D-CONTINUUM $X$ ADMITS A WHITNEY MAP FOR $C(X)$ IF AND ONLY IF IT IS METRIZABLE

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Abstract. The main purpose of this paper is to prove: a) a D-continuum $X$ admits a Whitney map for $C(X)$ if and only if it is metrizable, b) a continuum $X$ admits a Whitney map for $C^2(X)$ if and only if it is metrizable.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space $X$ is denoted by $w(X)$.

A generalized arc is a Hausdorff continuum with exactly two nonseparating points (end points) $x, y$. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

We say that a space $X$ is arcwise connected if for every pair $x, y$ of points of $X$ there exists a generalized arc $L$ with end points $x, y$.

Let $X$ be a space. We define its hyperspaces as the following sets:

- $2^X = \{ F \subseteq X : F \text{ is closed and nonempty} \}$,
- $C(X) = \{ F \in 2^X : F \text{ is connected} \}$,
- $C^2(X) = C(C(X))$,
- $X(n) = \{ F \in 2^X : F \text{ has at most } n \text{ points} \}$, $n \in \mathbb{N}$.

For any finitely many subsets $S_1, ..., S_n$, let

$$\langle S_1, ..., S_n \rangle = \left\{ F \in 2^X : F \subseteq \bigcup_{i=1}^n S_i, \text{ and } F \cap S_i \neq \emptyset, \text{ for each } i \right\}.$$
The topology on $2^X$ is the Vietoris topology, i.e., the topology with a base 
\[ \{ <U_1, \ldots, U_n> : U_i \text{ is an open subset of } X \text{ for each } i \text{ and each } n < \infty \} \], and 
$C(X), X(n)$ are subspaces of $2^X$. Moreover, $X(1)$ is homeomorphic to $X$.

Let $X$ and $Y$ be the spaces and let $f : X \to Y$ be a mapping. Define 
$2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [8, p. 170, Theorem 5.10] 
$2^f$ is continuous and $2^f(C(X)) \subset C(Y), 2^f(X(n)) \subset Y$. The restriction 
$2^f|C(X)$ is denoted by $C(f)$.

A continuum $X$ is called a $D$-continuum if for every pair $C, D$ of its 
disjoint non-degenerate subcontinua there exists a subcontinuum $E \subset X$ such 
that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$.

**Lemma 1.1.** [6, Lemma 2.3]. If $X$ is an arcwise connected continuum, 
then $X$ is a $D$-continuum.

**Lemma 1.2.** [6, Lemma 2.4]. If $X$ is a locally connected continuum, then 
$X$ is a $D$-continuum.

Let $\Lambda$ be a subspace of $2^X$. By a Whitney map for $\Lambda$ [9, p. 24, (0.50)] 
we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying

a) if $A, B \in \Lambda$ such that $A \subset B$ and $A \neq B$, then $g(A) < g(B)$ and

b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If $X$ is a metric continuum, then there exists a Whitney map for $2^X$ and 
$C(X)$ ([9, pp. 24-26], [3, p. 106]). On the other hand, if $X$ is non-metrizable, 
then it admits no Whitney map for $2^X$ [1]. It is known that there exist non-
metrizable continua which admit and ones which do not admit a Whitney 
map for $C(X)$ [1].

In the sequel we shall use the following theorem.

**Theorem 1.3.** [6, Theorem 3.3]. If a $D$-continuum $X$ admits a Whitney 
map for $C(X)$, then $C(X) \setminus X(1)$ is metrizable and $w(C(X) \setminus X(1)) \leq \aleph_0$.

It is known that if $X$ is a continuum, then $C(X)$ is arcwise connected [7, 
p. 1209, Theorem]. Hence, using Lemma 1.1 and Theorem 1.3, we obtain the 
following corollary.

**Corollary 1.4.** If $X$ is a continuum which admits a Whitney map for 
the hyperspace $C^2(X)$, then $C^2(X) \setminus C(X)(1)$ is metrizable and

$w(C^2(X) \setminus C(X)(1)) \leq \aleph_0$.

2. Main theorems

In this section we shall prove the main theorems of the paper, Theorems 2.2 and 2.6.

For this purpose we shall use the notion of a network of a topological space.
A family $\mathcal{N} = \{M_s : s \in S\}$ of subsets of a topological space $X$ is a network for $X$ if for every point $x \in X$ and any neighbourhood $U$ of $x$ there exists an $s \in S$ such that $x \in M_s \subseteq U$ [2, p. 170]. The network weight of a space $X$ is defined as the smallest cardinal number of the form $\text{card}(\mathcal{N})$, where $\mathcal{N}$ is a network for $X$; this cardinal number is denoted by $nw(X)$.

**Theorem 2.1.** [2, p. 171, Theorem 3.1.19]. For every compact space $X$ we have $nw(X) = w(X)$.

Now we shall prove the main theorem of this paper.

**Theorem 2.2.** A D-continuum $X$ admits a Whitney map for $C(X)$ if and only if it is metrizable.

**Proof.** If $X$ is metrizable, then $X$ admits a Whitney map ([3, p. 106], [9, pp. 24-26]). Conversely, suppose that $X$ admits a Whitney map for $C(X)$. By Theorem 1.3 we have that $C(X) \setminus X(1)$ is metrizable and $w(C(X) \setminus X(1)) \leq \aleph_0$. This means that there exists a countable base $B = \{B_i : i \in \mathbb{N}\}$ of $C(X) \setminus X(1)$. For each $B_i$ let $C_i = \{x \in X : x \in B, B \in B_i\}$, i.e., the union of all continua $B$ contained in $B_i$.

**Claim 1.** The family $\{C_i : i \in \mathbb{N}\}$ is a network of $X$. Let $x$ be a point of $X$ and let $U$ be an open subset of $X$ such that $x \in U$. There exists an open set $V$ such that $x \in V \subset CLV \subset U$. Let $K$ be a component of $CLV$ containing $x$. By Boundary Bumping Theorem [10, p. 73, Theorem 5.4] $K$ is non-degenerate and, consequently, $K \subseteq C(X) \setminus X(1)$. Now, $(U) \cap (C(X) \setminus X(1))$ is an intersection of $K$ in $C(X) \setminus X(1)$. It follows that there exists a $B_i \in B$ such that $K \subseteq B_i \subset (U) \cap (C(X) \setminus X(1))$. It is clear that $C_i \subset U$ and $x \in C_i$ since $x \in K$. Hence, the family $\{C_i : i \in \mathbb{N}\}$ is a network of $X$.

**Claim 2.** $nw(X) = \aleph_0$. Apply Claim 1 and the fact that $B$ is countable.

**Claim 3.** $w(X) = \aleph_0$. By Claim 2 we have $nw(X) = \aleph_0$. Moreover, by Theorem 2.1 $w(X) = \aleph_0$.

**Claim 4.** Finally, $X$ is metrizable.

Since each arcwise connected continuum is a D-continuum (Lemma 1.1) we have the following corollary which generalize Theorem 3.4 of the paper [5, p. 19].

**Corollary 2.3.** An arcwise connected continuum $X$ admits a Whitney map for $C(X)$ if and only if $X$ is metrizable.

An arboroid is a hereditarily unicoherent arcwise connected continuum. A metrizable arboroid is a dendroid. If $X$ is an arboroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in $X$ with endpoints $x$ and $y$.

A point $t$ of an arboroid $X$ is said to be a ramification point of $X$ if $t$ is the only common point of some three arcs such that it is the only common point of any two, and an end point of each of them.
If an arboroid $X$ has only one ramification point $t$, it is called a generalized fan with the top $t$. A metrizable generalized fan is called a fan.

The following corollary is a stronger result than Theorem 4.20 in [4] which states that a generalized fan $X$ admits a Whitney map for $C(X)$ if and only if it is metrizable.

**Corollary 2.4.** Let $X$ be an arboroid. Then $X$ admits a Whitney map for $C(X)$ if and only if it is metrizable.

**Proof.** Apply Corollary 2.3.

From Lemma 1.2 it follows that each locally connected continuum is a D-continuum. Thus, we have the following corollary of Theorem 2.2.

**Corollary 2.5.** A locally connected $X$ continuum admits a Whitney map for $C(X)$ if and only if it is metrizable.

The following theorem shows that the existence of a Whitney map for $C^2(X)$ is equivalent to metrizability of $X$.

**Theorem 2.6.** A continuum $X$ admits a Whitney map for $C^2(X)$ if and only if $X$ is metrizable.

**Proof.** From Corollary 1.4 it follows that if $X$ a continuum which admits a Whitney map for $C^2(X)$, then $C^2(X)\setminus C(X)(1)$ is metrizable and $w(C^2(X)\setminus C(X)(1)) \leq \aleph_0$. By Theorem 2.2 $w(C(X)) = \aleph_0$ since $C(X)$ is arcwise connected. This means that $w(X) = \aleph_0$ since $X$ is homeomorphic to $X(1) \subset C(X)$. Hence, $X$ is metrizable.

It is known [2, p. 171, Corollary 3.1.20] that if a compact space $X$ is the countable union of its subspaces $X_n, n \in \mathbb{N}$, such that $w(X_n) \leq \aleph_0$, then $w(X) \leq \aleph_0$. Using this fact and theorems proved in the previous section we obtain the following theorems.

**Theorem 2.7.** If a continuum $X$ is the countable union either of its $D$-subcontinua or of its arcwise connected subcontinua, then $X$ admits a Whitney map for $C(X)$ if and only if it is metrizable.

**Theorem 2.8.** If a compact space $X$ is the countable union of its subcontinua and admits a Whitney map for $C^2(X)$, then $X$ is metrizable.

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