FINITE NONABELIAN 2-GROUPS IN WHICH ANY TWO NONCOMMUTING ELEMENTS GENERATE A SUBGROUP OF MAXIMAL CLASS

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Abstract. We determine here the structure of the title groups. It turns out that such a group $G$ is either quasidihedral or $G=HZ(G)$, where $H$ is of maximal class or extraspecial and $U_1(Z(G)) \leq Z(H)$. This solves a problem stated by Berkovich. The corresponding problem for $p > 2$ is open but very difficult since the $p$-groups of maximal class are not classified for $p > 2$.

1. Introduction and known results

We determine here the structure of all finite nonabelian 2-groups in which any two noncommuting elements generate a subgroup of maximal class. More precisely, we prove the following result.

Theorem 1.1. Let $G$ be a finite nonabelian 2-group in which any two noncommuting elements generate a subgroup of maximal class. Then one of the following holds:

(a) $|G : H_2(G)| = 2$ and $H_2(G)$ is noncyclic (i.e., $G$ is quasidihedral but not dihedral);
(b) $G = HZ(G)$, where $H$ is of maximal class and $U_1(Z(G)) \leq Z(H)$;
(c) $G = HZ(G)$, where $H$ is extraspecial of order $\geq 2^5$ and $U_1(Z(G)) \leq Z(H)$.

Conversely, each group in (a), (b) and (c) satisfies the assumption of the theorem.

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We consider here only finite $p$-groups and our notation is standard. In particular, a 2-group $S$ is quasidihedral if $S$ has an abelian subgroup $T$ of exponent $> 2$ so that $|S : T| = 2$ and there is an involution in $S - T$ which inverts each element in $T$. It turns out that $T$ is a characteristic subgroup of $S$.

We state three known results which are used in the proof of Theorem 1.1.

**Proposition 1.2** (Berkovich [1, Lemma 4.2]). Let $G$ be a $p$-group with $|G'| = p$. Then $G = (A_1 \ast A_2 \ast \ldots \ast A_s)Z(G)$ ($\ast$ denotes a central product), where $A_1, A_2, \ldots, A_s$ are minimal nonabelian subgroups.

**Proposition 1.3** (Berkovich [1, §58] and Kazarin [2]). Let $G$ be a nonabelian 2-group all of whose cyclic subgroups of composite order are normal in $G$. Then we have either $|G : H_2(G)| = 2$ (and then $G$ is quasidihedral) or $|G'| = 2$ and the Frattini subgroup $\Phi(G)$ is cyclic.

**Proposition 1.4** (Janko [3, Proposition 2.3]). A 2-group is of maximal class if and only if $G$ is dihedral, semidihedral or generalized quaternion.

From Proposition 1.4 follows at once that if $G = \langle a, b \rangle$ is a 2-group of maximal class, then at least one of $a$ and $b$ is of order $\leq 4$ and $G$ possesses exactly one involution $z$ (where $(z) = Z(G)$) which is a square in $G$. We shall use freely this remark in the proof of Theorem 1.1.

2. **Proof of Theorem 1.1**

Let $G$ be a nonabelian 2-group in which any two noncommuting elements generate a subgroup of maximal class. We may assume that $G$ is not of maximal class.

(i) First we assume that $\exp(G) > 4$. Suppose for a moment that each element of order $\geq 8$ lies in $Z(G)$. Let $x, y \in G$ with $[x, y] \neq 1$. Since $(x, y)$ is of maximal class, we have in our case $(x, y) \cong D_8$ or $(x, y) \cong Q_8$. Let $k$ be an element of order 8 so that here $k \in Z(G)$. But then $kx$ and $ky$ are elements of order 8 with $[kx, ky] = [x, y] \neq 1$ and therefore $(kx, ky)$ is of maximal class, a contradiction. We have proved that $G$ possesses a cyclic subgroup $A$ of order $\geq 8$ such that $A \not\subseteq Z(G)$.

It is easy to see that any cyclic subgroup $X$ of order $\geq 8$ is normal in $G$. Indeed, let $g \in G$ so that $g$ either centralizes $X$ or $\langle X, g \rangle$ is of maximal class in which case $g$ normalizes $X$.

Let $y \in G$ be such that $[A, y] \neq 1$ and so $\langle A, y \rangle$ is of maximal class. Then $\langle A, y \rangle$ contains a subgroup of maximal class $\langle B, y \rangle$ of order $2^4$, where $B = \langle b \rangle \cong C_8$, $B \leq A$, and $y^2 \in \Omega_1(B) = \langle z \rangle$. We know that $B$ is normal in $G$. Set $M = C_G(B)$ so that $G/M \neq \{1\}$ is elementary abelian of order $\leq 4$. If $G/M \cong E_4$, then there is $l \in G - M$ such that $l^2 \in M$, $l^2$ centralizes $B$ and $b^l = bz$. But then $\langle b, l \rangle' = \langle z \rangle$ and so $\langle b, l \rangle$ is not of maximal class, a contradiction. Thus $|G : M| = 2$. 

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For each \( x \in G - M \), \( x^2 \in \langle z \rangle \). Indeed, \([b, x] \neq 1\) and so \( \langle b, x \rangle \) is of maximal class and therefore \( x^2 \in \Omega_1(B) = \langle z \rangle \). Consider \( G = G/\langle z \rangle \). Then all elements in \( G - M \) are involutions which implies that \( M/\langle z \rangle \) is abelian and for each \( m \in M \), \( m^y = m^{-1}z^\epsilon \), \( \epsilon = 0, 1 \).

Suppose that \( M \) is nonabelian. Then \( M' = \langle z \rangle \) and let \( m, n \in M \) with \([m, n] = z\). In that case (since \( \langle m, n \rangle \) is of maximal class), \( \langle m, n \rangle \cong D_8 \) or \( \cong Q_8 \). But then \( bm \) and \( bn \) are elements of order 8 with \([bm, bn] = [m, n] = z\) and so \( \langle bm, bn \rangle \) is of maximal class, a contradiction. Hence \( M \) is abelian. If \( M \) is cyclic, then \( \langle M, y \rangle = G \) is of maximal class, a contradiction. Thus, \( M \) is noncyclic abelian.

If all elements in \( G - M \) are involutions, then \( H_2(G) = M \) and we have obtained a group in part (a) of our theorem.

We may assume that not all elements in \( G - M \) are involutions and so we may suppose \( y^2 = z \). Let \( t \) be any involution in \( M - \langle z \rangle \) and assume that \( t \) is a square in \( M \), i.e., there is \( k \in M \) such that \( k^2 = t \). Since \( k^q = k^{-1}z^\epsilon \) (\( \epsilon = 0, 1 \)), \( \langle y, k \rangle \) is nonabelian. In that case \( \langle y, k \rangle \) is of maximal class containing two distinct involutions \( z \) and \( t \) which are squares in \( \langle y, k \rangle \), a contradiction. We have proved that \( M \) is abelian of type \((2^s, 2, ..., 2)\), \( s \geq 3 \).

Setting \( E = \Omega_1(M) \), we get \( M = \langle b' \rangle E \), where \( o(b') = 2^s \), \( |E| \geq 4 \), and \( \langle b' \rangle \cap E = \Omega_1(\langle b' \rangle) = \langle z \rangle \) since \( z \) is the unique involution in \( M \) which is a square in \( M \). Since \( (b')^y = (b')^{-1}z^y \) (\( y = 0, 1 \)), \( H = \langle b', y \rangle \) is of maximal class and \( G = HE \). For each \( t \in E \), we have either \( t^y = t \) or \( t^y = tz \). If \( y \) centralizes \( E \), then \( E = Z(G) \). If \( y \) does not centralize \( E \), then \( E_0 = CE(y) \) is of index 2 in \( E \). Let \( v \) be an element of order 4 in \( \langle b' \rangle \) and let \( u \in E - E_0 \). In that case
\[
(vu)^y = v^{-1}(uz) = (vu)(uz) = vu \text{ and } (vu)^2 = z,
\]
and so \( Z(G) = E_0(vu) \) with \( \Omega_1(Z(G)) = \langle z \rangle \). In any case we get \( G = HZ(G) \), \( Z(G) \supseteq Z(H) = \langle z \rangle \), and \( \Omega_1(Z(G)) \subseteq \langle z \rangle \). We have obtained a group in part (b) of our theorem.

(ii) We examine now the case \( \exp(G) = 4 \). Let \( \langle x \rangle \) be a cyclic subgroup of order 4 and \( y \in G \). Then either \( [x, y] = 1 \) or \( \langle x, y \rangle \cong D_8 \) or \( Q_8 \). In any case \( y \) normalizes \( \langle z \rangle \) and so each cyclic subgroup of order 4 is normal in \( G \).

We may use Proposition 1.3. It follows that either \( |G : H_2(G)| = 2 \) (and we get a group of part (a) of our theorem) or \( |G'| = 2 \) and \( \Phi(G) \) is cyclic.

Suppose that we are in the second case. Since \( G \) does not possess elements of order 8, we have \( |\Phi(G)| = 2 \) and then \( \Phi(G) = G' \). The fact \( |G'| = 2 \) implies that \( G = H_1 \ast H_2 \ast \ldots \ast H_n Z(G) \), where \( H_i \) (\( i = 1, ..., n \)) is minimal nonabelian (Proposition 1.2). In our case \( H_i \cong D_8 \) or \( Q_8 \) and so \( H = H_1 \ast H_2 \ast \ldots \ast H_n \) is extraspecial. Also, \( \Phi(G) = G' = H' = Z(H) \) implies that \( \Omega_1(Z(G)) \leq Z(H) \).

If \( n = 1 \), we have obtained a group of part (b) of our theorem and so we may assume that \( n > 1 \). In that case \( |H| \geq 2^5 \) and we have obtained a group of part (c) of our theorem.
It is necessary to prove only for groups (b) and (c) of our theorem that any two noncommuting elements generate a group of maximal class. Indeed, let \( h_1z_1 \) and \( h_2z_2 \) be any noncommuting elements in \( G \), where \( h_1, h_2 \in H \) and \( z_1, z_2 \in Z(G) \). Then \([h_1z_1, h_2z_2] = [h_1, h_2] \neq 1\) and so \( H_0 = \langle h_1, h_2 \rangle \leq H \) is a group of maximal class with \( H_0' \geq Z(H) \). On the other hand, a 2-group \( \langle h_1, h_2 \rangle \) is of maximal class if and only if \([h_1, h_2] \neq 1\), \( \langle [h_1, h_2] \rangle \) is normal in \( H_0 \) and \( h_1^2, h_2^2 \in \langle [h_1, h_2] \rangle \). Hence \( H_1 = \langle h_1z_1, h_2z_2 \rangle \) is of maximal class since \([h_1z_1, h_2z_2] = [h_1, h_2] \neq 1\), \( h_1z_1 \) and \( h_2z_2 \) normalize \( \langle [h_1z_1, h_2z_2] \rangle = \langle [h_1, h_2] \rangle \) and \( (h_1z_1)^2, (h_2z_2)^2 \) are contained in \( \langle [h_1, h_2] \rangle \) (noting that \( z_1^2, z_2^2 \in Z(H) \leq H_0' = H_1' \)).

References


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