The Euler characteristic of the symmetric product of a finite CW-complex*

KOSTADIN TRENČEVSKI†,‡

1 Institute of Mathematics, Ss. Cyril and Methodius University, P. O. Box 162, 1 000 Skopje, Macedonia

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Abstract. In this paper it is first shown that if $M$ is a 2-dimensional surface, then $M^{(m)}$ is orientable if and only if $M$ is orientable. Using the Macdonald’s result [5] the Betti numbers of $M^{(m)}$ are expressed explicitly, the Euler characteristic is found and it depends only on $m$ and the Euler characteristic of $M$. Moreover, the formula for the Euler characteristic is proved alternatively also without using the Macdonald’s results.

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1. Some preliminaries about symmetric products of manifolds

Let $M$ be an arbitrary set and $m$ a positive integer. In the $m$-fold Cartesian product $M^m$ we define a relation $\approx$ such that

$$(x_1, \ldots, x_m) \approx (y_1, \ldots, y_m) \iff \text{there exists a permutation } \theta : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\} \text{ such that } y_i = x_{\theta(i)},$$

$(1 \leq i \leq m)$.

This is a relation of equivalence and the class represented by $(x_1, \ldots, x_m)$ will be denoted by $[x_1, \ldots, x_m]$ and the set $M^m/\approx$ will be denoted by $M^{(m)}$. The set $M^{(m)}$ is called a symmetric product of $M$. Some authors call it a permutation product of $M$.

If $M$ is a topological space, then $M^{(m)}$ is also a topological space. The space $M^{(m)}$ was introduced quite early [9], and it was studied in [8, 4, 7, 1, 10], and in the Ph.D. thesis of Wagner [17]. Some recent results are obtained in [16, 2, 3, 11, 12]. If $M$ is an arbitrary connected manifold and $m > 1$, then it is proved in [9] that

$$\pi_1(M^{(m)}) \cong H_1(M, \mathbb{Z}).$$

(1)

Another important result [17] states that $(\mathbb{R}^n)^{(m)}$ is a manifold only for $n = 2$. If $n = 2$, then $(\mathbb{R}^2)^{(m)} = \mathbb{C}^{(m)}$ is homeomorphic to $\mathbb{C}^m$. Indeed, using that $\mathbb{C}$ is an algebraically closed field, it is obvious that the map $\varphi : \mathbb{C}^{(m)} \to \mathbb{C}^m$ defined by

$$\varphi[z_1, \ldots, z_m] = (\sigma_1(z_1, \ldots, z_m), \sigma_2(z_1, \ldots, z_m), \ldots, \sigma_m(z_1, \ldots, z_m))$$

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†Corresponding author. Email address: kostatre@pmf.ukim.mk (K. Trenčevski)

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is a bijection, where $\sigma_i (1 \leq i \leq m)$ is the $i$-th symmetric function of $z_1, \cdots, z_m$, i.e.

$$\sigma_i(z_1, \cdots, z_m) = \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq m} z_{j_1} \cdot z_{j_2} \cdots z_{j_i}.$$ 

The map $\varphi$ is also a homeomorphism. If $M$ is a 1-dimensional complex manifold, then $M^{(m)}$ is a complex manifold, too. For example, if $M$ is a 2-sphere, i.e. a complex manifold $\mathbb{C}P^1$, then $M^{(m)}$ is the projective complex space $\mathbb{C}P^m$. Using the symmetric products it is easy to see how $M^{(m)} = \mathbb{C}P^m$ decomposes into disjoint cells $\mathbb{C}^0, \mathbb{C}^1, \cdots, \mathbb{C}^m$. Let $\xi \in M$. Then we define $[x_1, \cdots, x_m] \in M_i$ if exactly $i$ of the elements $x_1, \cdots, x_m$ are equal to $\xi$. Thus

$$M^{(m)} = M_0 \cup M_1 \cup \cdots \cup M_m = (M \setminus \xi)^{(m)} \cup (M \setminus \xi)^{(m-1)} \cup \cdots \cup (M \setminus \xi)^{(0)} = \mathbb{C}^m \cup \mathbb{C}^{m-1} \cup \cdots \cup \mathbb{C}^0.$$ 

This theory about symmetric products has an important role in the theory of topological commutative vector valued groups [14, 15, 13].

The Poincaré polynomial of a symmetric product of a compact polyhedron is given in [5]. If $B_0, B_1, B_2, \cdots$ are the Betti numbers of a space $M$, then the Poincaré polynomial of the $m$th symmetric product of $M$ is the coefficient in front of $t^m$ in the power series expansion of

$$\frac{(1 + xt)^{B_1}(1 + x^3t)^{B_3} \cdots}{(1 - t)^{B_0}(1 - x^2t)^{B_2}(1 - x^4t)^{B_4} \cdots}.$$ 

Moreover, the homology of the symmetric products has been determined completely by J. Milgram [6].

In this paper some consequences of the results of Macdonald are given and an alternative proof for the expression of the Euler characteristic of the symmetric product of $M$ is given, too.

### 2. Some conclusions about the symmetric products

First, we prove the following theorem which gives a necessary and sufficient condition for the orientability of a 2-dimensional surface.

**Theorem 1.** Let $M$ be a 2-dimensional manifold. The manifold $M^{(m)}$ is orientable if and only if $M$ is orientable.

**Proof.** If $M$ is orientable, then $M$ is a complex manifold. Thus, $M^{(m)}$ is also a complex manifold and hence $M^{(m)}$ is orientable.

Assume that $M$ is non-orientable. Then there exist a non-orientable open subset $T$ of $M$ and an open subset $U$ which is homeomorphic to $\mathbb{R}^2$, such that $T \cap U = \emptyset$. Then $M^{(m)}$ contains the non-orientable submanifold $T \times U^{(m-1)} \cong T \times \mathbb{R}^{2m-2}$ with the same dimension and hence $M^{(m)}$ is not orientable. \( \square \)

For example, it is known that $(\mathbb{R}P^2)^{(m)} = \mathbb{R}P^{2m}$, and both spaces $\mathbb{R}P^2$ and $\mathbb{R}P^{2m}$ are not orientable.
Let $M$ be a finite CW complex and let us denote by

$$B_i = \dim(H_i(M, \mathbb{Z})), \quad i = 0, 1, 2, \ldots, n = \dim M$$

and let us denote the Betti numbers of the CW complex $M^{(m)}$ by

$$B_i^{(m)} = \dim(H_i(M^{(m)}, \mathbb{Z})), \quad i = 0, 1, 2, \ldots, nm.$$

As a direct consequence of the result of Macdonald [5] we have the following two theorems.

**Theorem 2.** The Betti numbers $B_i^{(m)}$, $i = 0, 1, 2, \ldots, nm$, are given explicitly by

$$B_i^{(m)} = (-1)^i \sum_{\alpha_0, \ldots, \alpha_n} \left\{ \frac{\alpha_0 - 1 + B_0}{\alpha_0} \right\} \left\{ \frac{\alpha_1 - 1 - B_1}{\alpha_1} \right\} \cdots \left\{ \frac{\alpha_n - 1 + (-1)^n B_n}{\alpha_n} \right\},$$

(3)

where the summation over $\alpha_0, \ldots, \alpha_n$ is under the following restrictions

$$0 \leq \alpha_0, \ldots, \alpha_n \leq m, \quad \alpha_0 + \cdots + \alpha_n = m, \quad \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = i.$$  

(4)

**Proof.** Using the equality

$$\left( \frac{\alpha - 1 + B}{\alpha} \right) = (-1)^\alpha \left( \frac{-B}{\alpha} \right),$$

for a positive integer $\alpha$ and integer $B$, we should prove that

$$B_i^{(m)} = (-1)^{i+m} \sum_{\alpha_0, \ldots, \alpha_n} \left\{ \frac{-B_0}{\alpha_0} \right\} \frac{B_1}{\alpha_1} \left\{ \frac{-B_2}{\alpha_2} \right\} \cdots \frac{(-1)^{n+1} B_n}{\alpha_n},$$

where the summation over $\alpha_1, \ldots, \alpha_n$ is given by (4). Since

$$\frac{(1 + xt)^{B_1}(1 + x^3t)^{B_3} \cdots}{(1 - t)^{B_0}(1 - x^2t)^{B_2}(1 - x^4t)^{B_4} \cdots} = \sum_{m=0}^{\infty} \sum_{i=0}^{mn} B_i^{(m)} x^i t^m,$$

it is sufficient to prove that

$$(1 - t)^{-B_0}(1 + xt)^{B_1}(1 - x^2t)^{-B_2}(1 + x^3t)^{B_3}(1 - x^4t)^{-B_4} \cdots$$

$$= \sum_{m=0}^{mn} \sum_{i=0}^{mn} \sum_{\alpha_0, \ldots, \alpha_n} (-1)^{i+m} x^i t^m \sum_{\alpha_0, \ldots, \alpha_n} \left\{ \frac{-B_0}{\alpha_0} \right\} \left\{ \frac{B_1}{\alpha_1} \right\} \cdots \frac{(-1)^{n+1} B_n}{\alpha_n},$$

where the summation over $\alpha_1, \ldots, \alpha_n$ is given by (4). Notice that $(-1)^{i+m} = (-1)^{\alpha_0 + \alpha_2 + \alpha_4 + \cdots}$. Applying the binomial formula for the powers of the left-hand side and choosing the summand $(-1)^{\alpha_0} \left( \frac{-B_0}{\alpha_0} \right)^\alpha_0$ from $(1 - t)^{-B_0}$, \( \left( \frac{B_1}{\alpha_1} \right) \), $\cdots$, $\left( \frac{(-1)^{n+1} B_n}{\alpha_n} \right)$ from $(1 + xt)^{B_1}$, $(1)^{\alpha_1} \left( \frac{-B_2}{\alpha_2} \right)^{\alpha_2} z^{2\alpha_2}$ from $(1 - x^2t)^{-B_2}$ and so on, and sum over $\alpha_0, \alpha_1, \alpha_2, \cdots$, according to (4), we obtain that the previous equality is true.
Theorem 3. Euler characteristics of $M$ and the symmetric product $M^{(m)}$ are related by

$$\chi(M^{(m)}) = \binom{m + \chi(M) - 1}{m}.$$  \hfill (5)

Proof. Let

$$\sum_{i=0}^{nm} (-1)^i B_i^{(m)} = \sum_{i=0}^{nm} \sum_{\alpha_0, \ldots, \alpha_n} (-1)^{\alpha_0} \left( \begin{array}{c} B_0 \\ \alpha_0 \end{array} \right) (-1)^{\alpha_1} \left( \begin{array}{c} B_1 \\ \alpha_1 \end{array} \right) \cdots (-1)^{\alpha_n} \left( \begin{array}{c} B_n \\ \alpha_n \end{array} \right),$$

where the summation over $\alpha_0, \ldots, \alpha_n$ satisfies conditions (4). Since $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = i$ and we sum over $i$, using that $\alpha_0 + \cdots + \alpha_n = m$ we obtain

$$\chi(M^{(m)}) = \sum_{i=0}^{nm} (-1)^i B_i^{(m)} = \sum_{\alpha_0 + \cdots + \alpha_n = m} (-1)^m \left( \begin{array}{c} B_0 \\ \alpha_0 \end{array} \right) \left( \begin{array}{c} B_1 \\ \alpha_1 \end{array} \right) \cdots \left( \begin{array}{c} B_n \\ \alpha_n \end{array} \right)$$

$$= (-1)^m \left( -\chi(M) + \sum_{i=0}^{m} (-1)^{i+1} B_i \right)$$

$$= (-1)^m \left( m + \chi(M) - 1 \right).$$

\hfill $\square$

Notice that Theorem 3 also follows from (2). Indeed if we put $x = -1$, then the Euler characteristic of $M^{(m)}$ is the coefficient in front of $t^m$. Indeed, the expression from (2) for $x = -1$ becomes

$$(1 - t)^{-B_0} (1 - t)^{-B_1} (1 - t)^{-B_2} \cdots = (1 - t)^{-\chi(M)} = \sum_{m=0}^{\infty} (-1)^m t^m \binom{-\chi(M)}{m}$$

$$= \sum_{m=0}^{\infty} t^m \binom{m + \chi(M) - 1}{m},$$

and hence the proof of Theorem 3 follows.

Theorem 4. If $M$ is a compact CW complex such that $\chi(M) = 0$, then $M^{(m)}$ cannot be decomposed into cells of type $C^i$.

Proof. If $\chi(M) = 0$, then $\chi(M^{(m)}) = 0$. If $M^{(m)}$ can be decomposed into cells of type $C^i$, then its Euler characteristic should be a positive number, which contradicts to $\chi(M^{(m)}) = 0$. \hfill $\square$

Notice that if $M$ is a torus, then it is a complex manifold, but its symmetric product cannot be decomposed into complex cells $C^0, C, C^2, \cdots$. 
3. Direct proof of the Theorem 3

In this section we present a direct proof of Theorem 3, without using the Macdonald’s results.

Here Euler characteristic $\chi$ will denote the number of even dimensional cells of type $\mathbb{R}^1$ minus the number of odd dimensional cells of the same type where all of the cells are disjoint.

**Proof.** Notice that (5) is trivially satisfied for $m = 1$, because $M^{(1)} = M$. So, without loss of generality, in the proof we assume that $m \geq 2$. First, let us prove formula (5) for the cells $C = \mathbb{R}^n$. The proof is by induction of $n$. If $n = 0$, i.e. $C$ is a point, then $C^{(m)}$ is also a point, and (5) is true because $\chi(C) = \chi(C^{(m)}) = 1$. Further, if $n = 1$, i.e. $C = \mathbb{R}$, then according to [17], $C^{(m)}$ is homeomorphic to the half space $H^m$ and (5) holds because $\chi(C) = -1$ and $\chi(C^{(m)}) = 0$. If $C = \mathbb{R}^2$, then $C^{(m)} = \mathbb{R}^{2m}$ and (5) is true, because $\chi(C) = \chi(C^{(m)}) = 1$.

Let us assume that $n \geq 3$. We should prove that $\chi((\mathbb{R}^n)^{(m)})$ is equal to $0$ if $n$ is an odd number and it is $1$ if $n$ is an even number, for $m > 1$. In all calculations we will neglect all sets whose Euler characteristic is $0$ according to the inductive assumption.

Assume that $n = 2k + 1$ and the statement is true for $2k$. We consider the space $\mathbb{R}^n$ as a disjoint union of a hyperplane $\Sigma$ and the corresponding two open half spaces $\Sigma_1$ and $\Sigma_2$. We prove that $\chi((\mathbb{R}^{2k+1})^{(m)}) = 0$ by induction of $m$. Assume that it is true for $2, 3, \ldots, m - 1$. According to the inductive assumption, it is sufficient to consider only those cells where only one point, $m$ points or no points are chosen in $\Sigma_1$ (also $\Sigma_2$). But if $m$ points are chosen in $\Sigma_1$ (resp. $\Sigma_2$), then in $\Sigma_2$ (resp. $\Sigma_1$) there is no one chosen point. Hence there we should consider only these six cases: 1) all points are chosen from $\Sigma_1$, 2) all points are chosen from $\Sigma_2$, 3) all points are chosen from $\Sigma$, 4) one point is chosen from $\Sigma_1$, one point from $\Sigma_2$ and the rest $m - 2$ points are chosen from $\Sigma$, 5) one point is chosen from $\Sigma_1$ and the rest $m - 1$ points are chosen from $\Sigma$, 6) one point is chosen from $\Sigma_2$ and the rest $m - 1$ points are chosen from $\Sigma$. Using the inductive assumption for $n - 1 = \dim \Sigma$, for the Euler characteristic of $(\mathbb{R}^n)^{(m)}$ we obtain the equality

$$
\chi((\mathbb{R}^n)^{(m)}) = \chi(\Sigma_1^{(m)}) + \chi(\Sigma_2^{(m)}) + \chi(\Sigma^{(m)}) + \chi(\Sigma_1 \times \Sigma_2) \times \chi(\Sigma^{(m-2)})
+ \chi(\Sigma_1) \times \chi(\Sigma^{(m-1)}) + \chi(\Sigma_2) \times \chi(\Sigma^{(m-1)})
= 2\chi((\mathbb{R}^n)^{(m)}) + 1 + 1 - 1 - 1
$$

and hence we obtain $\chi((\mathbb{R}^n)^{(m)}) = 0$.

Assume that $n = 2k$ and the statement is true for $2k - 1$. We consider the space $\mathbb{R}^n$ as a disjoint union of a hyperplane $\Sigma$ and two open half spaces $\Sigma_1$ and $\Sigma_2$ as in the previous case. We prove that $\chi((\mathbb{R}^{2k})^{(m)}) = 1$ by induction of $m$. Assume that it is true for $2, 3, \ldots, m - 1$. According to the inductive assumption, it is sufficient to consider only those cells where one point or no points are chosen in $\Sigma$. Hence we
obtain the following equality

$$
\chi((\mathbb{R}^n)^{(m)}) = \sum_{i=0}^{m} \chi(\Sigma_1^{(i)}) \cdot \chi(\Sigma_2^{(m-i)}) + \sum_{i=1}^{m-1} \chi(\Sigma_1^{(i)}) \cdot \chi(\Sigma_2^{(m-1-i)}) \cdot \chi(\Sigma) \\
= 2\chi((\mathbb{R}^n)^{(m)}) + (m-1) + m(-1)^{2k-1} = 2\chi((\mathbb{R}^n)^{(m)}) - 1,
$$

and hence $\chi((\mathbb{R}^n)^{(m)}) = 1$.

To complete the proof it is sufficient to prove that if (5) is true for arbitrary $m$ and disjoint sets $X_1$ and $X_2$ of cells of the CW complex $M$, then formula (5) is true for $X_1 \cup X_2$. Let $\chi(X_1) = p$, $\chi(X_2) = q$ and assume that

$$
\chi(X_1^{(i)}) = \binom{i + p - 1}{i} \text{ for } i = 0, 1, 2, \ldots,
$$

$$
\chi(X_2^{(i)}) = \binom{i + q - 1}{i} \text{ for } i = 0, 1, 2, \ldots.
$$

Then

$$
\chi((X_1 \cup X_2)^{(m)}) = \chi\left[X_1^{(m)} \cup (X_1^{(m-1)} \times X_2) \cup (X_1^{(m-2)} \times X_2^{(2)}) \cup \cdots \cup X_2^{(m)}\right] \\
= \sum_{s=0}^{m} \chi(X_1^{(s)}) \cdot \chi(X_2^{(m-s)}) \\
= \sum_{s=0}^{m} \binom{s + p - 1}{s} \binom{m - s + q - 1}{m - s} \\
= \sum_{s=0}^{m} (-1)^s \binom{-p}{s} (-1)^{m-s} \binom{-q}{m - s} \\
= (-1)^m \sum_{s=0}^{m} \binom{-p}{s} \binom{-q}{m - s} \\
= (-1)^m \binom{-p - q}{m} \\
= \binom{m + p + q - 1}{m}.
$$

References


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