AN EQUATION ON OPERATOR ALGEBRAS AND SEMISIMPLE $H^*$--ALGEBRAS

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Abstract. In this paper we prove the following result: Let $X$ be a Banach space over the real or complex field $F$ and let $L(X)$ be the algebra of all bounded linear operators on $X$. Suppose there exists an additive mapping $T : A(X) \rightarrow L(X)$, where $A(X) \subset L(X)$ is a standard operator algebra. Suppose that $T(A^3) = AT(A)A$ holds for all $A \in A(X)$. In this case $T$ is of the form $T(A) = \lambda A$ for any $A \in A(X)$ and some $\lambda \in F$. This result is applied to semisimple $H^*$--algebras.

This research is related to the work of Molnár [8] and is a continuation of our work [9, 10]. Throughout, $R$ will represent an associative ring with center $Z(R)$. A ring $R$ is n-torsion free, where $n > 1$ is an integer, if $nx = 0$, $x \in R$ implies $x = 0$. The commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that $R$ is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \rightarrow R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation $D$ is inner in case there exists $a \in R$, such that $D(x) = [a, x]$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [6] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein’s result can be found in [3]. Cusack [5] generalized Herstein’s result to 2–torsion free semiprime rings (see also [2] for an alternative proof).

2000 Mathematics Subject Classification. 16W10, 46K15, 39B05.

Key words and phrases. Prime ring, semiprime ring, Banach space, standard operator algebra, $H^*$--algebra, derivation, Jordan derivation, left (right) centralizer, left (right) Jordan centralizer.

This research has been supported by the Research Council of Slovenia.
An additive mapping \( T : R \to R \) is called a left centralizer in case \( T(xy) = T(x)y \) holds for all \( x, y \in R \).

The concept appears naturally in \( C^* \)-algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would write that \( T : \mathcal{R} \to \mathcal{R} \) is a homomorphism of a ring module \( \mathcal{R} \) into itself. For a semiprime ring \( \mathcal{R} \) all such homomorphisms are of the form \( T(x) = qx \) for all \( x \in \mathcal{R} \), where \( q \) is an element of Martindale right ring of quotients \( Q_r \) (see Chapter 2 in [2]). In case \( \mathcal{R} \) has the identity element \( T : \mathcal{R} \to \mathcal{R} \) is a left centralizer if \( T \) is of the form \( T(x) = ax \) for all \( x \in \mathcal{R} \) and some fixed element \( a \in \mathcal{R} \). An additive mapping \( T : \mathcal{R} \to \mathcal{R} \) is called a left Jordan centralizer in case \( T(x^2) = T(x)x + xT(x) \) holds for all \( x \in \mathcal{R} \). The definition of right centralizer and right Jordan centralizer should be self-explanatory. Following ideas from [4] Zalar [12] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Molnár [8] has proved that in case \( \mathcal{R} \) we have an additive mapping \( T : \mathcal{R} \to \mathcal{R} \), where \( \mathcal{R} \) is a semisimple \( H^* \)-algebra, satisfying the relation \( T(x^3) = T(x)x^2 + T(x^2)x \) for all \( x \in \mathcal{R} \), then \( T \) is a left (right ) centralizer. For the definition and for basic facts of \( H^* \)-algebras we refer to [1]. Vukman [9] has proved that in case there exists an additive mapping \( T : \mathcal{R} \to \mathcal{R} \), where \( \mathcal{R} \) is a 2-torsion free semiprime ring, satisfying the relation \( 2T(x^3) = T(x)x + xT(x) \) for all \( x \in \mathcal{R} \), then \( T \) is a left and also a right centralizer. Some result concerning centralizers in semiprime rings can be found in [10] and [11]. Let \( X \) be a normed space over the real or complex field \( F \), and let \( L(X) \) and \( F(X) \) denote the algebra of all bounded linear operators on \( X \) and the ideal of all finite rank operators in \( L(X) \), respectively. An algebra \( A(X) \subset L(X) \) is said to be standard in case \( F(X) \subset A(X) \). Let us point out that any standard algebra is prime, which is a consequence of Hahn-Banach theorem.

We are ready for our first result.

**THEOREM 1.** Let \( X \) be a Banach space over the real or complex field \( F \) and let \( A(X) \subset L(X) \) be a standard operator algebra. Suppose there exists an additive mapping \( T : A(X) \to L(X) \), such that \( T(A^3) = AT(A)A \) holds for all \( A \in A(X) \). In this case we have \( T(A) = \lambda A \) for any \( A \in A(X) \) and some \( \lambda \in F \).

**Proof.** We have the relation

\[
(1) \quad T(A^3) = AT(A)A, \quad \text{for all } A \in A(X).
\]

First we will consider the restriction of \( T \) on \( F(X) \). Let \( A \) be from \( F(X) \) and let \( P \in F(X) \), be a projection such that \( AP = PA = A \). From the relation (1) one obtains that \( T(P) = PT(P)P \) and \( T(P)P = PT(P) \) holds. Putting \( A + P \) for \( A \) in the relation above and applying the relation (1) we obtain
after some calculation
\[ 3T(A^2) + 3T(A) = PT(A)A + AT(A)P + AT(P)A + PT(P)A + AT(P)P + PT(A)P. \]

Putting \(-A\) for \(A\) in the above relation and comparing the relation so obtained with the above relation we obtain
\[ 3T(A^2) = PT(A)A + AT(A)P + AT(P)A, \]
and
\[ 3T(A) = PT(P)A + AT(P)P + PT(A)P. \]

Multiplying the above relation from both sides by \(P\), we obtain
\[ 2PT(A)P = PT(P)A + AT(P)P. \]

Combining the relations (3) and (4) we obtain \(2T(A) = PT(P)A + AT(P)P\). Now we have \(2T(A) = PT(P)A + AT(P)P = (PT(P)A + A(PT(P))P) = T(P)A + AT(P)\). Thus we have
\[ 2T(A) = AB + BA, \]
where \(B\) stands for \(T(P)\). Now we have \(2T(A)P = (AB + BA)P = ABP + BAP = APB + BA = AB + BA = 2T(A)\). We have therefore \(T(A)P = T(A)\). Similarly one obtains \(PT(A) = T(A)\). Now the relation (2) reduces to
\[ 3T(A^2) = T(A)A + AT(A) + ABA. \]

Combining (5) and (6) we obtain
\[ 0 = 6T(A^2) - 2T(A)A - 2AT(A) - 2ABA \]
\[ = 3(A^2B + BA^2) - (AB + BA)A - A(AB + BA) - 2ABA \]
\[ = 2(A^2B + BA^2) - 4ABA. \]

We have therefore \(A^2B + BA^2 = 2ABA\), which can be written according to the relation (5) in the form \(T(A^2) = ABA\), which reduces the relation (6) to
\[ 2T(A^2) = T(A)A + AT(A). \]

The relation (5) makes it possible to concluded that \(T\) maps \(F(X)\) into itself and that \(T\) is linear on \(F(X)\). Therefore we have a linear mapping \(T : F(X) \rightarrow F(X)\) satisfying the relation (7) for all \(A \in F(X)\). Since \(F(X)\) is prime one can conclude according to Theorem in [9] that \(T\) is a left and also a right centralizer. We intend to prove that there exists an operator \(C \in L(X)\), such that
\[ T(A) = CA, \text{ for all } A \in F(X) \]

For any fixed \(x \in X\) and \(f \in X^*\) we denote by \(x \otimes f\) an operator from \(F(X)\) defined by \((x \otimes f)y = f(y)x\), for all \(y \in X\). For any \(A \in L(X)\) we have \(A(x \otimes f) = ((Ax) \otimes f)\). Let us choose \(f\) and \(y\) such that \(f(y) = 1\) and...
define $Cx = T(x \otimes f)y$. Obviously, $C$ is linear. Using the fact that $T$ is left centralizer on $F(X)$ we obtain

\[(CA)x = C(Ax) = T((Ax) \otimes f)y = T(A(x \otimes f))y \]
\[= T(A)(x \otimes f)y = T(A)x, \quad x \in X.\]

We have therefore $T(A) = CA$ for any $A \in F(X)$. Since $T$ right centralizer on $F(X)$ we obtain $C(AB) = T(AB) = AT(B) = ACB$. We have therefore $[A, C] B = 0$ for any $A, B \in F(X)$ whence it follows that $[A, C] = 0$ for any $A \in F(X)$. Using closed graph theorem one can easily prove that $C$ is continuous. Since $C$ commutes with all operators from $F(X)$ one can conclude that $Cx = \lambda x$ holds for any $x \in X$ and some $\lambda \in F$, which gives together with the relation (8) that $T$ is of the form

\[T(A) = \lambda A\]

any $A \in F(X)$ and some $\lambda \in F$. It remains to prove that the above relation holds on $A(X)$ as well. Let us introduce $T_1 : A(X) \rightarrow L(X)$ by $T_1(A) = \lambda A$ and consider $T_0 = T - T_1$. The mapping $T_0$ is, obviously, additive and satisfies the relation (1). Besides, $T_0$ vanishes on $F(X)$. Let $A \in A(X)$, let $P$ be a one-dimensional projection and $S = A + PAP - (AP + PA)$. Since, obviously, $S - A \in F(X)$, we have $T_0(S) = T_0(A)$. Besides, $SP = PS = 0$. We have therefore the relation

\[T_0(A^3) = AT_0(A)A,\]

for all $A \in A(X)$. Applying the above relation we obtain

\[ST_0(S)S = T_0(S^3) = T_0(S^3 + P) = T_0((S + P)^3)\]
\[= (S + P)T_0(S + P)(S + P) = (S + P)T_0(S)(S + P)\]
\[= ST_0(S)S + PT_0(S)S + ST_0(S)P + PT_0(S)P.\]

We have therefore

\[PT_0(A)S + ST_0(A)P + PT_0(A)P = 0.\]

Multiplying the above relation from both sides by $P$ we obtain

\[PT_0(A)P = 0,\]

which reduces the relation (11) to

\[PT_0(A)S + ST_0(A)P = 0.\]

Right multiplication of the above relation by $P$ gives

\[ST_0(A)P = 0.\]

Applying (12) the relation (14) reduces to

\[AT_0(A)P - PAT_0(A)P = 0.\]
Putting in the above relation $A + B$ for $A$, where $A$ is from $A(X)$ and $B$ from $F(X)$, using the fact that $T_0$ vanishes on $F(X)$, and applying the relation (15), we obtain

$$0 = (A + B)T_0(A)P - P(A + B)T_0(A)P = BT_0(A)P - PB T_0(A)P$$

We have therefore proved that

$$BT_0(A)P - PB T_0(A)P = 0$$

holds for any $A \in A(X)$ and all $B \in F(X)$. Putting in the above relation $T_0(A)PB$ for $B$ and applying the relation (12), we obtain

$$(T_0(A)P)B(T_0(A)P) = 0, \text{ for all } B \in F(X),$$

whence it follows $T_0(A)P = 0$ by primeness of $F(X)$. Since $P$ is an arbitrary one-dimensional, one can conclude that $T_0(A) = 0$, for any $A \in A(X)$, which completes the proof of the theorem.

In the proof of Theorem 1 we used some ideas from Molnár’s paper [8]. Let us point out that in Theorem 1 we obtain as a result the continuity of $T$ under purely algebraic conditions concerning $T$, which means that Theorem 1 might be of some interest from the automatic continuity point of view.

**Theorem 2.** Let $A$ be a semisimple $H^*$-algebra and let $T : A \rightarrow A$ be such an additive mapping that $T (x^3) = xT (x)x$ holds for all $x \in A$. In this case $T$ is a left and a right centralizer.

**Proof.** The proof goes through using the same arguments as in the proof of Theorem in [8] with the exception that one has to use Theorem 1 instead of Lemma in [8].

Since in the formulation of the theorem above we have used only algebraic concepts, it would be interesting to study the relevant problem in a purely ring theoretical context. Let us point out that Vukman [9] has proved the following result. Let $R$ be a 2-torsion free semiprime ring and let $T : R \rightarrow R$ be an additive mapping. If $T(xy x) = xT(y)x$ holds for all $x, y \in R$, then $T$ is a left and a right centralizer. In the same paper one can find also a result which states that in case we have a 2-torsion free semiprime ring with the identity element and an additive mapping $T : R \rightarrow R$ satisfying the relation $T(x^3) = xT(x)x$ for all $x \in R$, then $T(x) = ax$ holds for all $x \in R$ and some $a \in Z(R)$.

**References**


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Received: 29.8.2004.