FINITE \( p \)-GROUPS WITH A UNIQUENESS CONDITION FOR NON-NORMAL SUBGROUPS

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Abstract. We determine up to isomorphism all finite \( p \)-groups \( G \) which possess non-normal subgroups and each non-normal subgroup is contained in exactly one maximal subgroup of \( G \). For \( p = 2 \) this problem was essentially more difficult and we obtain in that case two new infinite families of finite 2-groups.

We consider here only finite \( p \)-groups and our notation is standard. It is easy to see that minimal nonabelian \( p \)-groups and 2-groups of maximal class have the property that each non-normal subgroup is contained in exactly one maximal subgroup. It turns out that there are two further infinite families of 2-groups which also have this property. More precisely, we shall prove the following result which gives a complete classification of such \( p \)-groups.

**Theorem 1.** Let \( G \) be a finite \( p \)-group which possesses non-normal subgroups and we assume that each non-normal subgroup of \( G \) is contained in exactly one maximal subgroup. Then one of the following holds:

(a) \( G \) is minimal nonabelian;
(b) \( G \) is a 2-group of maximal class;
(c) \( G = \langle a, b \rangle \) is a non-metacyclic 2-group, where \( a^{2^n} = 1, n \geq 3, a(b) = 2 \) or 4, \( a^b = ak, k^2 = a^{-4}, [k, a] = 1, k^b = k^{-1} \) and we have either:
   (c1) \( b^2 \in \langle a^{2^{n-1}}, a^{2k} \rangle \cong E_4 \), in which case \( |G| = 2^{n+2}, \Phi(G) = \langle a^2 \rangle \times \langle a^{2k} \rangle \cong C_{2^{n-1}} \times C_2, Z(G) = \langle a^{2^{n-1}} \rangle \times \langle a^{2k} \rangle \cong E_4, \) and \( \langle a \rangle \times \langle a^{2k} \rangle \cong C_{2^n} \times C_2 \) is the unique abelian maximal subgroup of \( G \), or:

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(c2) \( b^2 \not\in \langle a^{2^n-1}, a^2 k \rangle \cong E_4 \), in which case \( o(b) = 4 \), \(|G| = 2^{n+3}\).

\( \Phi(G) = \langle a^2 \rangle \times \langle a^2 k \rangle \cong C_{2^n-1} \times C_2 \times C_2 \).

\( Z(G) = \langle a^{2^n-1} \rangle \times \langle a^2 k \rangle \times \langle b^2 \rangle \cong E_4 \) and \( \langle a \rangle \times \langle a^2 k \rangle \times \langle b^2 \rangle \cong C_{2^n-1} \times C_2 \times C_2 \) is the unique abelian maximal subgroup of \( G \).

In any case, \( G' = \langle k \rangle \cong C_{2^n-1} \), a centralizes \( \Phi(G) \), and \( b \) inverts each element of \( \Phi(G) \), and so each subgroup of \( \Phi(G) \) is normal in \( G \);

(d) \( G = \langle a, b \rangle \) is a splitting metacyclic 2-group, where \( a^{2^n} = b^4 = 1 \), \( n \geq 3 \), \( a^b = a^{-1} \). Let \( G = \langle a^2 \rangle \times \langle b^2 \rangle \cong C_{2^n-1} \times C_2 \).

Then \( Z(G) = \langle z \rangle \times \langle b^2 \rangle \cong E_4 \), \( G' = \langle a^2 \rangle \cong C_{2^n-1} \), and \( \langle a \rangle \times \langle b^2 \rangle \cong C_{2^n} \times C_2 \) is the unique abelian maximal subgroup of \( G \). Since \( a \) centralizes \( \Phi(G) \) and \( b \) inverts each element of \( \Phi(G) \), it follows that each subgroup of \( \Phi(G) \) is normal in \( G \).

To facilitate the proof of Theorem 1, we prove the following

**Lemma 2** (Y. Berkovich). Let \( G \) be a p-group, \( p > 2 \), such that all subgroups of \( \Phi(G) \) are normal in \( G \). Then \( \Phi(G) \leq Z(G) \).

**Proof.** By [1, Satz III, 7.12], \( \Phi(G) \) is abelian. Suppose that \( \Phi(G) \) is cyclic. Let \( U/\Phi(G) \) be a subgroup of order \( p \) in \( G/\Phi(G) \). Assume that \( U \) is nonabelian. Then \( U \cong M_{p^{\phi(p)}} \), so \( U = \Phi(G) \Omega_1(U) \), where \( \Omega_1(U) \) is a normal subgroup of type \( (p, p) \) in \( G \). In that case, \( \Omega_1(U) \) centralizes \( \Phi(G) \) so \( U \) is abelian, a contradiction. Let \( M = \{U < G \mid \Phi(G) < U, [U : \Phi(G)] = p \} \).

Then \( C_G(\Phi(G)) \geq \langle U < G \mid \Phi(G) < U, [U : \Phi(G)] = p \rangle \).

Now let \( \Phi(G) \) be noncyclic. Then \( \Phi(G) = Z_1 \times \cdots \times Z_n \), where \( Z_1, \ldots, Z_n \) are cyclic and \( n > 1 \). By induction on \( n \), \( \Phi(G/Z_i) \leq Z(G/Z_i) \) for all \( i \). Let \( f \in \Phi(G) \) and \( x \in G \). Then \( [f, x] \in Z_1 \cap \cdots \cap Z_n = \{1\} \) so \( f \in Z(G) \). It follows that \( \Phi(G) \leq Z(G) \).

**Proof of Theorem 1.** Let \( G \) be a p-group which possesses non-normal subgroups and we assume that each non-normal subgroup of \( G \) is contained in exactly one maximal subgroup. In particular, \( G \) is nonabelian with \( d(G) \geq 2 \) and so each subgroup of \( \Phi(G) \) must be normal in \( G \). Suppose that \( \Phi(G) \) is nonabelian. Then \( p = 2 \) and \( \Phi(G) \) is Hamiltonian, i.e., \( \Phi(G) = Q \times E \), where \( Q \cong Q_8 \) and \( exp(E) \leq 2 \).

But then \( E \) is normal in \( G \) and \( \Phi(G/E) = \Phi(G)/E \cong Q_8 \), contrary to a classical result of Burnside. Thus \( \Phi(G) \) is abelian and each subgroup of \( \Phi(G) \) is \( G \)-invariant.

If every cyclic subgroup of \( G \) is normal in \( G \), then every subgroup of \( G \) is normal in \( G \), a contradiction. Hence there is a non-normal cyclic subgroup \( \langle a \rangle \) of \( G \). In that case \( a \notin \Phi(G) \) but \( a^p \in \Phi(G) \) so that \( \langle a \rangle \Phi(G) \) must be the unique maximal subgroup of \( G \) containing \( \langle a \rangle \). It follows that \( d(G) = 2 \).

If \( \Phi(G) \leq Z(G) \), then each maximal subgroup of \( G \) is abelian and so \( G \) is minimal nonabelian which gives the possibility \( (a) \) of our theorem.

From now on we assume that \( \Phi(G) \not\leq Z(G) \). Set \( G = \langle a, b \rangle \). Then \([a, b] \neq 1 \) and \([a, b] \in \Phi(G) \). Therefore \( \langle [a, b] \rangle \) is normal in \( G \) and \( G/\langle [a, b] \rangle \)
is abelian which implies that $G' = \langle [a, b] \rangle \neq \{1\}$. If $|G'| = p$, then the fact $d(G) = 2$ forces that $G$ would be minimal nonabelian. But then $\Phi(G) \leq Z(G)$, a contradiction. Hence $G'$ is cyclic of order $\geq p^2$.

(i) First assume $p > 2$. By Lemma 2, $\Phi(G) \leq Z(G)$, a contradiction.

(ii) Now assume $p = 2$. If $\Phi(G)$ is cyclic, then (since $\Phi(G) = Z_2(G)$) $G$ has a cyclic subgroup of index 2. But $|G'| \geq 4$ and so $G$ is not isomorphic to $M_{2^r}$, $r \geq 4$, and so $G$ is of maximal class, which gives the possibility (b) of our theorem. From now on we shall assume that $\Phi(G)$ is not cyclic.

Set $G = \langle a, b \rangle$, $k = [a, b]$, and $\langle z \rangle = \Omega_2(\langle k \rangle)$ so that $G' = \langle k \rangle$, $o(k) \geq 4$, and $\langle z \rangle \leq Z(G)$. Since $\langle a^2 \rangle$ and $\langle b^2 \rangle$ (being contained in $\Phi(G)$) are normal in $G$, we have $\Phi(G) = \langle a^2 \rangle \langle b^2 \rangle \langle k \rangle$ and so the abelian subgroup $\Phi(G)$ is a product of three cyclic subgroups which implies $d(\Phi(G)) = 2$ or 3.

From $[a, b] = k$ follows $a^{-1}(b^{-1}ab) = k$ and $b^{-1}(a^{-1}ba) = k^{-1}$ and so

\begin{align*}
(1) & \quad a^b = ak, \\
(2) & \quad b^a = bk^{-1}.
\end{align*}

From (1) follows $(a^2)^b = (a^b)^2 = (ak)^2 = a kak = a^2k^a k$ and so

\begin{align*}
(3) & \quad (a^2)^b = a^2(k^a k).
\end{align*}

From (2) follows $(b^2)^a = (b^a)^2 = (bk^{-1})^2 = bk^{-1}bk^{-1} = b^2(k^{-1})^b k^{-1}$ and so

\begin{align*}
(4) & \quad (b^2)^a = b^2(k^b k)^{-1}.
\end{align*}

We also have

\[ a^2 = (a^2)^b = (a^2k^a k)^b = a^2k^a kk^ab k^b \]

and so

\begin{align*}
(5) & \quad kk^a k^b k^ab = 1.
\end{align*}

Finally, we compute (using (4))

\[ (ab)^2 = abab = a^2a^{-1}b^{-1}b^2ab = a^2(a^{-1}b^{-1}ab)(b^2)^a = a^2k^a k^b (k^b k)^{-1} = a^2k^2(k^{-1})^b, \]

and so

\begin{align*}
(6) & \quad (ab)^2 = a^2b^2(k^{-1})^b.
\end{align*}

Suppose that $G/\Phi(G)$ acts faithfully on $\langle k \rangle$. In that case $o(k) \geq 2^3$ and we may choose the generators $a, b \in G - \Phi(G)$ so that $k^a = k^{-1}$, $k^b = k z$ (where $\langle z \rangle = \Omega_2(\langle k \rangle)$). Using (3) and (4) we get $(a^2)^b = a^2$ (and so $a^2 \in Z(G)$) and $(b^2)^a = b^2k^{-2}z$. Since $k^a = k^{-1}$, we have $\langle k \rangle \cap \langle a \rangle \leq \langle z \rangle$. The subgroup $\langle b^2 \rangle$ (being contained in $\Phi(G)$) is normal in $G$ and so $k^{-2}z \in \langle b^2 \rangle$ and $k^2 \in \langle b^2 \rangle$ (since $z \in \langle k^2 \rangle$). We have $\langle b \rangle \cap \langle k \rangle = \langle k^2 \rangle$ since $k^b = k z \neq k$ and so $k \not\in \langle b \rangle$. If $b^2 \in \langle k^2 \rangle$, then $(b^2)^a = b^{-2}$ and on the other hand $(b^2)^a = b^2k^{-2}z$ and so $b^4 = k^2 z$. But $b^2 \in \langle k^2 \rangle$ implies $b^4 \in \langle k^4 \rangle$, a contradiction. Hence $b^2 \not\in \langle k^2 \rangle$ and so we can find an element $s \in \langle b^2 \rangle - \langle k \rangle$ such that $s^2 = k^{-2}$. Then
(sk)^2 = s^2k^2 = 1 and so sk is an involution in \( \Phi(G) \) which is not contained in \( \langle k \rangle \) and therefore sk \( \neq z \). But (sk)^b = skb = (sk)z and so \( \langle sk \rangle \) is not normal in G, a contradiction.

We have proved that \( G/\Phi(G) \) does not act faithfully on \( \langle k \rangle \). Then we can choose our generator \( a \in G - \Phi(G) \) so that \( k^a = k \). Using (3) we get \((a^2)^b = a^2k^2 \) and so \( 1 \neq k^2 \in \langle a^2 \rangle \) since \( \langle a^2 \rangle \) is normal in G. From (5) we get \((k^2)^b = k^{-2} \). Suppose that \( (k^2) = \langle a^2 \rangle \). Then we get \( a^{-z} = (a^2)^b = a^2k^2 \) and so \( k^2 = a^{-4} \), a contradiction. We have obtained:
\[
\begin{align*}
(7) & & k^a = k, & & (a^2)^b = a^2k^2, & & (k^2)^b = k^{-2}, & & \{1\} \neq \langle k^2 \rangle \trianglelefteq \langle a^2 \rangle, & & o(a) = 2^n, & & n \geq 3.
\end{align*}
\]

Suppose that \( k^b = kz \). Then (5) and (7) imply \( k^4 = 1 \) and so \( k^b = kz = k^{-1} \). It follows that we have to analyze the following three possibilities for the action of \( b \) on \( \langle k \rangle \): \( k^b = k^{-1}z \) with \( o(k) \geq 2^3 \), \( k^b = k \), and \( k^b = k^{-1} \).

**ii1** Suppose \( k^b = k^{-1}z \) with \( o(k) \geq 2^3 \). Then (4) gives \( (b^2)^a = b^2z \) and so \( z \in \langle b^2 \rangle \) (since \( \langle b^2 \rangle \) is normal in G) and \( \langle z \rangle \trianglelefteq \langle b^2 \rangle \) because \( b^2 \notin Z(G) \). Since (by (7)) \( \langle k^2 \rangle \trianglelefteq \langle a^2 \rangle \) and \( o(k^2) \geq 4 \), it follows \( o(a^2) \geq 2^3 \) and
\[
\langle z \rangle = \Omega_1(\langle k \rangle) = \Omega_1(\langle a \rangle) = \Omega_1(\langle b \rangle) \leq Z(G).
\]
From \( o(a^2) \geq 2^3 \) and \( k^2 \in \langle a^2 \rangle \), \( o(k^2) \geq 4 \), and \( (k^2)^b = k^{-2} \) follows \( (a^2)^b = a^{-2}z^e \) \((e = 0, 1)\) and \( C(a^2)(b) = \langle z \rangle \) so that \( \langle a^2 \rangle \trianglelefteq \langle b^2 \rangle \trianglelefteq \langle z \rangle \). Let \( v \) be an element of order 4 in \( \langle a^2 \rangle \) so that \( v^2 = z \) and \( v^b = v^{-1} = vz \). Let \( s \) be an element of order 4 in \( \langle b^2 \rangle \) so that \( s^2 = z \). We have \( (vs)^2 = v^2s^2 = 1 \) and so \( vs \) is an involution in \( \Phi(G) - \langle a \rangle \) but \( (vs)^b = (v^{-1}s)^z = (vs)^z \), a contradiction.

**ii2** Suppose \( k^b = k \) so that (5) and (7) imply \( k^4 = 1 \) and \( k^2 = z \). Then (4) and (7) imply \( (b^2)^a = b^2z \) and \( (a^2)^b = a^2z \). Also, \( \langle z \rangle \trianglelefteq \langle a^2 \rangle \) and \( \langle z \rangle \trianglelefteq \langle b^2 \rangle \) since \( \langle a^2 \rangle \) and \( \langle b^2 \rangle \) are normal in G, \( a^2 \notin Z(G) \) and \( b^2 \notin Z(G) \). If \( a^2 \in \langle b^2 \rangle \), then \( a^2 \in Z(G) \) and if \( b^2 \in \langle a^2 \rangle \), then \( b^2 \in Z(G) \). This is a contradiction. Hence \( D = \langle a^2 \rangle \cap \langle b^2 \rangle \geq \langle z \rangle \) and \( D \) is a proper subgroup of \( \langle a^2 \rangle \) and \( \langle b^2 \rangle \). Because of the symmetry, we may assume \( o(a) \geq o(b) \) so that \( |\langle a^2 \rangle|/D \geq |\langle b^2 \rangle|/D| = 2^n, u \geq 1 \). We set \( (b^2)^{2^n} = d \) so that \( D = \langle d \rangle \). We may choose an element \( a^e \in \langle a^2 \rangle - D \) such that \( (a^e)^{2^n} = a^{-1} \). Then \( (a^e)^{2^n} = 1 \) and \( (a^e)^{2^n} \cong C_{2^n} \) with \( \langle a^e \rangle \cap D = \{1\} \). On the other hand, \( (a^e)^{2^n} = (a^e)^{2^n} z = (a^e)^{2^n} z \), where \( z \in D \), a contradiction.

**ii3** Finally, suppose \( k^b = k^{-1} \). From (4) follows \( (b^2)^a = b^2 \) and so \( b^2 \in Z(G) \). By (7), \( (a^2)^b = a^2K^2, \langle k^2 \rangle \trianglelefteq \langle a^2 \rangle \), and so \( o(a^2) \geq 4 \). Also, \( (a^2)^b = a^2k, (a^2)^b = (a^2k^2)k^{-1} = a^2k \), and so \( a^2k \in Z(G) \).

**ii3a** First assume \( k \notin \langle a^2 \rangle \). We investigate for a moment the special case \( o(k) = 4 \), where \( k^2 = z, \langle z \rangle = \Omega_1(\langle k \rangle) = \Omega_1(\langle a \rangle) \) and \( (a^2)^b = a^2z \). If \( o(a^2) > 4 \), then take an element \( v \) of order 4 in \( \langle a^4 \rangle \) so that \( v^2 = z \) and \( v^b = v \). In that case \( (vk)^2 = v^2k^2 = 1 \) and so \( vk \) is an involution in \( \Phi(G) - \langle a^3 \rangle \) and \( (vk)^b = v^k = z \), a contradiction. Hence \( o(a^2) = 4 \), \( a^4 = z, k^2 = z = a^{-4}, (a^2)^b = a^2z = a^{-2}, \langle a^2, k \rangle \) is an abelian group of type
(4, 2) acted upon invertingly by $b$, and $a^2k$ is a central involution in $G$. Now suppose $o(k) \geq 8$. In that case $o(k^2) \geq 4$, $k^2 \in \langle a^2 \rangle$, $o(a^2) \geq 8$, and $b$ inverts $\langle k^2 \rangle$, which implies $(a^2)^b = a^{-2}z^\epsilon$, $\epsilon = 0, 1$. On the other hand, $(a^2)^b = a^2k^2$ and so $k^2 = a^{-4}z$. Let $v$ be an element of order 4 in $\langle a^4 \rangle$ so that $v^2 = z$ and $v^b = v^{-1} = vz$. Then we compute:

$$(a^2vk)^2 = a^4z^2 = z^{4+1}, \quad (a^2vk)^v = a^2k^2v^{-1}k^{-1} = (a^2vk)_z.$$ 

If $\epsilon = 1$, then $a^2vk$ is an involution in $\Phi(G) - \langle a^2 \rangle$ and $\langle a^2vk \rangle$ is not normal in $G$. Thus, $\epsilon = 0$, $(a^2)^h = a^{-2}$, $k^2 = a^{-4}$, $a^2k$ is an involution in $\Phi(G) - \langle a^2 \rangle$ and $b$ inverts each element of $\langle a^2, k \rangle = \langle a^2 \rangle \times \langle a^2k \rangle$, where $a^2k \in Z(G)$.

We have proved that in any case $k^2 = a^{-4}$, $o(a^2) \geq 4$, $o(k) \geq 4$, and $b$ inverts each element of the abelian group $\langle a^2, k \rangle = \langle a^2 \rangle \times \langle a^2k \rangle$, where $a^2k$ is an involution contained in $Z(G)$.

It remains to determine $b^2 \in Z(G)$. Suppose $o(b^2) \geq 4$ and let $\langle s \rangle$ be a cyclic subgroup of order 4 in $\langle b^2 \rangle$ so that $s \in Z(G)$. Obviously, $s \not\in \langle a^2, k \rangle$ since $Z(G) \cap \langle a^2, k \rangle = \langle z \rangle \times \langle a^2k \rangle \cong E_4$. Let $v$ be an element of order 4 in $\langle a^2 \rangle$ so that $v^2 = z$ and $v^b = v^{-1} = vz$. We have:

$$(vs)^b = v^{-1}s = (vs)z \quad \text{and} \quad (vs)^2 = v^2s^2 = zs^2.$$ 

If $s^2 = z$, then $vs$ is an involution in $\Phi(G) - \langle a^2, k \rangle$ and $vs \not\in Z(G)$, a contradiction. Hence $s^2 \neq z$ so that $\langle v, s \rangle = \langle v \rangle \times \langle s \rangle \cong C_4 \times C_4$. But $(vs)^b = (vs)z$, $(vs)^2 = zs^2 \neq z$, and so $(vs)$ is not normal in $G$, a contradiction. It follows that $o(b^2) \leq 2$. Hence we have either $b^2 \in \langle z, a^2k \rangle$, $\Phi(G) = \langle a^2, k \rangle = \langle a^2 \rangle \times \langle a^2k \rangle$, we have obtained the possibility (c1) of our theorem or $b^2$ is an involution in $\Phi(G) - \langle a^2, k \rangle$, $\Phi(G) = \langle a^2 \rangle \times \langle a^2k \rangle \times \langle b^2 \rangle$, and we have obtained the possibility (c2) of our theorem. Note that in both cases $a$ centralizes $\Phi(G)$ and $b$ inverts each element of $\Phi(G)$.

(ii3b) We assume $k \in \langle a^2 \rangle$. Since $o(k) \geq 4$, $k^b = k^{-1}$, $\langle a \rangle$ is normal in $G$, $o(a) \geq 8$, and $b$ induces on $\langle a \rangle$ an automorphism of order 2, we get $a^b = a^{-1}z^\epsilon$, $\epsilon = 0, 1$, where $\langle z \rangle = \langle a \rangle$ $\cong \langle a \rangle$. On the other hand, (1) gives $a^b = ak$ and so $k = a^{-2}z^\epsilon$, which gives $G^e = \langle k \rangle = \langle a^2 \rangle \cong C_{2^{n-1}}$, where $o(a) = 2^n$, $n \geq 3$, and $z = a^{2^{-n}}$.

Since $\Phi(G) = \langle a^2, b^2 \rangle$ and $\Phi(G)$ is noncyclic, we have $b^2 \not\in \langle a^2 \rangle$ and we know that $b^2 \in Z(G)$. Suppose $o(b^2) \geq 4$ and let $s$ be an element of order 4 in $\langle b^2 \rangle$. Let $v$ be an element of order 4 in $\langle a^2 \rangle$ so that $v^2 = z$ and $v^b = v^{-1} = vz$. Then

$$(vs)^b = v^{-1}s = (vs)z \quad \text{and} \quad (vs)^2 = v^2s^2 = zs^2.$$ 

If $s^2 = z$, then $vs$ is an involution in $\Phi(G) - \langle a^2 \rangle$ and $vs \not\in Z(G)$, a contradiction. Hence $s^2 \neq z$ so that $(v, s) = \langle v \rangle \times \langle s \rangle \cong C_4 \times C_4$. But $(vs)$ is not normal in $G$, a contradiction. Hence $b^2$ is an involution in $\Phi(G) - \langle a^2 \rangle$ and so $\Phi(G) = \langle a^2 \rangle \times \langle b^2 \rangle \cong C_{2^{n-1}} \times C_2$ and $Z(G) = \langle z \rangle \times \langle b^2 \rangle \cong E_4$. Also note that $a$ centralizes $\Phi(G)$ and $b$ inverts each element of $\Phi(G)$. We have obtained the possibility (d) of our theorem. \[\square\]
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