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Hardy-Hilbert's integral inequality in new kinds

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Abstract. New kinds of Hardy-Hilbert's integral inequalities are presented.
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1. Introduction

If $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leq \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(x)dx \right)^{1/p} \left(\int_0^\infty g^q(x)dx \right)^{1/q} \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible. Inequality (1) is called a Hardy-Hilbert's integral inequality (see [1]) and it is important in analysis and applications (cf. Mitrinovic et al.[2]), Hardy et al.[1] gave an inequality similar to (1) as:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy \leq pq \left(\int_0^\infty f^p(t)dt \right)^{1/p} \left(\int_0^\infty g^q(t)dt \right)^{1/q} \tag{2}$$

where the constant factor pq is the best possible.

Other mathematicians present generalizations or new kinds of (2) as follows:

Theorem 1 (see [3]). *If $\lambda > 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$ such that*

$$0 < \int_0^\infty t^{p-1-\lambda} f^p(t)dt < \infty \text{ and } 0 < \int_0^\infty t^{q-1-\lambda} g^q(t)dt < \infty,$$

then one has

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}}dxdy \leq \frac{pq}{\lambda} \left(\int_0^\infty t^{p-1-\lambda} f^p(t)dt \right)^{1/p} \left(\int_0^\infty t^{q-1-\lambda} g^q(t)dt \right)^{1/q}, \tag{3}$$

where the constant factor $\frac{pq}{\lambda}$ is the best possible.

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Theorem 2 (See [4]). *Suppose* $f,g \geq 0$, $0 < \int\limits_0^\infty f^2(x)dx < \infty$, $0 < \int\limits_0^\infty g^2(x)dx < \infty$. *Then*

$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}dxdy \leq c\left(\int\limits_0^\infty f^2(x)dx \int\limits_0^\infty g^2(x)dx\right)^{1/2}, \tag{4}$$

where $c = \sqrt{2}(\pi - 2\arctan\sqrt{2}) \approx 1.7408$.

The Gamma and Beta functions denoted by $\Gamma(p)$, $B(p,q)$, are defined by

$$\Gamma(p) = \int\limits_0^\infty t^{p-1}e^{-t}dt \quad , \quad p > 0 \tag{5}$$

and

$$B(p,q) = \int\limits_0^1 t^{p-1}(1-t)^{q-1}dt = \int\limits_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}dt \quad , \quad p > 0 \quad , \quad q > 0, \tag{6}$$

respectively.

2. Main results

Lemma 1. *Let* $p > 0$, $q > 0$. *Then*

$$\Gamma(p) = e\int\limits_0^1 \frac{x^{-p-1}e^{-1/x}}{(1-x)^{1-p}}dx \tag{7}$$

$$\Gamma(p) = e\int\limits_1^\infty \frac{e^{-x}}{(x-1)^{1-p}}dx \tag{8}$$

and

$$B(p,q) = \int\limits_1^\infty \frac{x^{-p-q}}{(x-1)^{1-p}}dx. \tag{9}$$

Proof. (7) and (8) follows by putting $t = x^{-1} - 1$, $x - 1$, in (5), respectively, and (8) follows by putting $x = 1/t$ in (6). $\square$

Theorem 3. *Assume that* $f,g \geq 0$, $\lambda > 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda}\max\{e^{x/y}, e^{y/x}\}}dxdy$$
$$\leq 2\frac{\Gamma(\lambda)}{e}\left(\int\limits_0^\infty t^{(1-\lambda)(p-1)-1}f^p(t)dt\right)^{1/p}\left(\int\limits_0^\infty t^{(1-\lambda)(q-1)-1}g^q(t)dt\right)^{1/q} \tag{10}$$

provided the integrals on the right-hand side do exist.

Proof.

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dxdy\\
&\quad=\int_0^\infty\int_0^\infty \frac{f(x)y^{(-1-\lambda)\frac{1}{p}}}{x^{(-1-\lambda)\frac{1}{q}}\,|x-y|^{(1-\lambda)/p}\left(\max\{e^{x/y},e^{y/x}\}\right)^{1/p}}\\
&\qquad\times\frac{g(y)x^{(-1-\lambda)\frac{1}{q}}}{y^{(-1-\lambda)\frac{1}{q}}\,|x-y|^{(1-\lambda)/q}\left(\max\{e^{x/y},e^{y/x}\}\right)^{1/q}}dxdy\\
&\quad\le\left(\int_0^\infty\int_0^\infty \frac{f^p(x)y^{-1-\lambda}}{x^{(-1-\lambda)\frac{p}{q}}\,|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dxdy\right)^{1/p}\\
&\qquad\times\left(\frac{g^q(y)x^{-1-\lambda}}{y^{(-1-\lambda)\frac{q}{p}}\,|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dxdy\right)^{1/q}\\
&\quad=M^{1/p}N^{1/q}.
\end{aligned}
$$

We first consider

$$
\begin{aligned}
M&=\int_0^\infty\int_0^\infty \frac{f^p(x)y^{-1-\lambda}}{x^{(-1-\lambda)\frac{p}{q}}\,|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dxdy\\
&=\int_0^\infty x^{(1+\lambda)(p-1)}f^p(x)\left(\int_0^\infty \frac{y^{-1-\lambda}}{|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dy\right)dx.
\end{aligned}
$$

Now,

$$\int_0^\infty \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dy=M_1+M_2,$$

where

$$M_1=\int_0^x \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dy,\quad x>0,$$

and

$$M_2=\int_x^\infty \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dy,\quad x>0.$$

Then

$$M_1 = \int_0^x \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dy = \int_0^x \frac{y^{-\lambda-1}}{(x-y)^{1-\lambda}e^{x/y}}dy$$
$$= x^{-1}\int_0^1 \frac{u^{-\lambda-1}e^{-1/u}}{(1-u)^{1-\lambda}}du, \quad (y/x=u)$$
$$= \frac{\Gamma(\lambda)}{e}x^{-1},$$

and

$$M_2 = \int_x^\infty \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dy = \int_x^\infty \frac{y^{-\lambda-1}}{(y-x)^{1-\lambda}e^{y/x}}dy$$
$$\le x^{-\lambda-1}\int_x^\infty \frac{1}{(y-x)^{1-\lambda}e^{y/x}}dy$$
$$= x^{-1}\int_1^\infty \frac{e^{-v}}{(v-1)^{1-\lambda}}dv, \quad y/x=v$$
$$= \frac{\Gamma(\lambda)}{e}x^{-1}.$$

Therefore,

$$\int_0^\infty \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dy \le 2\frac{\Gamma(\lambda)}{e}x^{-1},$$

and hence

$$M \le 2\frac{\Gamma(\lambda)}{e}\int_0^\infty x^{(1+\lambda)(p-1)-1}f^p(x)dx.$$

Similarly,

$$N \le 2\frac{\Gamma(\lambda)}{e}\int_0^\infty y^{(1+\lambda)(q-1)-1}g^q(y)dy.$$

Collecting the above estimations, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda}\max\{e^{x/y},e^{y/x}\}}dxdy$$
$$\le 2\frac{\Gamma(\lambda)}{e}\left(\int_0^\infty t^{(1-\lambda)(p-1)-1}f^p(t)dt\right)^{1/p}\left(\int_0^\infty t^{(1-\lambda)(q-1)-1}g^q(t)dt\right)^{1/q}. \quad (11)$$

□

Theorem 4. *Assume that* $f, g \geq 0,\ \lambda > 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dxdy \leq 2B(\lambda,\lambda) \left(\int_0^\infty t^\lambda f^p(t)dt \right)^{1/p} \left(\int_0^\infty t^\lambda g^q(t)dt \right)^{1/q}, \tag{12}$$

provided the integrals on the right-hand side do exist.

Proof.

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dxdy \\
&= \int_0^\infty \int_0^\infty \frac{f(x)}{|x-y|^{\frac{1-\lambda}{p}} \left(\max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}\right)^{1/p}} \\
&\quad \times \frac{g(y)}{|x-y|^{\frac{1-\lambda}{q}} \left(\max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}\right)^{1/q}} dxdy \\
&\leq \left(\int_0^\infty \int_0^\infty \frac{f^p(x)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dxdy \right)^{1/p} \\
&\quad \times \left(\int_0^\infty \int_0^\infty \frac{g^q(y)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dxdy \right)^{1/q} \\
&= A^{1/p} C^{1/q}.
\end{aligned}$$

First we estimate

$$\begin{aligned}
A &= \int_0^\infty \int_0^\infty \frac{f^p(x)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dxdy \\
&= \int_0^\infty f^p(x) \left(\int_0^\infty \frac{1}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dy \right) dx.
\end{aligned}$$

Now,

$$\int_0^\infty \frac{1}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dy = A_1 + A_2,$$

where

$$A_1 = \int_0^x \frac{1}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dy$$
$$= \int_0^x \frac{(\frac{y}{x})^{2\lambda}}{(x-y)^{1-\lambda}} dy$$
$$= x^\lambda \int_0^x \frac{(\frac{y}{x})^{\lambda+1} (\frac{y}{x})^{\lambda-1} \frac{1}{x}}{(1-\frac{y}{x})^{1-\lambda}} dy$$
$$\leq x^\lambda \int_0^x \frac{(\frac{y}{x})^{\lambda-1} \frac{1}{x}}{(1-\frac{y}{x})^{1-\lambda}} dy$$
$$= x^\lambda \int_0^1 \frac{u^{\lambda-1}}{(1-u)^{1-\lambda}} du$$
$$= B(\lambda, \lambda) x^\lambda,$$

and

$$A_2 = \int_x^\infty \frac{1}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dy$$
$$= \int_x^\infty \frac{(\frac{y}{x})^{-2\lambda}}{(y-x)^{1-\lambda}} dy$$
$$= x^\lambda \int_x^\infty \frac{(\frac{y}{x})^{-2\lambda} \frac{1}{x}}{(\frac{y}{x}-1)^{1-\lambda}} dy$$
$$= x^\lambda \int_1^\infty \frac{u^{-2\lambda}}{(u-1)^{1-\lambda}} du$$
$$= B(\lambda, \lambda) x^\lambda.$$

Therefore, we get

$$A \leq 2B(\lambda, \lambda) \int_0^\infty x^\lambda f^p(x) dx.$$

Similarly,

$$C \leq 2B(\lambda, \lambda) \int_0^\infty y^\lambda g^q(y) dy.$$

Collecting these estimates, we obtain

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda}\max\{(\frac{x}{y})^{2\lambda},(\frac{y}{x})^{2\lambda}\}}dxdy \leq 2B(\lambda,\lambda)\left(\int_0^\infty x^\lambda f^p(x)dx\right)^{1/p}\left(\int_0^\infty y^\lambda g^q(y)dy\right)^{1/q}.$$

□

Theorem 5. *Assume that* $f,g\geq 0$, $0<\lambda<2$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda(\min\{\frac{x}{y},\frac{y}{x}\})^{\frac{\lambda}{2}-1}}dxdy \leq B(\frac{\lambda}{2},\frac{\lambda}{2})\left(\int_0^\infty t^{1-\lambda}f^p(t)dt\right)^{1/p}\left(\int_0^\infty t^{1-\lambda}g^q(t)dt\right)^{1/q} \tag{13}$$

provided the integrals on the right-hand side do exist.

Proof.

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda(\min\{\frac{x}{y},\frac{y}{x}\})^{\frac{\lambda}{2}-1}}dxdy &= \int_0^\infty\int_0^\infty \frac{f(x)}{(x+y)^{\frac{\lambda}{p}}(\min\{\frac{x}{y},\frac{y}{x}\})^{(\frac{\lambda}{2}-1)\frac{1}{p}}}\\ &\quad\times\frac{g(y)}{(x+y)^{\frac{\lambda}{q}}(\min\{\frac{x}{y},\frac{y}{x}\})^{(\frac{\lambda}{2}-1)\frac{1}{q}}}dxdy\\ &\leq \left(\int_0^\infty\int_0^\infty \frac{f^p(x)}{(x+y)^\lambda(\min\{\frac{x}{y},\frac{y}{x}\})^{(\frac{\lambda}{2}-1)}}dxdy\right)^{1/p}\\ &\quad\times\left(\int_0^\infty\int_0^\infty \frac{g^q(y)}{(x+y)^\lambda(\min\{\frac{x}{y},\frac{y}{x}\})^{(\frac{\lambda}{2}-1)}}dxdy\right)^{1/q}\\ &= S^{1/p}T^{1/q}.\end{aligned}$$

We have

$$\begin{aligned}S &= \int_0^\infty\int_0^\infty \frac{f^p(x)}{(x+y)^\lambda(\min\{\frac{x}{y},\frac{y}{x}\})^{(\frac{\lambda}{2}-1)}}dxdy\\ &= \int_0^\infty f^p(x)\left(\int_0^\infty \frac{1}{(x+y)^\lambda(\min\{\frac{x}{y},\frac{y}{x}\})^{(\frac{\lambda}{2}-1)}}dy\right)dx\\ &= \int_0^\infty f^p(x)(S_1+S_2)dx, \qquad x>0,\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \int_0^x \frac{1}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)}} dy \\
&= \int_0^x \frac{1}{(x+y)^\lambda (\frac{y}{x})^{\frac{\lambda}{2}-1}} dy \\
&= x^{1-\lambda} \int_0^x \frac{(\frac{y}{x})^{2-\lambda} (\frac{y}{x})^{\frac{\lambda}{2}-1} \frac{1}{x}}{(1+\frac{y}{x})^\lambda} dy \\
&\le x^{1-\lambda} \int_0^x \frac{(\frac{y}{x})^{\frac{\lambda}{2}-1} \frac{1}{x}}{(1+\frac{y}{x})^\lambda} dy \\
&= x^{1-\lambda} \int_0^1 \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^\lambda} du \\
&= \frac{1}{2} B(\frac{\lambda}{2}, \frac{\lambda}{2}) x^{1-\lambda},
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \int_x^\infty \frac{1}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)}} dy \\
&= \int_x^\infty \frac{1}{(x+y)^\lambda (\frac{x}{y})^{\frac{\lambda}{2}-1}} dy \\
&= x^{1-\lambda} \int_x^\infty \frac{(\frac{y}{x})^{\frac{\lambda}{2}-1} \frac{1}{x}}{(1+\frac{y}{x})^\lambda} dy \\
&= \frac{1}{2} B(\frac{\lambda}{2}, \frac{\lambda}{2}) x^{1-\lambda}.
\end{aligned}$$

Therefore, we have

$$\int_0^\infty \frac{1}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)}} dy \le B(\frac{\lambda}{2}, \frac{\lambda}{2}) x^{1-\lambda},$$

which implies

$$S \le B(\frac{\lambda}{2}, \frac{\lambda}{2}) \int_0^\infty x^{1-\lambda} f^p(x) dx.$$

Similarly,

$$T \le B(\frac{\lambda}{2}, \frac{\lambda}{2}) \int_0^\infty y^{1-\lambda} g^q(y) dy.$$

Summarizing the above we obtain

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda (\min\{\frac{x}{y},\frac{y}{x}\})^{\frac{\lambda}{2}-1}} dxdy \leq B(\frac{\lambda}{2},\frac{\lambda}{2}) \left(\int_0^\infty t^{1-\lambda} f^p(t)dt \right)^{1/p} \times \left(\int_0^\infty t^{1-\lambda} g^q(t)dt \right)^{1/q}.$$

□

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