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A note on generalized absolute summability factors

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Abstract. In this paper, a general theorem on $|A,\delta|_k$- summability factors of infinite series has been proved under weaker conditions.

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1. Introduction

Rhoades and Savas [4] recently have obtained sufficient conditions for the series $\sum a_n\lambda_n$ to be absolutely summable of order k by a triangular matrix.

In this paper we generalize the result of Rhoades and Savas under weaker conditions for $|A,\delta|_k, k \geq 1, 0 \leq \delta < 1/k$.

A positive sequence $\{b_n\}$ is said to be almost increasing if there exists an increasing sequence $\{c_n\}$ and positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$, (see, [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = e^{(-1)^n}n$.

Let A be a lower triangular matrix, $\{s_n\}$ a sequence. Then

$$A_n := \sum_{\nu=0}^{n} a_{n\nu}s_\nu.$$

A series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}|A_n - A_{n-1}|^k < \infty. \tag{1}$$

and it is said to be summable $|A,\delta|_k, k \geq 1$ and $\delta \geq 0$ if (see,[2])

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|A_n - A_{n-1}|^k < \infty. \tag{2}$$

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We may associate with A two lower triangular matrices $\overline{A}$ and $\hat{A}$ defined as follows:

$$\bar{a}_{n\nu} = \sum_{r=\nu}^{n} a_{nr}, \qquad n, \nu = 0, 1, 2, \dots,$$

and

$$\hat{a}_{n\nu} = \bar{a}_{n\nu} - \bar{a}_{n-1,\nu}, \qquad n = 1, 2, 3, \dots.$$

With $s_n := \sum_{i=0}^{n} \lambda_i a_i$.

$$\begin{aligned} y_n &:= \sum_{i=0}^{n} a_{ni} s_i = \sum_{i=0}^{n} a_{ni} \sum_{\nu=0}^{i} \lambda_\nu a_\nu \\ &= \sum_{\nu=0}^{n} \lambda_\nu a_\nu \sum_{i=\nu}^{n} a_{ni} = \sum_{\nu=0}^{n} \bar{a}_{n\nu} \lambda_\nu a_\nu \end{aligned}$$

and

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} (\bar{a}_{n\nu} - \bar{a}_{n-1,\nu}) \lambda_\nu a_\nu = \sum_{\nu=0}^{n} \hat{a}_{n\nu} \lambda_\nu a_\nu. \tag{3}$$

Theorem 1. *Let A be a lower triangular matrix satisfying*

(i) $\bar{a}_{n0} = 1, n = 0, 1, \dots,,$

(ii) $a_{n-1,\nu} \geq a_{n\nu}$ for $n \geq \nu + 1$, *and*

(iii) $na_{nn} \asymp O(1)$

(iv) $\displaystyle\sum_{\nu=1}^{n-1} a_{\nu\nu} |\hat{a}_{n\nu+1}| = O\Big(a_{nn}\Big),$

(v) $\displaystyle\sum_{n=\nu+1}^{m+1} n^{\delta k} |\Delta_\nu \hat{a}_{n\nu}| = O\Big(\nu^{\delta k} a_{\nu\nu}\Big)$ *and*

(vi) $\displaystyle\sum_{n=\nu+1}^{m+1} n^{\delta k} |\hat{a}_{n\nu+1}| = O\Big(\nu^{\delta k}\Big).$

If $\{X_n\}$ is an almost increasing sequence such that

(vii) $\lambda_m X_m = O(1),$

(viii) $\displaystyle\sum_{n=1}^{m} (nX_n) |\Delta^2 \lambda_n| = O(1),$ *and*

(ix) $\displaystyle\sum_{n=1}^{m} n^{\delta k} a_{nn} |t_n|^k = O(X_m),$ *where* $\displaystyle t_n := \frac{1}{n+1} \sum_{k=1}^{n} k a_k,$

then the series $\sum a_n \lambda_n$ is summable $|A, \delta|_k, k \geq 1, 0 \leq \delta < 1/k$.

Lemma 1 (see [4]). *If (X_n) is an almost increasing sequence, then under the conditions of the theorem we have that*

(i) $\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty$ *and*

(ii) $nX_n|\Delta\lambda_n| = O(1)$.

Proof. From (3) we may write

$$\begin{aligned}
Y_n &= \sum_{\nu=1}^{n} \Big(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\Big)\nu a_\nu \\
&= \sum_{\nu=1}^{n} \Big(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\Big)\Big[\sum_{r=1}^{\nu} ra_r - \sum_{r=1}^{\nu-1} ra_r\Big] \\
&= \sum_{\nu=1}^{n-1} \Delta_\nu\Big(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\Big)\sum_{r=1}^{\nu} ra_r + \frac{\hat{a}_{nn}\lambda_n}{n}\sum_{\nu=1}^{n} \nu a_\nu \\
&= \sum_{\nu=1}^{n-1} (\Delta_\nu \hat{a}_{n\nu})\lambda_\nu \frac{\nu+1}{\nu} t_\nu + \sum_{\nu=1}^{n-1} \hat{a}_{n,\nu+1}(\Delta\lambda_\nu)\frac{\nu+1}{\nu}t_\nu \\
&\quad + \sum_{\nu=1}^{n-1} \hat{a}_{n,\nu+1}\lambda_{\nu+1}\frac{1}{\nu}t_\nu + \frac{(n+1)a_{nn}\lambda_n t_n}{n} \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \quad \text{say.}
\end{aligned}$$

To finish the proof it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|T_{nr}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

Using Hölder's inequality and (iii),

$$\begin{aligned}
I_1 := \sum_{n=1}^{m} n^{\delta k+k-1}|T_{n1}|^k &= \sum_{n=1}^{m} n^{\delta k+k-1}\Big|\sum_{\nu=1}^{n-1} \Delta_\nu \hat{a}_{n\nu}\lambda_\nu \frac{\nu+1}{\nu}t_\nu\Big|^k \\
&= O(1)\sum_{n=1}^{m+1} n^{\delta k+k-1}\Big(\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}||\lambda_\nu||t_\nu|\Big)^k \\
&= O(1)\sum_{n=1}^{m+1} n^{\delta k+k-1}\Big(\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}||\lambda_\nu|^k|t_\nu|^k\Big)\Big(\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}|\Big)^{k-1}.
\end{aligned}$$

Using the fact that, from (vii), $\{\lambda_n\}$ is bounded, and condition (i) of Lemma 1, and (v)

$$\begin{aligned}
I_1 &= O(1)\sum_{n=1}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|\lambda_\nu|^k|t_\nu|^k|\Delta_\nu\hat{a}_{n\nu}| \\
&= O(1)\sum_{n=1}^{m+1} n^{\delta k}(na_{nn})^{k-1}\Big(\sum_{\nu=1}^{n-1}|\lambda_\nu|^{k-1}|\lambda_\nu||\Delta_\nu\hat{a}_{n\nu}||t_\nu|^k\Big) \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu||t_\nu|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}(na_{nn})^{k-1}|\Delta_\nu\hat{a}_{n\nu}| \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu||t_\nu|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}|\Delta_\nu\hat{a}_{n\nu}| \\
&= O(1)\sum_{\nu=1}^{m}\nu^{\delta k}|\lambda_\nu|a_{\nu\nu}|t_\nu|^k \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu|\Big[\sum_{r=1}^{\nu}a_{rr}|t_r|^k r^{\delta k}-\sum_{r=1}^{\nu-1}a_{rr}|t_r|^k r^{\delta k}\Big] \\
&= O(1)\Big[\sum_{\nu=1}^{m-1}\Delta(|\lambda_\nu|)\sum_{r=1}^{\nu}a_{rr}|t_r|^k r^{\delta k}+|\lambda_m|\sum_{r=1}^{m}a_{rr}|t_r|^k r^{\delta k}\Big] \\
&= O(1)\sum_{\nu=1}^{m-1}|\Delta\lambda_\nu|X_\nu+O(1)|\lambda_m|X_m \\
&= O(1).
\end{aligned}$$

Using Hölder's inequality, (iii), and (iv),

$$\begin{aligned}
I_2 := \sum_{n=2}^{m+1} n^{\delta k+k-1}|T_{n2}|^k &= \sum_{n=2}^{m+1} n^{\delta k+k-1}\Big|\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}(\Delta\lambda_\nu)\frac{\nu+1}{\nu}t_\nu\Big|^k \\
\leq \ &\sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{\nu=1}^{n-1}|\hat{a}_{n,\nu+1}||\Delta\lambda_\nu|\frac{\nu+1}{\nu}|t_\nu|\Big]^k \\
= \ &O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{\nu=1}^{n-1}|\hat{a}_{n,\nu+1}||\Delta\lambda_\nu||t_\nu|\Big]^k \\
= \ &O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{\nu=1}^{n-1}(\nu)|\Delta\lambda_\nu||t_\nu|a_{\nu\nu}|\hat{a}_{n,\nu+1}|\Big]^k \\
= \ &O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\sum_{\nu=1}^{n-1}(\nu|\Delta\lambda_\nu|)^k|t_\nu|^k a_{\nu\nu}|\hat{a}_{n,\nu+1}|\Big] \\
&\times\Big[\sum_{\nu=1}^{n-1}a_{\nu\nu}|\hat{a}_{n,\nu+1}|\Big]^{k-1}
\end{aligned}$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}(\nu|\Delta\lambda_\nu|)^k|t_\nu|^k a_{\nu\nu}|\hat{a}_{n,\nu+1}|$$
$$= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}(\nu|\Delta\lambda_\nu|)^{k-1}(\nu|\Delta\lambda_\nu|)a_{\nu\nu}|\hat{a}_{n,\nu+1}||t_\nu|^k$$

Conclusion (ii) of Lemma 1 implies that $\nu|\Delta\lambda_\nu| = O(1)$. Therefore, using (iii), (v) and (vi)

$$I_2 := O(1)\sum_{\nu=1}^{m}\nu|\Delta\lambda_\nu|a_{\nu\nu}|t_\nu|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}(na_{nn})^{k-1}|\hat{a}_{\nu\nu+1}|$$
$$= O(1)\sum_{\nu=1}^{m}\nu|\Delta\lambda_\nu|a_{\nu\nu}|t_\nu|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}|\hat{a}_{n,\nu+1}|.$$

Therefore,

$$I_2 := O1)\sum_{\nu=1}^{m}\nu^{\delta k}\nu|\Delta\lambda_\nu|a_{\nu\nu}|t_\nu|^k.$$

Using summation by parts and (ix),

$$I_2 = O(1)\sum_{\nu=1}^{m}\nu|\Delta\lambda_\nu|\Big[\sum_{r=1}^{\nu}a_{rr}|t_r|^k r^{\delta k} - \sum_{r=1}^{\nu-1}a_{rr}|t_r|^k r^{\delta k}\Big]$$
$$= O(1)\sum_{\nu=1}^{m-1}|\Delta(\nu\Delta\lambda_\nu)|X_\nu + O(1).$$

But

$$\Delta(\nu\Delta\lambda_\nu) = \nu\Delta\lambda_\nu - (\nu+1)\Delta\lambda_{\nu+1} = \nu\Delta^2\lambda_\nu - \Delta\lambda_{\nu+1}.$$

Using (viii) and property (i) from Lemma 1, and the fact that $\{X_n\}$ is almost increasing,

$$I_2 = O(1)\sum_{\nu=1}^{m-1}\nu|\Delta^2\lambda_\nu|X_\nu + O(1)\sum_{\nu=1}^{m-1}|\Delta\lambda_{\nu+1}|X_{\nu+1} = O(1).$$

Using (iii), Hölder's inequality, (iv), summation by parts, property (i) of Lemma 1, (vi), (vii) and (ix)

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}|T_{n3}|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\Big|\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}\lambda_{\nu+1}\frac{1}{\nu}t_\nu\Big|^k$$
$$\le \sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|\frac{\hat{a}_{n,\nu+1}}{\nu}|t_\nu|\Big]^k$$
$$= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}||\hat{a}_{n,\nu+1}||t_\nu|a_{\nu\nu}\Big]^k$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^k a_{\nu\nu}|t_\nu|^k|\hat{a}_{n,\nu+1}|\Big] \\
&\quad\times\Big[\sum_{\nu=1}^{n-1} a_{\nu\nu}|\hat{a}_{n,\nu+1}|\Big]^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^{k-1}|\lambda_{\nu+1}|a_{\nu\nu}|t_\nu|^k|\hat{a}_{n,\nu+1}| \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_{\nu+1}||t_\nu|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}|\hat{a}_{n,\nu+1}| \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_{\nu+1}|a_{\nu\nu}|t_\nu|^k\nu^{\delta k} \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_{\nu+1}|\Big[\sum_{r=1}^{\nu} a_{rr}|t_r|^k r^{\delta k}-\sum_{r=1}^{\nu-1} a_{rr}|t_r|^k r^{\delta k}\Big] \\
&= O(1)\Big[\sum_{\nu=1}^{m-1}|\Delta\lambda_{\nu+1}|\sum_{r=1}^{\nu} a_{rr}|t_r|^k r^{\delta k}+|\lambda_{m+1}|\sum_{r=1}^{\nu} a_{rr}|t_r|^k r^{\delta k}\Big] \\
&= O(1)\sum_{\nu=1}^{m-1}|\Delta\lambda_{\nu+1}|X_\nu + O(1)|\lambda_{\nu+1}|X_m \\
&= O(1).
\end{aligned}
$$

Finally, using (iii), summation by parts, property (i) of Lemma 1 and (vii),

$$
\begin{aligned}
\sum_{n=1}^{m} n^{\delta k+k-1}|T_{n4}|^k &= \sum_{n=1}^{m} n^{\delta k+k-1}\Big|\frac{(n+1)a_{nn}\lambda_n t_n}{n}\Big|^k \\
&= O(1)\sum_{n=1}^{m} n^{\delta k+k-1}|a_{nn}|^k|\lambda_n|^k|t_n|^k \\
&= O(1)\sum_{n=1}^{m} n^{\delta k}(na_{nn})^{k-1}a_{nn}|\lambda_n|^{k-1}|\lambda_n||t_n|^k \\
&= O(1)\sum_{n=1}^{m} n^{\delta k}a_{nn}|\lambda_n||t_n|^k,
\end{aligned}
$$

as in the proof of I_1. □

Setting $\delta = 0$ in the theorem yields the following corollary.

Corollary 1. *Let A be a triangle satisfying conditions (i)-(iv) of Theorem 1 and let $\{X_n\}$ be an almost increasing sequence satisfying conditions (vii)-(viii). If*

(ix) $\sum_{n=1}^m a_{nn}|t_n|^k = O(X_m)$,

then the series $\sum a_n\lambda_n$ is summable $|A|_k, k \geq 1$.

Corollary 2. *Let $\{p_n\}$ be a positive sequence such that $P_n := \sum_{k=0}^{n} p_k \to \infty$, and satisfies*

(i) $np_n \asymp O(P_n)$,

(ii) $$\sum_{n=\nu+1}^{m+1} n^{\delta k} \left|\frac{p_n}{P_n P_{n-1}}\right| = O\Big(\frac{\nu^{\delta k}}{P_\nu}\Big).$$

If $\{X_n\}$ is an almost increasing sequence such that

(iii) $\lambda_m X_m = O(1)$,

(iv) $$\sum_{n=1}^{m} nX_n|\Delta^2\lambda_n| = O(1), \quad \textit{and}$$

(v) $$\sum_{n=1}^{\infty} n^{\delta k-1}|t_n|^k = O(X_m),$$

then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p, \delta|_k, k \geq 1$ for $0 \leq \delta < 1/k$.

Proof. Conditions (iii) and (iv) of Corollary 2 are conditions (vii) and (viii) of Theorem 1, respectively.

Conditions (i), (ii) and (iv) of Theorem 1 are automatically satisfied for any weighted mean method. Condition (iii) and (ix) of Theorem 1 become conditions (i) and (v) of Corollary 2 and conditions (v) and (vi) of Theorem 1 become condition (ii) of Corollary 2. □

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