

Second order parameter–uniform convergence for a finite difference method for a singularly perturbed linear reaction–diffusion system*

PARAMASIVAM MATHIYAZHAGAN¹, VALARMATHI SIGAMANI¹ AND JOHN J. H. MILLER^{2,†}

¹ *Department of Mathematics, Bishop Heber College, Tiruchirappalli-620 017, Tamil Nadu, India*

² *Institute for Numerical Computation and Analysis, Dublin, Ireland*

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Abstract. A singularly perturbed linear system of second order ordinary differential equations of reaction-diffusion type with given boundary conditions is considered. The leading term of each equation is multiplied by a small positive parameter. These singular perturbation parameters are assumed to be distinct. The components of the solution exhibit overlapping layers. Shishkin piecewise–uniform meshes are introduced, which are used in conjunction with a classical finite difference discretisation, to construct a numerical method for solving this problem. It is proved that the numerical approximations obtained with this method are essentially second order convergent uniformly with respect to all of the parameters.

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1. Introduction

The following two-point boundary value problem is considered for the singularly perturbed linear system of second order differential equations

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x), \quad x \in (0, 1), \quad \vec{u}(0) \text{ and } \vec{u}(1) \text{ given.} \quad (1)$$

Here \vec{u} is a column n –vector, E and $A(x)$ are $n \times n$ matrices, $E = \text{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ with $0 < \varepsilon_i \leq 1$ for all $i = 1, \dots, n$. The ε_i are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \dots < \varepsilon_n.$$

Cases with some of the parameters coincident are not considered here.

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†Corresponding author. *Email addresses:* sivambhcedu@gmail.com (M. Paramasivam), valarmathi07@gmail.com (S. Valarmathi), jm@incaireland.org (J. J. H. Miller)

The problem can also be written in the operator form

$$\vec{L}\vec{u} = \vec{f}, \quad \vec{u}(0) \text{ and } \vec{u}(1) \text{ given,}$$

where the operator \vec{L} is defined by

$$\vec{L} = -ED^2 + A \quad \text{and} \quad D^2 = \frac{d^2}{dx^2}.$$

For all $x \in [0, 1]$, it is assumed that the components $a_{ij}(x)$ of $A(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{\substack{j \neq i \\ j=1}}^n |a_{ij}(x)| \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad a_{ij}(x) \leq 0 \quad \text{for } i \neq j \quad (2)$$

and, for some α ,

$$0 < \alpha < \min_{\substack{x \in [0, 1] \\ 1 \leq i \leq n}} \left(\sum_{j=1}^n a_{ij}(x) \right). \quad (3)$$

The required smoothness of the problem data is assumed wherever necessary. It is also assumed, without loss of generality, that

$$\max_{1 \leq i \leq n} \sqrt{\varepsilon_i} \leq \frac{\sqrt{\alpha}}{6}. \quad (4)$$

The norms $\|\vec{V}\| = \max_{1 \leq k \leq n} |V_k|$ for any n -vector \vec{V} , $\|y\| = \sup_{0 \leq x \leq 1} |y(x)|$ for any scalar-valued function y and $\|\vec{y}\| = \max_{1 \leq k \leq n} \|y_k\|$ for any vector-valued function \vec{y} are introduced. Throughout the paper C denotes a generic positive constant, which is independent of x and of all singular perturbation and discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense.

For a general introduction to parameter-uniform numerical methods for singular perturbation problems, see [8], [6] and [1]. Parameter-uniform numerical methods for various special cases of (1) are examined in, for example, [4] and [5]. For the problem (1) itself parameter-uniform numerical methods of first and second order are considered in [3]. However, the present paper differs from [3] in two important ways. First of all, the meshes, and hence the numerical methods, used are different from those in [3]; the transition points between meshes of differing resolution are defined in a similar but different manner. The piecewise-uniform Shishkin meshes $M_{\vec{b}}$ in the present paper have the elegant property that they reduce to uniform meshes whenever $\vec{b} = \vec{0}$. Secondly, the proofs given here do not require the use of Green's function techniques, as is the case in [3]. The significance of this is that it is more likely that such techniques can be extended in future to problems in higher dimensions and to nonlinear problems, than is the case for proofs depending on Green's functions. It is also satisfying to demonstrate that the methods of proof pioneered by G. I. Shishkin can be extended successfully to systems of this kind.

The plan of the paper is as follows. In the next section both standard and novel bounds on the smooth and singular components of the exact solution are obtained. The sharp estimates for the singular component in Lemma 7 are proved by mathematical induction, while interesting orderings of the points $x_{i,j}^{(s)}$ are established in Lemma 5. In Section 4 piecewise-uniform Shishkin meshes are introduced, the discrete problem is defined and the discrete maximum principle and discrete stability properties are established. In Section 6 expressions for the local truncation errors and some standard estimates are stated. In Section 7 parameter-uniform estimates for the local truncation error of the smooth and singular components are obtained in a sequence of lemmas and theorems. The section culminates with the statement and proof of the essentially second order parameter-uniform error estimate.

2. Standard analytical results

The operator \vec{L} satisfies the following maximum principle

Lemma 1. *Let $A(x)$ satisfy (2) and (3). Let $\vec{\psi}$ be any function in the domain of \vec{L} such that $\vec{\psi}(0) \geq \vec{0}$ and $\vec{\psi}(1) \geq \vec{0}$. Then $\vec{L}\vec{\psi}(x) \geq \vec{0}$ for all $x \in (0, 1)$ implies that $\vec{\psi}(x) \geq \vec{0}$ for all $x \in [0, 1]$.*

Proof. Let i^*, x^* be such that $\psi_{i^*}(x^*) = \min_{i,x} \psi_i(x)$ and assume that the lemma is false. Then $\psi_{i^*}(x^*) < 0$. From the hypotheses we have $x^* \notin \{0, 1\}$ and $\psi_{i^*}''(x^*) \geq 0$. Thus

$$(\vec{L}\vec{\psi}(x^*))_{i^*} = -\varepsilon_{i^*}\psi_{i^*}''(x^*) + \sum_{j=1}^n a_{i^*,j}(x^*)\psi_j(x^*) < 0,$$

which contradicts the assumption and proves the result for \vec{L} . □

Let $\tilde{A}(x)$ be any principal sub-matrix of $A(x)$ and $\tilde{\vec{L}}$ the corresponding operator. To see that any $\tilde{\vec{L}}$ satisfies the same maximum principle as \vec{L} , it suffices to observe that the elements of $\tilde{A}(x)$ satisfy *a fortiori* the same inequalities as those of $A(x)$.

Lemma 2. *Let $A(x)$ satisfy (2) and (3). If $\vec{\psi}$ is any function in the domain of \vec{L} , then for each i , $1 \leq i \leq n$,*

$$|\psi_i(x)| \leq \max \left\{ \|\vec{\psi}(0)\|, \|\vec{\psi}(1)\|, \frac{1}{\alpha} \|\vec{L}\vec{\psi}\| \right\}, \quad x \in [0, 1].$$

Proof. Define the two functions

$$\vec{\theta}^\pm(x) = \max \left\{ \|\vec{\psi}(0)\|, \|\vec{\psi}(1)\|, \frac{1}{\alpha} \|\vec{L}\vec{\psi}\| \right\} \vec{e} \pm \vec{\psi}(x)$$

where $\vec{e} = (1, \dots, 1)^T$ is the unit column vector. Using the properties of A it is not hard to verify that $\vec{\theta}^\pm(0) \geq \vec{0}$, $\vec{\theta}^\pm(1) \geq \vec{0}$ and $\vec{L}\vec{\theta}^\pm(x) \geq \vec{0}$. It follows from Lemma 1 that $\vec{\theta}^\pm(x) \geq \vec{0}$ for all $x \in [0, 1]$. □

Standard estimates of the exact solution and its derivatives are contained in the following lemma.

Lemma 3. *Let $A(x)$ satisfy (2) and (3) and let \vec{u} be the exact solution of (1). Then, for each $i = 1 \dots n$, $x \in [0, 1]$ and $k = 0, 1, 2$,*

$$|u_i^{(k)}(x)| \leq C\varepsilon_i^{-\frac{k}{2}} (||\vec{u}(0)|| + ||\vec{u}(1)|| + ||\vec{f}'||)$$

$$|u_i^{(3)}(x)| \leq C\varepsilon_1^{-\frac{1}{2}}\varepsilon_i^{-1} (||\vec{u}(0)|| + ||\vec{u}(1)|| + ||\vec{f}'|| + \sqrt{\varepsilon_1}||\vec{f}''||)$$

and

$$|u_i^{(4)}(x)| \leq C\varepsilon_1^{-1}\varepsilon_i^{-1} (||\vec{u}(0)|| + ||\vec{u}(1)|| + ||\vec{f}'|| + \varepsilon_1||\vec{f}''||).$$

Proof. The bound on \vec{u} is an immediate consequence of Lemma 2 and the differential equation.

To bound $u_i'(x)$, for all i and any x , consider an interval $N_x = [a, a + \sqrt{\varepsilon_i}]$ such that $x \in N_x$. Then, by the mean value theorem, for some $y \in N_x$,

$$u_i'(y) = \frac{u_i(a + \sqrt{\varepsilon_i}) - u_i(a)}{\sqrt{\varepsilon_i}}$$

and it follows that

$$|u_i'(y)| \leq 2\varepsilon_i^{-\frac{1}{2}} ||u_i||.$$

Now

$$\vec{u}'(x) = \vec{u}'(y) + \int_y^x \vec{u}''(s)ds = \vec{u}'(y) + E^{-1} \int_y^x (-\vec{f}'(s) + A(s)\vec{u}(s))ds$$

and so

$$|u_i'(x)| \leq |u_i'(y)| + C\varepsilon_i^{-1} (||f_i|| + ||\vec{u}||) \int_y^x ds \leq C\varepsilon_i^{-\frac{1}{2}} (||f_i|| + ||\vec{u}||)$$

from which the required bound follows.

Rewriting and differentiating the differential equation gives $\vec{u}'' = E^{-1}(A\vec{u} - \vec{f})$, $\vec{u}^{(3)} = E^{-1}(A\vec{u}' + A'\vec{u} - \vec{f}')$, $\vec{u}^{(4)} = E^{-1}(A\vec{u}'' + 2A'\vec{u}' + A''\vec{u} - \vec{f}'')$ and the bounds on u_i'' , $u_i^{(3)}$, $u_i^{(4)}$ follow. □

The reduced solution \vec{u}_0 of (1) is the solution of the reduced equation $A\vec{u}_0 = \vec{f}$. The Shishkin decomposition of the exact solution \vec{u} of (1) is $\vec{u} = \vec{v} + \vec{w}$ where the smooth component \vec{v} is the solution of $\vec{L}\vec{v} = \vec{f}$ with $\vec{v}(0) = \vec{u}_0(0)$ and $\vec{v}(1) = \vec{u}_0(1)$ and the singular component \vec{w} is the solution of $\vec{L}\vec{w} = \vec{0}$ with $\vec{w}(0) = \vec{u}(0) - \vec{v}(0)$ and $\vec{w}(1) = \vec{u}(1) - \vec{v}(1)$. For convenience the left and right boundary layers of \vec{w} are separated using the further decomposition $\vec{w} = \vec{w}^l + \vec{w}^r$ where $\vec{L}\vec{w}^l = \vec{0}$, $\vec{w}^l(0) = \vec{u}(0) - \vec{v}(0)$, $\vec{w}^l(1) = \vec{0}$ and $\vec{L}\vec{w}^r = \vec{0}$, $\vec{w}^r(0) = \vec{0}$, $\vec{w}^r(1) = \vec{u}(1) - \vec{v}(1)$.

Bounds on the smooth component and its derivatives are contained in

Lemma 4. *Let $A(x)$ satisfy (2) and (3). Then the smooth component \vec{v} and its derivatives satisfy, for all $x \in [0, 1]$, $i = 1, \dots, n$ and $k = 0, \dots, 4$,*

$$|v_i^{(k)}(x)| \leq C(1 + \varepsilon_i^{1-\frac{k}{2}}).$$

Proof. The bound on \vec{v} is an immediate consequence of the defining equations for \vec{v} and Lemma 2.

The bounds on \vec{v}' and \vec{v}'' are found as follows. Differentiating twice the equation for \vec{v} , it is not hard to see that \vec{v}'' satisfies

$$\vec{L}\vec{v}'' = \vec{g} \text{ where } \vec{g} = \vec{f}'' - A''\vec{v} - 2A'\vec{v}'. \tag{5}$$

Also the defining equations for \vec{v} yield at $x = 0, x = 1$

$$\vec{v}''(0) = \vec{0}, \quad \vec{v}''(1) = \vec{0}. \tag{6}$$

Applying Lemma 2 to \vec{v}'' then gives

$$\|\vec{v}''\| \leq C(1 + \|\vec{v}'\|). \tag{7}$$

Choosing i^*, x^* such that $1 \leq i^* \leq n, x^* \in (0, 1)$ and

$$v'_{i^*}(x^*) = \|\vec{v}'\| \tag{8}$$

and using a Taylor expansion it follows that, for any $y \in [0, 1 - x^*]$ and some $\eta, x^* < \eta < x^* + y$,

$$v_{i^*}(x^* + y) = v_{i^*}(x^*) + y v'_{i^*}(x^*) + \frac{y^2}{2} v''_{i^*}(\eta). \tag{9}$$

Rearranging (9) yields

$$v'_{i^*}(x^*) = \frac{v_{i^*}(x^* + y) - v_{i^*}(x^*)}{y} - \frac{y}{2} v''_{i^*}(\eta) \tag{10}$$

and so, from (8) and (10),

$$\|\vec{v}'\| \leq \frac{2}{y} \|\vec{v}\| + \frac{y}{2} \|\vec{v}''\|. \tag{11}$$

Using (11), (7) and the bound on \vec{v} yields

$$(1 - \frac{Cy}{2}) \|\vec{v}''\| \leq C(1 + \frac{2}{y}). \tag{12}$$

Choosing $y = \min(\frac{1}{C}, 1 - x^*)$, (12) then gives $\|\vec{v}''\| \leq C$ and (11) gives $\|\vec{v}'\| \leq C$ as required. The bounds on $\vec{v}^{(3)}, \vec{v}^{(4)}$ are obtained by a similar argument. \square

3. Improved estimates

The layer functions $B_i^l, B_i^r, B_i, i = 1, \dots, n,$ associated with the solution \vec{u} , are defined on $[0, 1]$ by

$$B_i^l(x) = e^{-x\sqrt{\alpha/\varepsilon_i}}, B_i^r(x) = B_i^l(1 - x), B_i(x) = B_i^l(x) + B_i^r(x).$$

The following elementary properties of these layer functions, for all $1 \leq i < j \leq n$ and $0 \leq x < y \leq 1$, should be noted:

- (a) $B_i^l(x) < B_j^l(x), B_i^l(x) > B_i^l(y), 0 < B_i^l(x) \leq 1.$
- (b) $B_i^r(x) < B_j^r(x), B_i^r(x) < B_i^r(y), 0 < B_i^r(x) \leq 1.$
- (c) $B_i(x)$ is monotone decreasing (increasing) for increasing $x \in [0, \frac{1}{2}][(\frac{1}{2}, 1]).$
- (d) $B_i(x) \leq 2B_i^l(x)$ for $x \in [0, \frac{1}{2}].$

Definition 1. For $B_i^l, B_j^l,$ each $i, j, 1 \leq i \neq j \leq n$ and each $s, s > 0,$ the point $x_{i,j}^{(s)}$ is defined by

$$\frac{B_i^l(x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^l(x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{13}$$

It is remarked that

$$\frac{B_i^r(1 - x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^r(1 - x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{14}$$

In the next lemma the existence and uniqueness of the points $x_{i,j}^{(s)}$ are shown. Various properties are also established.

Lemma 5. For all i, j such that $1 \leq i < j \leq n$ and $0 < s \leq 3/2,$ the points $x_{i,j}^{(s)}$ exist, are uniquely defined and satisfy the following inequalities

$$\frac{B_i^l(x)}{\varepsilon_i^s} > \frac{B_j^l(x)}{\varepsilon_j^s}, x \in [0, x_{i,j}^{(s)}), \frac{B_i^l(x)}{\varepsilon_i^s} < \frac{B_j^l(x)}{\varepsilon_j^s}, x \in (x_{i,j}^{(s)}, 1]. \tag{15}$$

Moreover,

$$x_{i,j}^{(s)} < x_{i+1,j}^{(s)}, \text{ if } i + 1 < j \text{ and } x_{i,j}^{(s)} < x_{i,j+1}^{(s)}, \text{ if } i < j. \tag{16}$$

Also

$$x_{i,j}^{(s)} < 2s \frac{\sqrt{\varepsilon_j}}{\sqrt{\alpha}} \text{ and } x_{i,j}^{(s)} \in (0, \frac{1}{2}) \text{ if } i < j. \tag{17}$$

Analogous results hold for the B_i^r, B_j^r and the points $1 - x_{i,j}^{(s)}.$

Proof. Existence, uniqueness and (15) follow from the observation that the ratio of the two sides of (13), namely

$$\frac{B_i^l(x)}{\varepsilon_i^s} \frac{\varepsilon_j^s}{B_j^l(x)} = \frac{\varepsilon_j^s}{\varepsilon_i^s} \exp(-\sqrt{\alpha}x(\frac{1}{\sqrt{\varepsilon_i}} - \frac{1}{\sqrt{\varepsilon_j}})),$$

is monotonically decreasing from the value $\frac{\varepsilon_j^s}{\varepsilon_i^s} > 1$ as x increases from 0.

The point $x_{i,j}^{(s)}$ is the unique point x at which this ratio has the value 1. Rearranging (13), and using the inequality $\ln x < x - 1$ for all $x > 1$, gives

$$x_{i,j}^{(s)} = 2s \left[\frac{\ln(\frac{1}{\sqrt{\varepsilon_i}}) - \ln(\frac{1}{\sqrt{\varepsilon_j}})}{\sqrt{\alpha}(\frac{1}{\sqrt{\varepsilon_i}} - \frac{1}{\sqrt{\varepsilon_j}})} \right] = \frac{2s \ln(\frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}})}{\sqrt{\alpha}(\frac{1}{\sqrt{\varepsilon_i}} - \frac{1}{\sqrt{\varepsilon_j}})} < 2s \frac{\sqrt{\varepsilon_j}}{\sqrt{\alpha}}, \tag{18}$$

which is the first part of (17). The second part follows immediately from this and (4).

To prove (16), writing $\sqrt{\varepsilon_k} = \exp(-p_k)$, for some $p_k > 0$ and all k , it follows that

$$x_{i,j}^{(s)} = \frac{2s(p_i - p_j)}{\sqrt{\alpha}(\exp p_i - \exp p_j)}.$$

The inequality $x_{i,j}^{(s)} < x_{i+1,j}^{(s)}$ is equivalent to

$$\frac{p_i - p_j}{\exp p_i - \exp p_j} < \frac{p_{i+1} - p_j}{\exp p_{i+1} - \exp p_j},$$

which can be written in the form

$$(p_{i+1} - p_j) \exp(p_i - p_j) + (p_i - p_{i+1}) - (p_i - p_j) \exp(p_{i+1} - p_j) > 0.$$

With $a = p_i - p_j$ and $b = p_{i+1} - p_j$ it is not hard to see that $a > b > 0$ and $a - b = p_i - p_{i+1}$. Moreover, the previous inequality is then equivalent to

$$\frac{\exp a - 1}{a} > \frac{\exp b - 1}{b},$$

which is true because $a > b$ and proves the first part of (16). The second part is proved by a similar argument.

The analogous results for the B_i^r, B_j^r and the points $1 - x_{i,j}^{(s)}$ are proved by a similar argument. □

In the following lemma sharper estimates of the smooth component are presented.

Lemma 6. *Let $A(x)$ satisfy (2) and (3). Then the smooth component \vec{v} of the solution \vec{u} of (1) satisfies for $i = 1, \dots, n, k = 0, 1, 2, 3$ and $x \in \bar{\Omega}$*

$$|v_i^{(k)}(x)| \leq C \left(1 + \sum_{q=i}^n \frac{B_q(x)}{\varepsilon_q^{\frac{k}{2}-1}} \right).$$

Proof. Define a barrier function

$$\vec{\psi}^\pm(x) = C[1 + B_n(x)]\vec{e} \pm \vec{v}^{(k)}(x), \quad k = 0, 1, 2 \quad \text{and} \quad x \in \bar{\Omega}.$$

Using Lemma 1, we find that $\vec{L}\vec{\psi}^\pm(x) \geq \vec{0}$ and $\vec{\psi}^\pm(0) \geq \vec{0}$, $\vec{\psi}^\pm(1) \geq \vec{0}$ for proper choices of the constant C .

Thus using Lemma 4 we conclude that for $k = 0, 1, 2$,

$$|v_i^{(k)}(x)| \leq C[1 + B_n(x)], \quad x \in \bar{\Omega}. \tag{19}$$

Consider the system of equations (5), (6) satisfied by \vec{v}'' and note that $\|\vec{g}'\| \leq C$ from Lemma 4.

For convenience let \vec{p} denote \vec{v}'' then

$$\vec{L}\vec{p} = \vec{g}, \quad \vec{p}(0) = \vec{0}, \quad \vec{p}(1) = \vec{0}. \tag{20}$$

Let \vec{q} and \vec{r} be the smooth and singular components of \vec{p} given by

$$\vec{L}\vec{q} = \vec{g}, \quad \vec{q}(0) = A(0)^{-1}\vec{g}(0), \quad \vec{q}(1) = A(1)^{-1}\vec{g}(1)$$

and

$$\vec{L}\vec{r} = \vec{0}, \quad \vec{r}(0) = -\vec{q}(0), \quad \vec{r}(1) = -\vec{q}(1).$$

Using Lemmas 4 and 7 we have, for $i = 1, \dots, n$ and $x \in \bar{\Omega}$,

$$\begin{aligned} |q'_i(x)| &\leq C, \\ |r'_i(x)| &\leq C \left[\frac{B_i(x)}{\sqrt{\varepsilon_i}} + \dots + \frac{B_n(x)}{\sqrt{\varepsilon_n}} \right]. \end{aligned}$$

Hence, for $x \in \bar{\Omega}$ and $i = 1, \dots, n$,

$$|v_i'''(x)| = |p'_i(x)| \leq C \left[1 + \frac{B_i(x)}{\sqrt{\varepsilon_i}} + \dots + \frac{B_n(x)}{\sqrt{\varepsilon_n}} \right]. \tag{21}$$

From (19) and (21), we find that for $k = 0, 1, 2, 3$ and $x \in \bar{\Omega}$,

$$|v_i^{(k)}(x)| \leq C \left[1 + \varepsilon_i^{1-\frac{k}{2}} B_i(x) + \dots + \varepsilon_n^{1-\frac{k}{2}} B_n(x) \right]. \quad \square$$

Remark 1. *It is interesting to note that the above estimate reduces to the estimate of the smooth component of the solution of the scalar problem given in [6] when $n = 1$.*

Bounds on the singular components \vec{w}^l, \vec{w}^r of \vec{u} and their derivatives are contained in

Lemma 7. *Let $A(x)$ satisfy (2) and (3). Then there exists a constant C , such that for each $x \in [0, 1]$ and $i = 1, \dots, n$,*

$$\begin{aligned} |w_i^l(x)| &\leq C B_n^l(x), \quad |w_i^{l'}(x)| \leq C \sum_{q=i}^n \frac{B_q^l(x)}{\sqrt{\varepsilon_q}}, \\ |w_i^{l''}(x)| &\leq C \sum_{q=i}^n \frac{B_q^l(x)}{\varepsilon_q}, \quad |w_i^{l,(3)}(x)| \leq C \sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q^{3/2}}, \\ |\varepsilon_i w_i^{l,(4)}(x)| &\leq C \sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q}. \end{aligned}$$

Analogous results hold for w_i^r and their derivatives.

Proof. First we obtain the bound on \vec{w}^l . We define the two functions $\vec{\theta}^\pm = CB_n^l \vec{e} \pm \vec{w}^l$. Then clearly $\vec{\theta}^\pm(0) \geq \vec{0}$, $\vec{\theta}^\pm(1) \geq \vec{0}$ and $L\vec{\theta}^\pm = CL(B_n^l \vec{e})$. Then, for $i = 1, \dots, n$, $(L\vec{\theta}^\pm)_i = C(\sum_{j=1}^n a_{i,j} - \alpha_{\varepsilon_n^{\pm}})B_n^l > 0$. By Lemma 1, $\vec{\theta}^\pm \geq \vec{0}$, which leads to the required bound on \vec{w}^l .

Assuming, for the moment, the bounds on the first and second derivatives $w_i^{l'}$ and $w_i^{l''}$, the system of differential equations satisfied by \vec{w}^l is differentiated twice to get

$$-E\vec{w}^{l,(4)} + A\vec{w}^{l,''} + 2A'\vec{w}^{l'} + A''\vec{w}^l = \vec{0}.$$

The required bounds on the $w_i^{l,(4)}$ follow from those on w_i^l , $w_i^{l'}$ and $w_i^{l''}$. It remains therefore to establish the bounds on $w_i^{l'}, w_i^{l''}$ and $w_i^{l'''}$, for which the following mathematical induction argument is used. It is assumed that the bounds hold for all systems up to order $n - 1$. It is then shown that the bounds hold for order n . The induction argument is completed by observing that the bounds for the scalar case $n = 1$ are proved in [6].

It is now shown that under the induction hypothesis the required bounds hold for $w_i^{l'}, w_i^{l''}$ and $w_i^{l'''}$. The bounds when $i = n$ are established first. The differential equation for w_n^l gives $\varepsilon_n w_n^{l,''} = (A\vec{w}^l)_n$ and the required bound on $w_n^{l,''}$ follows at once from that for \vec{w}^l . For $w_n^{l'}$ it is seen from the bounds in Lemma 3, applied to the system satisfied by \vec{w}^l , that $|w_i^{l'}(x)| \leq C\varepsilon_i^{-\frac{1}{2}}$. In particular, $|w_n^{l'}(0)| \leq C\varepsilon_n^{-\frac{1}{2}}$ and $|w_n^{l'}(1)| \leq C\varepsilon_n^{-\frac{1}{2}}$. It is also not hard to verify that $\vec{L}\vec{w}^{l'} = -A'\vec{w}^l$. Using these results, the inequalities $\varepsilon_i < \varepsilon_n$, $i < n$, and the properties of A , it follows that the two barrier functions $\vec{\theta}^\pm = CE^{-\frac{1}{2}}B_n^l \vec{e} \pm \vec{w}^{l'}$ satisfy the inequalities $\vec{\theta}^\pm(0) \geq \vec{0}$, $\vec{\theta}^\pm(1) \geq \vec{0}$ and $\vec{L}\vec{\theta}^\pm \geq \vec{0}$. It follows from Lemma 1 that $\vec{\theta}^\pm \geq \vec{0}$ and in particular that its n^{th} component satisfies $|w_n^{l'}(x)| \leq C\varepsilon_n^{-\frac{1}{2}}B_n^l(x)$ as required.

Now, consider

$$-\varepsilon_n w_n^{l,''}(x) + a_{n1}(x)w_1^l(x) + a_{n2}(x)w_2^l(x) + \dots + a_{nn}(x)w_n^l(x) = f_n(x). \quad (22)$$

Differentiating (22) once, we get

$$\begin{aligned} -\varepsilon_n w_n^{l,(3)}(x) &= f_n'(x) - \sum_{j=1}^n (a_{nj}(x)w_j^l(x))' \\ |w_n^{l,(3)}(x)| &\leq C\varepsilon_n^{-1} \left[1 + \sum_{j=1}^n |w_j^{l'}(x)| \right] \\ &\leq C\varepsilon_n^{-1} \left[\frac{B_1^l(x)}{\sqrt{\varepsilon_1}} + \dots + \frac{B_n^l(x)}{\sqrt{\varepsilon_n}} \right] \\ &\leq C \sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q^{3/2}}. \end{aligned}$$

To bound $w_i^{l'}$, $w_i^{l''}$ and $w_i^{l,(3)}$ for $1 \leq i \leq n - 1$ introduce $\vec{w}^l = (w_1^l, \dots, w_{n-1}^l)$.

Then, taking the first $n - 1$ equations satisfied by \vec{w}^l , it follows that

$$-\tilde{E}\vec{w}^{l,\prime\prime} + \tilde{A}\vec{w}^l = \vec{g},$$

where \tilde{E} , \tilde{A} is the matrix obtained by deleting the last row and column from E , A , respectively, and the components of \vec{g} are $g_i = -a_{i,n}w_n^l$ for $1 \leq i \leq n - 1$. Using the bounds already obtained for w_n^l , $w_n^{l,\prime}$, $w_n^{l,\prime\prime}$ and $w_n^{l,\prime\prime\prime}$, it is seen that \vec{g} is bounded by $CB_n^l(x)$, \vec{g}' by $C\frac{B_n^l(x)}{\sqrt{\varepsilon_n}}$, \vec{g}'' by $C\frac{B_n^l(x)}{\varepsilon_n}$ and \vec{g}''' by $C\sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q^{3/2}}$. The boundary

conditions for \vec{w}^l are $\vec{w}^l(0) = \vec{u}(0) - \vec{u}^0(0)$, $\vec{w}^l(1) = \vec{0}$, where \vec{u}^0 is the solution of the reduced problem $\vec{u}^0 = A^{-1}\vec{f}$, and are bounded by $C(\|\vec{u}(0)\| + \|\vec{f}(0)\|)$ and $C(\|\vec{u}(1)\| + \|\vec{f}(1)\|)$. Now decompose \vec{w}^l into smooth and singular components to get

$$\vec{w}^l = \vec{q} + \vec{r}, \quad \vec{w}^{l,\prime} = \vec{q}' + \vec{r}'.$$

Applying Lemma 4 to \vec{q} and using the bounds on the inhomogeneous term \vec{g} and its derivatives \vec{g}' , \vec{g}'' and $\vec{g}^{(3)}$ it follows that

$$|\vec{q}'(x)| \leq C\frac{B_n^l(x)}{\sqrt{\varepsilon_n}}, \quad |\vec{q}''(x)| \leq C\frac{B_n^l(x)}{\varepsilon_n}$$

and

$$|\vec{q}'''(x)| \leq C\sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q^{3/2}}.$$

Using mathematical induction, assume that the result holds for all systems with $n - 1$ equations. Then Lemma 7 applies to \vec{r} and so, for $i = 1, \dots, n - 1$,

$$|r_i'(x)| \leq C\sum_{q=i}^{n-1} \frac{B_q^l(x)}{\sqrt{\varepsilon_q}}, \quad |r_i''(x)| \leq C\sum_{q=i}^{n-1} \frac{B_q^l(x)}{\varepsilon_q}, \quad |r_i'''(x)| \leq C\sum_{q=1}^{n-1} \frac{B_q^l(x)}{\varepsilon_q^{3/2}}.$$

Combining the bounds for the derivatives of q_i and r_i , it follows that

$$|w_i^{l,\prime}(x)| \leq C\sum_{q=i}^n \frac{B_q^l(x)}{\sqrt{\varepsilon_q}}, \quad |w_i^{l,\prime\prime}(x)| \leq C\sum_{q=i}^n \frac{B_q^l(x)}{\varepsilon_q}, \quad |w_i^{l,\prime\prime\prime}(x)| \leq C\sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q^{3/2}}.$$

Thus, the bounds on $w_i^{l,\prime}$, $w_i^{l,\prime\prime}$ and $w_i^{l,\prime\prime\prime}$ hold for a system with n equations, as required. A similar proof of the analogous results for the right boundary layer functions holds. □

4. The Shishkin mesh

A piecewise uniform mesh with N mesh-intervals and mesh-points $\{x_i\}_{i=0}^N$ is now constructed by dividing the interval $[0, 1]$ into $2n + 1$ sub-intervals as follows

$$[0, \tau_1] \cup \dots \cup (\tau_{n-1}, \tau_n] \cup (\tau_n, 1 - \tau_n] \cup (1 - \tau_n, 1 - \tau_{n-1}] \cup \dots \cup (1 - \tau_1, 1].$$

The n parameters τ_k , which determine the points separating the uniform meshes, are defined by

$$\tau_n = \min \left\{ \frac{1}{4}, 2 \frac{\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N \right\} \tag{23}$$

and for $k = 1, \dots, n - 1$

$$\tau_k = \min \left\{ \frac{\tau_{k+1}}{2}, 2 \frac{\sqrt{\varepsilon_k}}{\sqrt{\alpha}} \ln N \right\}. \tag{24}$$

Clearly

$$0 < \tau_1 < \dots < \tau_n \leq \frac{1}{4}, \quad \frac{3}{4} \leq 1 - \tau_n < \dots < 1 - \tau_1 < 1.$$

Then, on the sub-interval $(\tau_n, 1 - \tau_n]$ a uniform mesh with $\frac{N}{2}$ mesh-intervals is placed, on each of the sub-intervals $(\tau_k, \tau_{k+1}]$ and $(1 - \tau_{k+1}, 1 - \tau_k]$, $k = 1, \dots, n - 1$, a uniform mesh of $\frac{N}{2^{n-k+2}}$ mesh-intervals is placed and on both of the sub-intervals $[0, \tau_1]$ and $(1 - \tau_1, 1]$ a uniform mesh of $\frac{N}{2^{n+1}}$ mesh-intervals is placed. In practice it is convenient to take

$$N = 2^{n+p+1} \tag{25}$$

for some natural number p . It follows that in the sub-interval $[\tau_{k-1}, \tau_k]$ there are $N/2^{n-k+3} = 2^{k+p-2}$ mesh-intervals. This construction leads to a class of 2^n piecewise uniform Shishkin meshes $M_{\vec{b}}$, where \vec{b} denotes an n -vector with $b_i = 0$ if $\tau_i = \frac{\tau_{i+1}}{2}$ and $b_i = 1$ otherwise. From the above construction it clear that the only points at which the meshsize can change are in a subset $J_{\vec{b}}$ of the set of transition points $T_{\vec{b}} = \{\tau_k\}_{k=1}^n \cup \{1 - \tau_k\}_{k=1}^n$. It is not hard to see that the change in the meshsize at each point τ_k is $2^{n-k+3}(d_k - d_{k-1})$, where $d_k = \frac{\tau_{k+1}}{2} - \tau_k$ for $1 \leq k \leq n$, with the conventions $d_0 = 0$, $\tau_{n+1} = 1/2$. Notice that $d_k \geq 0$ and that $b_k = 0$ if and only if $d_k = 0$. It follows that $M_{\vec{b}}$ is a classical uniform mesh when $\vec{b} = \vec{0}$.

The following notation is now introduced: $H_j = x_{j+1} - x_j$, $h_j = x_j - x_{j-1}$, $\delta_j = x_{j+1} - x_{j-1}$, $J_{\vec{b}} = \{x_j : H_j - h_j \neq 0\}$. Clearly, $J_{\vec{b}}$ is the set of points at which the meshsize changes and $J_{\vec{b}} \subset T_{\vec{b}}$. Note that, in general, $J_{\vec{b}}$ is a proper subset of $T_{\vec{b}}$. Moreover, if $b_k = 0$ then $H_k \leq h_k$ and if $b_k = b_{k-1} = 0$ then $H_k = h_k$. In the latter case, it follows that the meshsize does not change at τ_k or $1 - \tau_k$.

It is not hard to see also that

$$\tau_k \leq C \sqrt{\varepsilon_k} \ln N, \quad 1 \leq k \leq n, \tag{26}$$

$$h_k = 2^{n-k+3} N^{-1} (\tau_k - \tau_{k-1}), \quad H_k = 2^{n-k+2} N^{-1} (\tau_{k+1} - \tau_k), \tag{27}$$

$$\delta_j = H_j + h_j \leq C \max\{H_j, h_j\}, \quad 1 \leq j \leq N - 1, \tag{28}$$

$$\tau_k = 2^{-(j-k+1)} \tau_{j+1} \text{ when } b_k = \dots = b_j = 0, \quad 1 \leq k < j \leq n \tag{29}$$

and

$$B_k^l(\tau_k) = B_k^r(1 - \tau_k) = N^{-2} \text{ when } b_k = 1. \tag{30}$$

The geometrical results in the following lemma are used later.

Lemma 8. *Assume that $b_k = 1$. Then the following inequalities hold*

$$x_{k-1,k}^{(s)} \leq \tau_k - h_k \text{ for } 1 < k \leq n, \tag{31}$$

$$\frac{B_i^l(\tau_k)}{\sqrt{\varepsilon_i}} < \frac{1}{\sqrt{\varepsilon_k}} \text{ for } 1 \leq i, k \leq n, \tag{32}$$

$$B_q^l(\tau_k - h_k) \leq CB_q^l(\tau_k) \text{ for } 1 \leq k \leq q \leq n. \tag{33}$$

Proof. To verify (31) note that by Lemma 5

$$x_{k-1,k}^{(s)} < 2s \frac{\sqrt{\varepsilon_k}}{\sqrt{\alpha}} = \frac{s\tau_k}{\ln N} = \frac{s\tau_k}{(n+p+1)\ln 2} \leq \frac{\tau_k}{2}.$$

Also,

$$h_k = \frac{2^{n-k+3}(\tau_k - \tau_{k-1})}{N} = 2^{2-k-p}(\tau_k - \tau_{k-1}) \leq \frac{\tau_k - \tau_{k-1}}{2} < \frac{\tau_k}{2}.$$

It follows that $x_{k-1,k}^{(s)} + h_k \leq \tau_k$ as required.

To verify (32) note that if $i \geq k$ the result is trivial. On the other hand, if $i < k$, by (31) and Lemma 5,

$$\frac{B_i^l(\tau_k)}{\sqrt{\varepsilon_i}} \leq \frac{B_i^l(x_{i,k}^{(1)})}{\sqrt{\varepsilon_i}} < \frac{B_k^l(x_{i,k}^{(1)})}{\sqrt{\varepsilon_k}} \leq \frac{1}{\sqrt{\varepsilon_k}}.$$

Finally, to verify (33) note that

$$h_k = (\tau_k - \tau_{k-1})2^{n-k+3}N^{-1} \leq \tau_k 2^{n-k+3}N^{-1} = \frac{\sqrt{\varepsilon_k}}{\sqrt{\alpha}} 2^{n-k+4}N^{-1} \ln N$$

and

$$e^{2^{n-k+4}N^{-1} \ln N} = (N^{\frac{1}{N}})^{2^{n-k+4}} \leq C,$$

so

$$\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_q}} h_k \leq \frac{\sqrt{\varepsilon_k}}{\sqrt{\varepsilon_q}} 2^{n-k+4}N^{-1} \ln N \leq 2^{n-k+4}N^{-1} \ln N \leq C$$

since $k \leq q$. It follows that

$$B_q^l(\tau_k - h_k) = B_q^l(\tau_k) e^{\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_q}} h_k} \leq CB_q^l(\tau_k)$$

as required. □

5. The discrete problem

In this section a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (1), which is shown later to be

essentially second order parameter-uniform convergent. In the scalar case, when $n = 1$, this result is well known. In [2] a similar numerical analysis is done for a first order system of n equations with coincident perturbation parameters. For the general case considered here, the error analysis is based on an extension of the techniques employed in [7]. It is assumed henceforth that the problem data satisfy whatever smoothness conditions are required.

The discrete two-point boundary value problem is now defined on any mesh $M_{\bar{h}}$ by the finite difference method

$$-E\delta^2\vec{U}(x) + A(x)\vec{U}(x) = \vec{f}(x), \quad \vec{U}(0) = \vec{u}(0), \quad \vec{U}(1) = \vec{u}(1). \tag{34}$$

This is used to compute numerical approximations to the exact solution of (1). Note that (34) can also be written in the operator form

$$\vec{L}^N\vec{U} = \vec{f}, \quad \vec{U}(0) = \vec{u}(0), \quad \vec{U}(1) = \vec{u}(1)$$

where

$$\vec{L}^N = -E\delta^2 + A$$

and δ^2 , D^+ and D^- are the difference operators

$$\delta^2\vec{U}(x_j) = \frac{D^+\vec{U}(x_j) - D^-\vec{U}(x_j)}{\bar{h}_j},$$

$$D^+\vec{U}(x_j) = \frac{\vec{U}(x_{j+1}) - \vec{U}(x_j)}{h_{j+1}}$$

and

$$D^-\vec{U}(x_j) = \frac{\vec{U}(x_j) - \vec{U}(x_{j-1}))}{h_j}$$

with $\bar{h}_j = \frac{h_j + h_{j+1}}{2}$, $h_j = x_j - x_{j-1}$.

The following discrete results are analogous to those for the continuous case.

Lemma 9. *Let $A(x)$ satisfy (2) and (3). Then, for any mesh function $\vec{\Psi}$, the inequalities $\vec{\Psi}(0) \geq \vec{0}$, $\vec{\Psi}(1) \geq \vec{0}$ and $\vec{L}^N\vec{\Psi}(x_j) \geq \vec{0}$ for $1 \leq j \leq N - 1$ imply that $\vec{\Psi}(x_j) \geq \vec{0}$ for $0 \leq j \leq N$.*

Proof. Let i^*, j^* be such that $\Psi_{i^*}(x_{j^*}) = \min_{i,j} \Psi_i(x_j)$ and assume that the lemma is false. Then $\Psi_{i^*}(x_{j^*}) < 0$. From the hypotheses we have $j^* \neq 0, N$ and $\Psi_{i^*}(x_{j^*}) - \Psi_{i^*}(x_{j^*-1}) \leq 0$, $\Psi_{i^*}(x_{j^*+1}) - \Psi_{i^*}(x_{j^*}) \geq 0$, so $\delta^2\Psi_{i^*}(x_{j^*}) > 0$. It follows that

$$\left(\vec{L}^N\vec{\Psi}(x_{j^*})\right)_{i^*} = -\varepsilon_{i^*}\delta^2\Psi_{i^*}(x_{j^*}) + \sum_{k=1}^n a_{i^*,k}(x_{j^*})\Psi_k(x_{j^*}) < 0,$$

which is a contradiction, as required. □

An immediate consequence of this is the following discrete stability result.

Lemma 10. *Let $A(x)$ satisfy (2) and (3). Then, for any mesh function $\vec{\Psi}$,*

$$\|\vec{\Psi}(x_j)\| \leq \max \left\{ \|\vec{\Psi}(0)\|, \|\vec{\Psi}(1)\|, \frac{1}{\alpha} \|\vec{L}^N \vec{\Psi}\| \right\}, \quad 0 \leq j \leq N.$$

Proof. Define the two functions

$$\vec{\Theta}^\pm(x_j) = \max \{ \|\vec{\Psi}(0)\|, \|\vec{\Psi}(1)\|, \frac{1}{\alpha} \|\vec{L}^N \vec{\Psi}\| \} \vec{e}^\pm \pm \vec{\Psi}(x_j)$$

where $\vec{e} = (1, \dots, 1)$ is the unit vector. Using the properties of A it is not hard to verify that $\vec{\Theta}^\pm(0) \geq \vec{0}$, $\vec{\Theta}^\pm(1) \geq \vec{0}$ and $\vec{L}^N \vec{\Theta}^\pm(x_j) \geq \vec{0}$. It follows from Lemma 9 that $\vec{\Theta}^\pm(x_j) \geq \vec{0}$ for all $0 \leq j \leq N$. \square

The following comparison result will be used in the proof of the error estimate.

Lemma 11. *Assume that the mesh functions $\vec{\Phi}$ and \vec{Z} satisfy, for $j = 1, \dots, N-1$,*

$$\|\vec{Z}(0)\| \leq \vec{\Phi}(0), \quad \|\vec{Z}(1)\| \leq \vec{\Phi}(1), \quad \|\vec{L}^N \vec{Z}(x_j)\| \leq \vec{L}^N \vec{\Phi}(x_j).$$

Then, for $j = 0, \dots, N$,

$$\|\vec{Z}(x_j)\| \leq \vec{\Phi}(x_j).$$

Proof. Define the two mesh functions $\vec{\Psi}^\pm$ by

$$\vec{\Psi}^\pm = \vec{\Phi} \pm \vec{Z}.$$

Then $\vec{\Psi}^\pm$ satisfies, for $j = 1, \dots, N-1$,

$$\vec{\Psi}^\pm(0) = \vec{\Psi}^\pm(1) = \vec{0}, \quad \vec{L}^N \vec{\Psi}^\pm(x_j) \geq \vec{0}.$$

The result follows from an application of Lemma 9. \square

6. The local truncation error

From Lemma 10, it is seen that in order to bound the error $\|\vec{U} - \vec{u}\|$ it suffices to bound $\vec{L}^N(\vec{U} - \vec{u})$. But this expression satisfies

$$\begin{aligned} \vec{L}^N(\vec{U} - \vec{u}) &= \vec{L}^N(\vec{U}) - \vec{L}^N(\vec{u}) = \vec{f} - \vec{L}^N(\vec{u}) = \vec{L}(\vec{u}) - \vec{L}^N(\vec{u}) \\ &= (\vec{L} - \vec{L}^N)\vec{u} = -E(\delta^2 - D^2)\vec{u} \end{aligned}$$

which is the local truncation of the second derivative. Let \vec{V}, \vec{W} be the discrete analogues of \vec{v}, \vec{w} , respectively. Then, similarly,

$$\vec{L}^N(\vec{V} - \vec{v}) = -E(\delta^2 - D^2)\vec{v}, \quad \vec{L}^N(\vec{W} - \vec{w}) = -E(\delta^2 - D^2)\vec{w}.$$

By the triangle inequality,

$$\|\vec{L}^N(\vec{U} - \vec{u})\| \leq \|\vec{L}^N(\vec{V} - \vec{v})\| + \|\vec{L}^N(\vec{W} - \vec{w})\|. \quad (35)$$

Thus, the smooth and singular components of the local truncation error can be treated separately. In view of this it is noted that, for any smooth function ψ , the following three distinct estimates of the local truncation error of its second derivative hold:

for $x_j \in M_{\bar{b}}$

$$|(\delta^2 - D^2)\psi(x_j)| \leq C \max_{s \in I_j} |\psi''(s)| \tag{36}$$

and

$$|(\delta^2 - D^2)\psi(x_j)| \leq C\delta_j \max_{s \in I_j} |\psi^{(3)}(s)|, \tag{37}$$

for $x_j \notin J_{\bar{b}}$

$$|(\delta^2 - D^2)\psi(x_j)| \leq C\delta_j^2 \max_{s \in I_j} |\psi^{(4)}(s)|, \tag{38}$$

for $\tau_k \in J_{\bar{b}}$

$$|(\delta^2 - D^2)\psi(\tau_k)| \leq C(|H_k - h_k| \cdot |\psi^{(3)}(\tau_k)| + \delta_k^2 \max_{s \in I_k} |\psi^{(4)}(s)|). \tag{39}$$

Here $I_j = [x_{j-1}, x_{j+1}]$.

7. Error estimate

The proof of the error estimate is broken into two parts. In the first a theorem concerning the smooth part of the error is proved. Then the singular part of the error is considered. A barrier function is now constructed, which is used in both parts of the proof.

For each $\tau_k \in T_{\bar{b}}$, introduce the piecewise linear polynomial

$$\theta_k(x) = \begin{cases} \frac{x}{\tau_k}, & 0 \leq x \leq \tau_k \\ 1, & \tau_k < x < 1 - \tau_k \\ \frac{1-x}{\tau_k}, & 1 - \tau_k \leq x \leq 1. \end{cases}$$

It is not hard to verify that, for each $x_j = \tau_k \in T_{\bar{b}}$,

$$\vec{L}^N(\theta_k(x_j)\vec{e})_i \geq \begin{cases} \alpha + \frac{2\varepsilon_i}{\tau_k(H_k + h_k)}, & \text{if } x_j = \tau_k \in J_{\bar{b}} \\ \alpha\theta_k(x_j), & \text{if } x_j \notin J_{\bar{b}}. \end{cases}$$

On the Shishkin mesh $M_{\bar{b}}$ define the barrier function $\vec{\Phi}$ by

$$\vec{\Phi}(x_j) = C N^{-2}(\ln N)^3 [1 + \sum_{k \in I_{\bar{b}}} \theta_k(x_j)]\vec{e}, \tag{40}$$

where C is any sufficiently large constant.

Then $\vec{\Phi}$ satisfies

$$0 \leq \Phi_i(x_j) \leq C N^{-2}(\ln N)^3, \quad 1 \leq i \leq n. \tag{41}$$

Also, for $x_j \notin J_{\vec{b}}$,

$$(\vec{L}^N \vec{\Phi}(x_j))_i \geq CN^{-2}(\ln N)^3 \tag{42}$$

and, for $\tau_k \in J_{\vec{b}}$,

$$(\vec{L}^N \vec{\Phi}(\tau_k))_i \geq C(1 + \frac{\varepsilon_i}{\sqrt{\varepsilon_k}(H_k + h_k)})(N^{-1} \ln N)^2,$$

from which it follows that, for $\tau_k \in J_{\vec{b}}$ and $H_k \geq h_k$,

$$(\vec{L}^N \vec{\Phi}(\tau_k))_i \geq C(N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_k}\sqrt{\varepsilon_{k+1}}}N^{-1} \ln N) \tag{43}$$

and, for $\tau_k \in J_{\vec{b}}$ and $H_k \leq h_k$,

$$(\vec{L}^N \vec{\Phi}(\tau_k))_i \geq C(N^{-2} + \frac{\varepsilon_i}{\varepsilon_k}N^{-1} \ln N). \tag{44}$$

The following theorem gives the error estimate for the smooth component.

Theorem 1. *Let $A(x)$ satisfy (2) and (3). Let \vec{v} denote the smooth component of the exact solution from (1) and \vec{V} the smooth component of the discrete solution from (34). Then*

$$\|\vec{V} - \vec{v}\| \leq CN^{-2}(\ln N)^3. \tag{45}$$

Proof. An application of Lemma 11 is made, using the above barrier function. To prove the theorem it suffices to show that the ratio

$$R(v_i(x_j)) = \frac{|\varepsilon_i(\delta^2 - D^2)v_i(x_j)|}{|(\vec{L}^N \vec{\Phi}(x_j))_i|}, \quad x_j \in M_{\vec{b}}$$

satisfies

$$R(v_i(x_j)) \leq C. \tag{46}$$

For $x_j \notin J_{\vec{b}}$ the bound (46) follows immediately from Lemma 4, (38), (28) and (42). Now assume that $x_j = \tau_k \in J_{\vec{b}}$. The required estimates of the denominator of $R(v_i(\tau_k))$ are (43) and (44). The numerator is bounded above using Lemma 6, (32) and (37). The cases $b_k = 1$ and $b_k = 0$ are treated separately and the inequalities (26), (27), (28), (30) and (33) are used systematically.

Suppose first that $b_k = 1$, then there are four possible subcases:

$$\begin{aligned} i \leq k, \quad H_k \geq h_k, \quad R(v_i(\tau_k)) &\leq C\varepsilon_{k+1}. \\ &H_k \leq h_k, \quad R(v_i(\tau_k)) \leq C\varepsilon_k. \\ i > k, \quad H_k \geq h_k, \quad R(v_i(\tau_k)) &\leq C\varepsilon_{k+1} \frac{\sqrt{\varepsilon_k}}{\sqrt{\varepsilon_i}}. \\ &H_k \leq h_k, \quad R(v_i(\tau_k)) \leq C\varepsilon_k \frac{\sqrt{\varepsilon_k}}{\sqrt{\varepsilon_i}}. \end{aligned} \tag{47}$$

Secondly, if $b_k = 0$, then $b_{k-1} = 1$, because otherwise $\tau_k \notin J_{\bar{b}}$ and furthermore $H_k \leq h_k$. There are two possible subcases:

$$\begin{aligned} i &\leq k - 1, \quad H_k \leq h_k, \quad R(v_i(\tau_k)) \leq C\varepsilon_k. \\ i &> k - 1, \quad H_k \leq h_k, \quad R(v_i(\tau_k)) \leq C\varepsilon_i. \end{aligned} \tag{48}$$

In all six subcases, because of the ordering of the ε_i , it is clear that condition (46) is fulfilled. This concludes the proof. \square

Before the singular part of the error is estimated the following lemmas are established.

Lemma 12. *Let $A(x)$ satisfy (2) and (3). Then, on each mesh $M_{\bar{b}}$, for $1 \leq i \leq n$ and $1 \leq j \leq N$, the following estimates hold*

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \leq C \frac{\delta_j^2}{\varepsilon_1} \text{ for } x_j \notin J_{\bar{b}}. \tag{49}$$

An analogous result holds for the w_i^r .

Proof. When $x_j \notin J_{\bar{b}}$, from (38) and Lemma 7, it follows that

$$\begin{aligned} |\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| &\leq C\delta_j^2 \max_{s \in I_j} |\varepsilon_i w_i^{l,(4)}(s)| \\ &\leq C\delta_j^2 \max_{s \in I_j} \sum_{q=1}^n \frac{B_q^l(s)}{\varepsilon_q} \leq \frac{C\delta_j^2}{\varepsilon_1} \end{aligned}$$

as required. \square

In what follows fourth degree polynomials of the form

$$p_{i;\theta}(x) = \sum_{k=0}^4 \frac{(x - x_\theta)^k}{k!} w_i^{l,(k)}(x_\theta)$$

are used, where θ denotes a pair of integers separated by a comma.

Lemma 13. *Let $A(x)$ satisfy (2) and (3) and assume that $M_{\bar{b}}$ is such that $b_k = 1$ for some k , $1 \leq k \leq n - 1$. Then, for each i, j , $1 \leq i \leq n$, $1 \leq j \leq N$ there exists a decomposition*

$$w_i^l = \sum_{q=1}^{k+1} w_{i,q}$$

for which the following estimates hold for each q and r , $1 \leq q \leq k$, $0 \leq r \leq 2$,

$$|\varepsilon_i w_{i,q}^{(r+2)}(x_j)| \leq C\varepsilon_q^{-\frac{r}{2}} B_q^l(x_j)$$

and

$$|\varepsilon_i w_{i,k+1}^{(3)}(x_j)| \leq C \sum_{q=k+1}^n \frac{B_q^l(x_j)}{\sqrt{\varepsilon_q}}, \quad |\varepsilon_i w_{i,k+1}^{(4)}(x_j)| \leq C \sum_{q=k+1}^n \frac{B_q^l(x_j)}{\varepsilon_q}.$$

Furthermore, for $x_j \notin J_{\bar{b}}$,

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \leq C(B_k^l(x_{j-1}) + \frac{\delta_j^2}{\varepsilon_{k+1}}) \tag{50}$$

and, for $\tau_k \in J_{\bar{b}}$,

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(\tau_k)| \leq C(B_k^l(\tau_k - h_k) + \frac{\delta_k}{\sqrt{\varepsilon_{k+1}}}). \tag{51}$$

Analogous results hold for the w_i^r and their derivatives.

Proof. Consider the decomposition

$$w_i^l = \sum_{m=1}^{k+1} w_{i,m},$$

where the components are defined by

$$w_{i,k+1} = \begin{cases} p_{i;k,k+1}, & \text{on } [0, x_{k,k+1}^{(1)}) \\ w_i^l, & \text{otherwise} \end{cases}$$

and for each $m, k \geq m \geq 2$,

$$w_{i,m} = \begin{cases} p_{i;m-1,m}, & \text{on } [0, x_{m-1,m}^{(1)}) \\ w_i^l - \sum_{q=m+1}^{k+1} w_{i,q}, & \text{otherwise} \end{cases}$$

and

$$w_{i,1} = w_i^l - \sum_{q=2}^{k+1} w_{i,q} \text{ on } [0, 1].$$

From the above definitions it follows that, for each $m, 1 \leq m \leq k, w_{i,m} = 0$ on $[x_{m,m+1}^{(1)}, 1]$.

To establish the bounds on the fourth derivatives it is seen that:

for $x \in [x_{k,k+1}^{(1)}, 1]$, Lemma 7 and $x \geq x_{k,k+1}^{(1)}$ imply that

$$|\varepsilon_i w_{i,k+1}^{(4)}(x)| = |\varepsilon_i w_i^{l,(4)}(x)| \leq C \sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q} \leq C \sum_{q=k+1}^n \frac{B_q^l(x)}{\varepsilon_q};$$

for $x \in [0, x_{k,k+1}^{(1)}]$, Lemma 7 and $x \leq x_{k,k+1}^{(1)}$ imply that

$$\begin{aligned} |\varepsilon_i w_{i,k+1}^{(4)}(x)| &= |\varepsilon_i w_i^{l,(4)}(x_{k,k+1}^{(1)})| \leq C \sum_{q=1}^n \frac{B_q^l(x_{k,k+1}^{(1)})}{\varepsilon_q} \\ &\leq C \sum_{q=k+1}^n \frac{B_q^l(x_{k,k+1}^{(1)})}{\varepsilon_q} \leq C \sum_{q=k+1}^n \frac{B_q^l(x)}{\varepsilon_q}; \end{aligned}$$

and for each $m = k, \dots, 2$, it follows that

for $x \in [x_{m,m+1}^{(1)}, 1]$, $w_{i,m}^{(4)} = 0$;

for $x \in [x_{m-1,m}^{(1)}, x_{m,m+1}^{(1)}]$, Lemma 7 implies that

$$|\varepsilon_i w_{i,m}^{(4)}(x)| \leq |\varepsilon_i w_i^{l,(4)}(x)| + \sum_{q=m+1}^{k+1} |\varepsilon_i w_{i,q}^{(4)}(x)| \leq C \sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q} \leq C \frac{B_m^l(x)}{\varepsilon_m};$$

for $x \in [0, x_{m-1,m}^{(1)}]$, Lemma 7 and $x \leq x_{m-1,m}^{(1)}$ imply that

$$\begin{aligned} |\varepsilon_i w_{i,m}^{(4)}(x)| &= |\varepsilon_i w_i^{l,(4)}(x_{m-1,m}^{(1)})| \leq C \sum_{q=1}^n \frac{B_q^l(x_{m-1,m}^{(1)})}{\varepsilon_q} \\ &\leq C \frac{B_m^l(x_{m-1,m}^{(1)})}{\varepsilon_m} \leq C \frac{B_m^l(x)}{\varepsilon_m}; \end{aligned}$$

for $x \in [x_{1,2}^{(1)}, 1]$, $w_{i,1}^{(4)} = 0$;

for $x \in [0, x_{1,2}^{(1)}]$, Lemma 7 implies that

$$|\varepsilon_i w_{i,1}^{(4)}(x)| \leq |\varepsilon_i w_i^{l,(4)}(x)| + \sum_{q=2}^{k+1} |\varepsilon_i w_{i,q}^{(4)}(x)| \leq C \sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q} \leq C \frac{B_1^l(x)}{\varepsilon_1}.$$

For the bounds on the second and third derivatives note that, for each m , $1 \leq m \leq k$:

for $x \in [x_{m,m+1}^{(1)}, 1]$, $w_{i,m}'' = 0 = w_{i,m}^{(3)}$;

for $x \in [0, x_{m,m+1}^{(1)}]$,

$$\int_x^{x_{m,m+1}^{(1)}} \varepsilon_i w_{i,m}^{(4)}(s) ds = \varepsilon_i w_{i,m}^{(3)}(x_{m,m+1}^{(1)}) - \varepsilon_i w_{i,m}^{(3)}(x) = -\varepsilon_i w_{i,m}^{(3)}(x)$$

and so

$$|\varepsilon_i w_{i,m}^{(3)}(x)| \leq \int_x^{x_{m,m+1}^{(1)}} |\varepsilon_i w_{i,m}^{(4)}(s)| ds \leq \frac{C}{\varepsilon_m} \int_x^{x_{m,m+1}^{(1)}} B_m^l(s) ds \leq C \frac{B_m^l(x)}{\sqrt{\varepsilon_m}}.$$

In a similar way, it can be shown that

$$|\varepsilon_i w_{i,m}''(x)| \leq C B_m^l(x).$$

Using the above decomposition yields

$$|\varepsilon_i (\delta^2 - D^2) w_i^l(x_j)| \leq \sum_{q=1}^k |\varepsilon_i (\delta^2 - D^2) w_{i,q}(x_j)| + |\varepsilon_i (\delta^2 - D^2) w_{i,k+1}(x_j)|.$$

For $x_j \notin J_{\bar{b}}$, applying (38) to the last term and (36) to all other terms on the right-hand side, it follows that

$$|\varepsilon_i (\delta^2 - D^2) w_i^l(x_j)| \leq C \left(\sum_{q=1}^k \max_{s \in I_j} |\varepsilon_i w_{i,q}''(s)| + \delta_j^2 \max_{s \in I_j} |\varepsilon_i w_{i,k+1}^{(4)}(s)| \right).$$

Then (50) is obtained by using the bounds on the derivatives obtained in the first part of the lemma.

On the other hand, for $x_j = \tau_k \in J_{\bar{b}}$, applying (37) to the last term and (36) to the other terms, (51) is obtained by a similar argument. The proof for the w_i^r and their derivatives is similar. \square

In what follows third degree polynomials of the form

$$p_{i;\theta}^*(x) = \sum_{k=0}^3 \frac{(x - y_\theta)^k}{k!} w_i^{l,(k)}(y_\theta)$$

are used, where θ denotes a pair of integers separated by a comma.

Lemma 14. *Let $A(x)$ satisfy (2) and (3) and assume that $M_{\bar{b}}$ is such that $b_k = 1$ for some k , $1 \leq k \leq n - 1$. Then, for each i, j , $1 \leq i \leq n$, $1 \leq j \leq N$ there exists a decomposition*

$$w_i^l = \sum_{m=1}^{k+1} w_{i,m},$$

for which the following estimates hold for each m , $1 \leq m \leq k$,

$$|w_{i,m}''(x_j)| \leq C \frac{B_m^l(x_j)}{\varepsilon_m}, \quad |w_{i,m}^{(3)}(x_j)| \leq C \frac{B_m^l(x_j)}{\varepsilon_m^{3/2}}$$

and

$$|w_{i,k+1}^{(3)}(x_j)| \leq C \sum_{q=k+1}^n \frac{B_q^l(x_j)}{\varepsilon_q^{3/2}}.$$

Furthermore,

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \leq C\varepsilon_i \left(\sum_{q=1}^k \frac{B_q^l(x_{j-1})}{\varepsilon_q} + \frac{\delta_j}{\varepsilon_{k+1}^{3/2}} \right). \tag{52}$$

Analogous results hold for the w_i^r and their derivatives.

Proof. The proof is similar to that of Lemma 13 with the points $x_{i,j}^{(1)}$ replaced by the points $x_{i,j}^{(3/2)}$. Consider the decomposition

$$w_i^l = \sum_{m=1}^{k+1} w_{i,m},$$

where the components are defined by

$$w_{i,k+1} = \begin{cases} p_{i;k,k+1}^*, & \text{on } [0, x_{k,k+1}^{(3/2)}) \\ w_i^l, & \text{otherwise} \end{cases}$$

and for each $m, k \geq m \geq 2$,

$$w_{i,m} = \begin{cases} p_{i;m-1,m}^* & \text{on } [0, x_{m-1,m}^{(3/2)}) \\ w_i^l - \sum_{q=m+1}^{k+1} w_{i,q}, & \text{otherwise} \end{cases}$$

and

$$w_{i,1} = w_i^l - \sum_{q=2}^{k+1} w_{i,q} \text{ on } [0, 1].$$

From the above definitions it follows that, for each $m, 1 \leq m \leq k, w_{i,m} = 0$ on $[x_{m,m+1}^{(3/2)}, 1]$.

To establish the bounds on the third derivatives it is seen that:

for $x \in [x_{k,k+1}^{(3/2)}, 1]$, Lemma 7 and $x \geq x_{k,k+1}^{(3/2)}$ imply that

$$|w_{i,k+1}^{(3)}(x)| = |w_i^{l,(3)}(x)| \leq C \sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q^{3/2}} \leq C \sum_{q=k+1}^n \frac{B_q^l(x)}{\varepsilon_q^{3/2}};$$

for $x \in [0, x_{k,k+1}^{(3/2)}]$, Lemma 7 and $x \leq x_{k,k+1}^{(3/2)}$ imply that

$$\begin{aligned} |w_{i,k+1}^{(3)}(x)| &= |w_i^{l,(3)}(x_{k,k+1}^{(3/2)})| \leq C \sum_{q=1}^n \frac{B_q^l(x_{k,k+1}^{(3/2)})}{\varepsilon_q^{3/2}} \\ &\leq C \sum_{q=k+1}^n \frac{B_q^l(x_{k,k+1}^{(3/2)})}{\varepsilon_q^{3/2}} \leq C \sum_{q=k+1}^n \frac{B_q^l(x)}{\varepsilon_q^{3/2}}; \end{aligned}$$

and for each $m = k, \dots, 2$, it follows that

for $x \in [x_{m,m+1}^{(3/2)}, 1], w_{i,m}^{(3)} = 0$;

for $x \in [x_{m-1,m}^{(3/2)}, x_{m,m+1}^{(3/2)}]$, Lemma 7 implies that

$$|w_{i,m}^{(3)}(x)| \leq |w_i^{l,(3)}(x)| + \sum_{q=m+1}^{k+1} |w_{i,q}^{(3)}(x)| \leq C \sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q^{3/2}} \leq C \frac{B_m^l(x)}{\varepsilon_m^{3/2}};$$

for $x \in [0, x_{m-1,m}^{(3/2)}]$, Lemma 7 and $x \leq x_{m-1,m}^{(3/2)}$ imply that

$$\begin{aligned} |w_{i,m}^{(3)}(x)| &= |w_i^{l,(3)}(x_{m-1,m}^{(3/2)})| \leq C \sum_{q=1}^n \frac{B_q^l(x_{m-1,m}^{(3/2)})}{\varepsilon_q^{3/2}} \\ &\leq C \frac{B_m^l(x_{m-1,m}^{(3/2)})}{\varepsilon_m^{3/2}} \leq C \frac{B_m^l(x)}{\varepsilon_m^{3/2}}; \end{aligned}$$

for $x \in [x_{1,2}^{(3/2)}, 1], w_{i,1}^{(3)} = 0$;

for $x \in [0, x_{1,2}^{(3/2)}]$, Lemma 7 implies that

$$|w_{i,1}^{(3)}(x)| \leq |w_i^{l,(3)}(x)| + \sum_{q=2}^{k+1} |w_{i,q}^{(3)}(x)| \leq C \sum_{q=1}^n \frac{B_q^l(x)}{\varepsilon_q^{3/2}} \leq C \frac{B_1^l(x)}{\varepsilon_1^{3/2}}.$$

For the bounds on the second derivatives note that, for each $m, 1 \leq m \leq k$:

for $x \in [x_{m,m+1}^{(3/2)}, 1]$, $w_{i,m}'' = 0$;
 for $x \in [0, x_{m,m+1}^{(3/2)}]$,

$$\int_x^{x_{m,m+1}^{(3/2)}} w_{i,m}^{(3)}(s) ds = w_{i,m}''(x_{m,m+1}^{(3/2)}) - w_{i,m}''(x) = -w_{i,m}''(x)$$

and so

$$|w_{i,m}''(x)| \leq \int_x^{x_{m,m+1}^{(3/2)}} |w_{i,m}^{(3)}(s)| ds \leq \frac{C}{\varepsilon_m^{3/2}} \int_x^{x_{m,m+1}^{(3/2)}} B_m^l(s) ds \leq C \frac{B_m^l(x)}{\varepsilon_m}.$$

Finally, since

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \leq \sum_{m=1}^k |\varepsilon_i(\delta^2 - D^2)w_{i,m}(x_j)| + |\varepsilon_i(\delta^2 - D^2)w_{i,k+1}(x_j)|,$$

using (37) on the last term and (36) on all other terms on the right-hand side, it follows that

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \leq C \left(\sum_{m=1}^k \max_{s \in I_j} |\varepsilon_i w_{i,m}''(s)| + \delta_j \max_{s \in I_j} |\varepsilon_i w_{i,k+1}^{(3)}(s)| \right).$$

The desired result follows by applying the bounds on the derivatives obtained in the first part of the lemma. The proof for the w_i^r and their derivatives is similar. \square

Lemma 15. *Let $A(x)$ satisfy (2) and (3). Then, on each mesh $M_{\vec{v}}$, the following estimate holds for $i = 1, \dots, n$ and each $j = 1, \dots, N$,*

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \leq CB_n^l(x_{j-1}).$$

An analogous result holds for the w_i^r .

Proof. From (36) and Lemma 7, for each $i = 1, \dots, n$ and $j = 1, \dots, N$, it follows that

$$\begin{aligned} |\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| &\leq C \max_{s \in I_j} |\varepsilon_i w_i^{l, ''}(s)| \\ &\leq C \varepsilon_i \sum_{q=i}^n \frac{B_q^l(x_{j-1})}{\varepsilon_q} \leq CB_n^l(x_{j-1}). \end{aligned}$$

The proof for the w_i^r and their derivatives is similar. \square

The following theorem provides the error estimate for the singular component.

Theorem 2. *Let $A(x)$ satisfy (2) and (3). Let \vec{w} denote the singular component of the exact solution from (1) and \vec{W} the singular component of the discrete solution from (34). Then*

$$\|\vec{W} - \vec{w}\| \leq C N^{-2} (\ln N)^3. \tag{53}$$

Proof. Since $\vec{w} = \vec{w}^l + \vec{w}^r$, it suffices to prove the result for \vec{w}^l and \vec{w}^r separately. Here it is proved for \vec{w}^l by an application of Lemma 11. A similar proof holds for \vec{w}^r .

The proof is in two parts.

First assume that $x_j \notin J_{\vec{b}}$. Each open subinterval (τ_k, τ_{k+1}) is treated separately.

First, consider $x_j \in (0, \tau_1)$. Then, on each mesh $M_{\vec{b}}$, $\delta_j \leq CN^{-1}\tau_1$ and the result follows from (26) and Lemma 12.

Secondly, consider $x_j \in (\tau_1, \tau_2)$, then $\tau_1 \leq x_{j-1}$ and $\delta_j \leq CN^{-1}\tau_2$. The 2^n possible meshes are divided into subclasses of two types. On the meshes $M_{\vec{b}}$ with $b_1 = 0$ the result follows from (26), (29) and Lemma 12. On the meshes $M_{\vec{b}}$ with $b_1 = 1$ the result follows from (26), (30) and Lemma 13.

Thirdly, in the general case $x_j \in (\tau_m, \tau_{m+1})$ for $2 \leq m \leq n - 1$, it follows that $\tau_m \leq x_{j-1}$ and $\delta_j \leq CN^{-1}\tau_{m+1}$. Then $M_{\vec{b}}$ is divided into subclasses of three types: $M_{\vec{b}}^0 = \{M_{\vec{b}} : b_1 = \dots = b_m = 0\}$, $M_{\vec{b}}^r = \{M_{\vec{b}} : b_r = 1, b_{r+1} = \dots = b_m = 0 \text{ for some } 1 \leq r \leq m - 1\}$ and $M_{\vec{b}}^m = \{M_{\vec{b}} : b_m = 1\}$. On $M_{\vec{b}}^0$ the result follows from (26), (29) and Lemma 12; on $M_{\vec{b}}^r$ from (26), (29), (30) and Lemma 13; on $M_{\vec{b}}^m$ from (26), (30) and Lemma 13.

Finally, for $x_j \in (\tau_n, 1)$, $\tau_n \leq x_{j-1}$ and $\delta_j \leq CN^{-1}$. Then $M_{\vec{b}}$ is divided into subclasses of three types: $M_{\vec{b}}^0 = \{M_{\vec{b}} : b_1 = \dots = b_n = 0\}$, $M_{\vec{b}}^r = \{M_{\vec{b}} : b_r = 1, b_{r+1} = \dots = b_n = 0 \text{ for some } 1 \leq r \leq n - 1\}$ and $M_{\vec{b}}^n = \{M_{\vec{b}} : b_n = 1\}$. On $M_{\vec{b}}^0$ the result follows from (26), (29) and Lemma 12; on $M_{\vec{b}}^r$ from (26), (29), (30) and Lemma 13; on $M_{\vec{b}}^n$ from (30) and Lemma 15.

Now assume that $x_j = \tau_k \in J_{\vec{b}}$. Analogously to the proof of Theorem 1, the ratio $R(w_i^l(\tau_k))$ is introduced in order to facilitate the use of Lemma 11. To complete the proof it suffices to show that the ratio

$$R(w_i^l(x_j)) = \frac{|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)|}{|(\vec{L}^N \vec{\Phi}(x_j))_i|}, \quad x_j \in M_{\vec{b}}$$

satisfies

$$R(w_i^l(\tau_k)) \leq C. \tag{54}$$

The required estimates of the denominator of $R(w_i^l(\tau_k))$ are (43) and (44). The numerator is bounded above using Lemmas 13 and 14. The cases $b_k = 1$ and $b_k = 0$ are treated separately and the inequalities (26), (27), (28), (30) and (33) are used systematically.

Suppose first that $b_k = 1$, then there are four possible subcases:

$$\begin{aligned} \text{Lemma 14, } i \leq k, H_k \geq h_k, R(w_i^l(\tau_k)) &\leq \frac{C(\frac{\varepsilon_i}{\varepsilon_k} N^{-2} + \frac{\varepsilon_i}{\varepsilon_{k+1}} N^{-1} \ln N)}{C(N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_k} \sqrt{\varepsilon_{k+1}}} N^{-1} \ln N)}. \\ H_k \leq h_k, R(w_i^l(\tau_k)) &\leq \frac{C(\frac{\varepsilon_i}{\varepsilon_k} N^{-2} + \frac{\varepsilon_i \sqrt{\varepsilon_k}}{\varepsilon_{k+1}^{3/2}} N^{-1} \ln N)}{C(N^{-2} + \frac{\varepsilon_i}{\varepsilon_k} N^{-1} \ln N)}. \end{aligned} \tag{55}$$

$$\begin{aligned} \text{Lemma 13, } i > k, H_k \geq h_k, R(w_i^l(\tau_k)) &\leq \frac{C(N^{-2} + N^{-1} \ln N)}{C(N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_k} \sqrt{\varepsilon_{k+1}}} N^{-1} \ln N)}. \\ H_k \leq h_k, R(w_i^l(\tau_k)) &\leq \frac{C(N^{-2} + \frac{\sqrt{\varepsilon_k}}{\sqrt{\varepsilon_{k+1}}} N^{-1} \ln N)}{C(N^{-2} + N^{-1} \ln N)}. \end{aligned}$$

Secondly, if $b_k = 0$, then $b_{k-1} = 1$, because otherwise $\tau_k \notin J_{\vec{b}}$, and furthermore $H_k \leq h_k$. There are two possible subcases:

$$\text{Lemma 14, } i \leq k - 1, H_k \leq h_k, R(w_i^l(\tau_k)) \leq \frac{C(\frac{-\varepsilon_i}{\varepsilon_{k-1}} N^{-2} + \frac{\varepsilon_i}{\varepsilon_k} N^{-1} \ln N)}{C(N^{-2} + \frac{\varepsilon_i}{\varepsilon_k} N^{-1} \ln N)}. \tag{56}$$

$$\text{Lemma 13, } i > k - 1, H_k \leq h_k, R(w_i^l(\tau_k)) \leq \frac{C(N^{-2} + N^{-1} \ln N)}{C(N^{-2} + N^{-1} \ln N)}.$$

In all six subcases, because of the ordering of the ε_i , it is clear that condition (54) is fulfilled. This concludes the proof. \square

The following theorem gives the required essentially second order parameter-uniform error estimate.

Theorem 3. *Let $A(x)$ satisfy (2) and (3). Let \vec{u} denote the exact solution from (1) and \vec{U} the discrete solution from (34). Then*

$$\|\vec{U} - \vec{u}\| \leq C N^{-2} (\ln N)^3. \tag{57}$$

Proof. An application of the triangle inequality and the results of Theorems 1 and 2 lead immediately to the required result. \square

8. Numerical results

The above numerical method is applied to the following singularly perturbed boundary value problem

Example 1.

$$\left. \begin{aligned} -\varepsilon_1 u_1''(x) + 5u_1(x) - u_2(x) - u_3(x) &= x^2 \\ -\varepsilon_2 u_2''(x) - u_1(x) + (5+x)u_2(x) - u_3(x) &= e^{-x} \\ -\varepsilon_3 u_3''(x) - (1+x)u_1(x) - u_2(x) + (5+x)u_3(x) &= 1+x \end{aligned} \right\} \tag{58}$$

for $x \in (0, 1)$ and $\vec{u}(0) = \vec{0}$, $\vec{u}(1) = \vec{0}$. For various values of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\alpha = 2.0$ and $N = 2^r$, $r = 8, \dots, 13$ the computed order of $\vec{\varepsilon}$ -uniform convergence and the computed $\vec{\varepsilon}$ -uniform error constant are found using the general methodology from [1]. The results are presented in Table 1.

η	Number of mesh points N				
	512	1024	2048	4096	8192
0.100E+01	0.284E-05	0.711E-06	0.178E-06	0.421E-07	0.192E-06
0.100E-02	0.519E-03	0.242E-03	0.109E-03	0.414E-04	0.146E-04
0.100E-05	0.519E-03	0.242E-03	0.109E-03	0.414E-04	0.146E-04
0.100E-08	0.519E-03	0.242E-03	0.109E-03	0.414E-04	0.146E-04
0.100E-11	0.519E-03	0.242E-03	0.109E-03	0.414E-04	0.146E-04
0.100E-14	0.519E-03	0.242E-03	0.109E-03	0.414E-04	0.146E-04
0.100E-17	0.519E-03	0.242E-03	0.109E-03	0.414E-04	0.145E-04
D^N	0.519E-03	0.242E-03	0.109E-03	0.414E-04	0.146E-04
p^N	0.110E+01	0.115E+01	0.140E+01	0.150E+01	
C_p^N	0.432E+00	0.432E+00	0.417E+00	0.339E+00	0.257E+00
Computed order of $\vec{\varepsilon}$ -uniform convergence = $0.110E + 01$					
Computed $\vec{\varepsilon}$ -uniform error constant = $0.432E + 00$					

Table 1. Values of $D_\varepsilon^N, D^N, p^N, p^*$, and C_p^N for $\varepsilon_1 = \frac{\eta}{16}, \varepsilon_2 = \frac{\eta}{4}, \varepsilon_3 = \eta$ and various values of N with $\alpha = 2.0$

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