of \( G \) provided every subgroup of \( G \) containing \( H \) properly, has exponent \( > 2^k \).

7. Let \( H \) be a subgroup of exponent 4 in a 2-group \( G \) such that every subgroup of \( G \) properly containing \( H \), has exponent \( > 4 \). Study the structure of \( G \) provided \( |H| \leq 2^5 \).

8. Classify the nonmetacyclic \( p \)-groups \( G \) containing a normal subgroup \( R \) of order \( p \) such that \( G/R \) is metacyclic.

9. Let \( H \) be a maximal member of the set of subgroups of exponent \( \exp(H) \) in a \( p \)-group \( G \). Study the structure of \( G \) provided \( H \) is extraspecial.

10. Let a nonmetacyclic 2-group \( G = BC \), where \( B \) and \( C \) are cyclic. (i) Describe the maximal subgroup of \( G \) that is not generated by two elements (see Remark 18). (ii) Find all possible numbers of involutions in \( G \). (iii) Does there exist \( A < G \) such that \( |A/\overline{\Omega}_2(A)| = p^5 \)? If so, study its structure and embedding in \( G \). (iv) Is it true that \( \overline{\Omega}_2(A) = \overline{\Omega}^2(A) \) for all \( A < G \)?

11. Classify the 2-groups \( G \) such that \( \Omega_k(G) \) is metacyclic, for \( k > 2 \).\(^{14} \)

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References


\(^{14}\)This question was solved by Janko [Jan4] for \( k = 2 \).


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