ON SQUARES OF IRREDUCIBLE CHARACTERS

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Abstract. We study the finite groups $G$ with a faithful irreducible character whose square is a linear combination of algebraically conjugate irreducible characters of $G$. In conclusion, we offer another proof of one theorem of Isaacs-Zisser.

There are a few papers treating the finite groups possessing an irreducible character whose powers are linear combinations of appropriate irreducible characters, for example, [BC] and [IZ]. Our note is inspired by these two papers, especially, the second one.

In what follows, $G$ is a finite group. We use standard notation of finite group theory (see [BZ]). Recall that if $\chi \in \text{Irr}(G)$, then the generalized character $\chi^{(2)}$ (see [BZ, Chapter 4]) is defined as follows:

$$\chi^{(2)}(g) = \chi(g^2) \quad (g \in G).$$

Next, $\text{Char}(G)$ denotes the set of characters of a group $G$ and, if $\theta$ is a generalized character of $G$, then

$$\text{Irr}(\theta) = \{ \chi \in \text{Irr}(G) \mid \langle \theta, \chi \rangle \neq 0 \}.$$

The quasikernel $Z(\chi)$ of $\chi \in \text{Char}(G)$ is defined as follows:

$$Z(\chi) = \{ g \in G \mid |\chi(g)| = \chi(1) \}.$$

It is known that $Z(\chi)$ is a normal subgroup of $G$ containing $\ker(\chi)$ and $Z(G/\ker(\chi)) \leq Z(\chi)/\ker(\chi)$ with equality if, in addition, $\chi \in \text{Irr}(G)$. In what follows, we use freely results stated in this paragraph.

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Professor Zhmud (1918–2007) died in 29 December 2007. This note was prepared by Y. Berkovich, based on a letter dated March 14, 2000.
Let $\chi \in \text{Irr}(G)$. It is known that 
\begin{equation}
\text{(1)} \quad \text{Irr}(\chi^{(2)}) \subseteq \text{Irr}(\chi^2).
\end{equation}
Indeed, $\chi^{(2)} = \chi^2 - 2\Lambda^2\chi$ (see formula (22) in [BZ, §4.6]) so it suffices to show that $\text{Irr}(\Lambda^2\chi) \subseteq \text{Irr}(\chi^2)$. Next, $\chi^2 = \Lambda^2\chi + \theta$, where $\theta$ is the exterior square of $\chi$ (see [BZ, Lemma 4.16, formula (17)]) so $\text{Irr}(\Lambda^2\chi) \subseteq \text{Irr}(\chi^2)$, as desired. Therefore, if $\text{Irr}(\chi^2) = \{\psi_1, \ldots, \psi_n\}$ and 
\begin{equation}
\text{(2)} \quad \chi^2 = \sum_{j=1}^n a_j\psi_j, \text{ where all } a_j \text{ are positive integers},
\end{equation}
then, by (1), we have 
\begin{equation}
\text{(3)} \quad \chi^{(2)} = \sum_{j=1}^n b_j\psi_j, \text{ where all } b_j \text{ are integers}.
\end{equation}
Set 
\begin{equation}
\text{(4)} \quad a = a_1 + \cdots + a_n, \quad b = b_1 + \cdots + b_n.
\end{equation}

Let $\epsilon$ be a $|G|$-th primitive root of 1, $G = \text{Gal}(Q(\epsilon)/Q)$, where $Q$ is the field of rational numbers. The group $G$ acts in the natural way on the set $\text{Irr}(G)$ as follows: if $\chi \in \text{Irr}(G)$ and $\sigma \in G$, then $(\sigma\chi)(g) = \sigma(\chi(g))$ for all $g \in G$ (see [BZ, Chapter 3]). Characters $\psi, \psi' \in \text{Irr}(G)$ are said to be algebraically conjugate if $\psi' = \sigma\psi$ for some $\sigma \in G$. In that case, as it is easy to check, $\psi(1) = \psi'(1)$, $\ker(\psi) = \ker(\psi')$ and $\text{Z}(\psi) = \text{Z}(\psi')$.\footnote{Let us prove the second equality. Take $g \in \ker(\psi)$. We have $\psi'(g) = \sigma(\psi(g)) = \sigma(\psi(1)) = \psi(1)$ so $\ker(\psi) \leq \ker(\psi')$. Since $\sigma^{-1} \in G$, the reverse inclusion holds as well.}

**Definition 1.** A group $G$ with an irreducible character $\chi$ possesses a property $\mathcal{A}$, if it satisfies the following conditions:
\begin{itemize}
\item[(A1)] $|G|$ is even.
\item[(A2)] $\chi$ is faithful.
\item[(A3)] $n \geq 2$ and $\psi_j \ (j = 1, \ldots, n)$ are algebraically conjugate with $\psi = \psi_1$ (see decomposition (2)), i.e., $\psi_j = \sigma_j\psi$ for some $\sigma_j \in G \ (j = 1, \ldots, n)$.
\item[(A4)] $\chi$ and $\psi_1, \ldots, \psi_n$ satisfy (2) and (3).
\end{itemize}

Our main result is the following

**Theorem 2.** If the group $G$ satisfies condition $\mathcal{A}$, then the following assertions hold (here, as in part (A3) of the definition, $\psi = \psi_1$):
\begin{itemize}
\item[(a)] $G$ is nonabelian and $\chi(1) > 1$.
\item[(b)] $G$ has only one involution $u$.
\item[(c)] $\ker(\psi) = \langle u \rangle$.
\item[(d)] $\text{Z}(\psi)$ is abelian.
\item[(e)] $\text{Sylow } 2\text{-subgroups of } G$ are cyclic.
\end{itemize}
(f) \( G = P \cdot N \), a semidirect product, where \( P \in \text{Syl}_2(G) \) and \( \{1\} < N \triangleleft G \).

(g) \( \chi(1) = \psi(1) \) and \( b = 1 \).

(h) \( \chi_N \in \text{Irr}(N) \).

(i) All \( (\psi_j)_N \) are irreducible and nonreal for \( j = 1, \ldots, n \).

(j) If \( w \in G \) with \( \alpha(w) = 8 \), then \( \psi(w) = m\sqrt{-1} \), where \( m \) is an integer dividing \( \chi(1) \).

**Proof.** (a) It follows from (2) and (4) that \( \chi(1)^2 = a\psi(1) \) so \( \chi(1) > 1 \) since \( a \geq n > 1 \). Therefore, \( G \) is nonabelian.

(b) Let \( u \) be an involution in \( G \). Since \( \psi(u) \) is a rational integer, it follows that \( \psi_j(u) = \psi(u) \) since the \( \psi_j \)'s are algebraically conjugate. Therefore, \( \chi(\psi(u) = b\psi(u) \) (see (3) and (4)). Since \( \chi(\psi(u) = \chi(u^2) = \chi(1) \), we get

\[
\chi(1) = b\psi(u). \tag{5}
\]

On the other hand, \( \chi(1) = \chi(\psi(u) = b\psi(1) \) so, taking into account that \( b \neq 0 \), we get \( \psi(u) = \psi(1) \), by (5), i.e., \( u \in \ker(\psi) = \ker(\psi_j) \) for \( j = 1, \ldots, n \). Therefore, it follows from (2) that \( \ker(\chi^2) = \ker(\psi) \) so \( u \in \ker(\chi^2) \), i.e., \( \chi(u)^2 = \chi(1)^2 \). In that case, \( \chi(u) = \pm \chi(1) \), and we obtain

\[
\chi(u) = -\chi(1) \tag{6}
\]

since our character \( \chi \) is faithful. It follows that \( u \in \text{Z}(\chi) = \text{Z}(G) \) since \( \chi \) is faithful, i.e., \( \text{Z}(G) \) contains all involutions of \( G \). Since \( \text{Z}(G) \) is cyclic (\( \chi \) is faithful), we conclude that \( u \) is the unique involution in \( G \), and (b) is proven.

(c) As we have proved in (b) (see the sentence after formula (5)), \( u \in \ker(\psi) \). It suffices to show that \( u \) is the unique nonidentity element of \( \ker(\psi) \).

Take \( x \in \ker(\psi)^G \). Then \( \psi_j(x) = \psi(1) \) for all \( j \) so, by (2), \( x \in \ker(\chi^2) \) so that, again by (2), we have

\[
\chi(1)^2 = \chi(x)^2 = a\psi(1), \quad \text{and hence} \quad \chi(x) = -\chi(1)
\]

since \( \chi \) is faithful. In particular, \( x \in \text{Z}(\chi) \) so that \( \chi(x^2) = \chi(1) \) and \( x^2 \in \ker(\chi) = \{1\} \), i.e., \( x \) is an involution. It follows from this and (b), that \( x = u \).

Thus \( \ker(\psi) = \{1, u\} = \langle u \rangle \), as required.

(d) It follows from (c), that \( \text{Z}(\psi) \) is abelian since \( \text{Z}(\psi)/\ker(\psi) \) is cyclic (in view of irreducibility of \( \psi \)) and \( |\ker(\psi)| = 2 \).

(e) Let \( P \in \text{Syl}_2(G) \). By (b), \( P \) is either cyclic or generalized quaternion. Assume, by way of contradiction, that \( P \) is generalized quaternion. Take in \( P \) an element \( v \) of order 4. Then, by (b),

\[
v^2 = u. \tag{7}
\]

Let \( j \in \{1, \ldots, n\} \). Then, by (A2), \( \psi_j = \sigma_j\psi \) so that \( \psi_j(v) = (\sigma_j\psi)(v) = \sigma_j(\psi(v)) \). We have \( \sigma_j(\epsilon) = \epsilon^{\nu_j} \) for some rational integer \( \nu_j \) such that \( \text{GCD}(\nu_j, |G|) = 1 \) (recall that \( \epsilon \) is the primitive \( |G| \)-th root of 1 chosen above); then \( \nu_j \) is odd in view of (A1). Setting \( \nu_j = 2\lambda_j + 1 \) and taking into account
that $\psi(v)$ is a sum of powers of $\epsilon$, we get

$$\psi_j(v) = \sigma_j(\psi(v)) = \psi(v^{\epsilon_j}) = \psi(v^{2^{\lambda_j+1}}) = \psi(u^{\lambda_j}v) = \psi(v)$$

since $u \in \ker(\psi)$, by (c). Thus

(8) $$\psi_j(v) = \psi(v) \quad (j = 1, \ldots, n).$$

Then, by (6), (8), (3) and (4), we obtain

$$-\chi(1) = \chi(u) = \chi(v^2) = \chi^{(2)}(v) = \sum_{j=1}^{n} b_j \psi_j(v) = b \psi(v).$$

On the other hand, $\chi(1) = \chi^{(2)}(1) = b \psi(1)$ so $\psi(v) = -\psi(1)$. It follows from this and (8), that

(9) $$\psi_j(v) = -\psi(1) \quad (j = 1, \ldots, n).$$

It follows from (9) that, if $T$ is a representation of $G$ affording the character $\psi_j$, then $T(v) = -I_{\psi(1)}$, where $I_{\psi(1)}$ is a $\psi(1) \times \psi(1)$ identity matrix. Therefore, by (2), we get

$$\chi(v^2) = \sum_{j=1}^{n} a_j \psi_j(v) = -a \psi(1) = -\chi(1)^2,$$

and we conclude that

(10) $$\chi(v) = ci\chi(1),$$

where $c = \pm 1$ and $i = \sqrt{-1}$. It follows from $|\chi(v)| = \chi(1)$ that $v \in Z(\chi) = Z(G)$, a contradiction, since the center of $P$, which is a generalized quaternion group, has order 2. Thus, $P \in \text{Syl}_2(G)$ is cyclic.

(f) By (e), $G$ is 2-nilpotent so $G = P \cdot N$, and $N > \{1\}$ since $G$ is nonabelian.

(g) Since $G/N \cong P$ is cyclic, then $\chi$ is not ramified over $N$ (Burnside; see [BZ, Exercise 7 in Chapter 7]) so we get the following Clifford decomposition:

(11) $$\chi_N = \sum_{k=1}^{l} \phi_k.$$

It follows from (11) that

(12) $$\chi^{(2)}_N = \sum_{k=1}^{l} \phi_k^{(2)}.$$
Since $|N|$ is odd, it follows that $\phi_k^{(2)}$ are distinct irreducible characters of $N$ for all $k$, and $(\chi^{(2)})_N$ is a character of $N$. By (9), we have

$$(\chi^{(2)})_N = \sum_{j=1}^{n} b_j (\psi_j)_N.$$  

Let $\phi_1 = \phi$. Since

$$1 = \langle (\chi^{(2)})_N, \phi^{(2)} \rangle = \sum_{j=1}^{n} b_j \langle (\psi_j)_N, \phi^{(2)} \rangle,$$

we get $\langle (\psi_s)_N, \phi^{(2)} \rangle \neq 0$ for some $s \in \{1, \ldots, n\}$. This means that $\phi^{(2)} \in \text{Irr}(\psi_s)_N$. By Clifford’s theorem, $\text{Irr}(\psi_s)_N$ is a $G$-orbit of $\phi^{(2)}$, i.e.,

$$\text{Irr}(\psi_s)_N = \{\phi_1^{(2)}, \ldots, \phi_i^{(2)}\},$$

where $\phi_1 = \phi$.

We conclude that $(\psi_s)_N = \sum_{k=1}^{t} \phi_k^{(2)}$ so, by (12), $(\chi^{(2)})_N = (\psi_s)_N$. It follows, in particular, that $\chi(1) = \psi_s(1) = \psi(1)$. Since $\chi(1) = \chi^{(2)}(1) = b\psi(1)$, we get $b = 1$, completing the proof.

(h) It follows from (2) and (3) that

$$\chi^2 - \chi^{(2)} = \sum_{j=1}^{n} (a_j - b_j)\psi_j.$$  

Since

$$\frac{1}{2}(\chi^2 - \chi^{(2)}) = \Lambda^2 \chi \in \text{Char}(G)$$

(see formula (22) in [BZ, §4.6]), we get $a_j \equiv b_j \pmod{2}$ for $j = 1, \ldots, n$. Summing up all $j$, one obtains $a \equiv b \pmod{2}$ so $a$ is odd since $b = 1$, by (g). As we have noticed in the proof of (a), $\chi(1)^2 = a\psi(1)$. Therefore, since $\chi(1) = \psi(1)$, by (g), we have $\chi(1) = a$, and hence $\chi(1)$ is odd. It follows from (11) that $l = |\text{Irr}(\chi_N)| = |G : I_G(\phi)|$, where $I_G(\phi)$ is the inertia group of $\phi$ in $G$, and $\chi(1) = l\phi(1)$ so $l$ is odd. Since $I_G(\phi) \supseteq N$ and $|G : N| = |P|$ is a power of $2$, we get $l = 1$, i.e., $\chi_N \in \text{Irr}(N)$, proving (h).

(i) Since $l = 1$, there exists $s \in \{1, \ldots, n\}$ such that $(\psi_s)_N = \phi^{(2)} \in \text{Irr}(N)$. Since all $\psi_j$ are algebraically conjugate, we get $(\psi_j)_N \in \text{Irr}(N)$. If $(\psi_j)_N$ is real for some $j$, it is the principal character $1_G$ of $G$ since $|N|$ is odd. It follows that $\psi(1) = \psi_j(1) = 1$. Then, by (g), $\chi(1) = 1$, a contradiction. Thus, all $\psi_j$ are not real.

(j) Let $w \in G$ be of order 8. Setting $v = w^2$, we get $o(v) = 4$. By (b), $v^2 = u$. It follows from (10) that $c_i \chi(1) = \chi(v) = \chi(w^2) = \chi^{(2)}(w)$ so, in view

Indeed, assume that $\phi_i^{(2)} = \phi_j^{(3)}$. Then, for $x \in N$ we have $\phi_i(x^2) = \phi_i^{(2)}(x) = \phi_j^{(2)}(x^2)$, and so $\phi_i = \phi_j$ since $\{x^2 \mid x \in N\} = N$. This proves the first assertion. Now the second assertion is obvious.
of (9), we get

\[ \sum_{j=1}^{n} b_j \psi_j(w) = \chi^{(2)}(w) = ci\chi(1) \]

(here \( i = \sqrt{-1} \)). Recall, that \( \psi_j(w) = \psi(w^\nu_j) \), where \( \nu_j \) is odd integer (see the proof of (e)). It follows from \( o(w) = 8 \) that \( \nu_j \in \{1, 3, 5, 7\} \). Since \( u \in \text{ker}(\psi) \) and \( \psi(v) = -\psi(1) \) (see (9)), we have (consider a representation of \( G \) affording the character \( \psi \); see two line after formula (9))

\[
\begin{align*}
\psi(w^3) &= \psi(vw) = -\psi(w), \\
\psi(w^5) &= \psi(uw) = \psi(w), \\
\psi(w^7) &= \psi(uvw) = -\psi(w).
\end{align*}
\]

Thus, \( \psi_j(w) \in \{\psi(w), -\psi(w)\} \). Setting \( \psi_j(w) = c_j \psi(1) \), where \( c_j = \pm 1 \), one can rewrite (13) in the form

\[ ci\chi(1) = d\psi(w), \]

where \( d = \sum_{j=1}^{n} b_j c_j \) is a rational integer. Note that \( \overline{\psi(w)} = \psi(w^{-1}) = \psi(w^7) = -\psi(w) \).

Therefore, \( \psi(w) = im \), where \( m \) is a real number. Now, (14) yields \( \chi(1) = cdm \). Therefore, \( m = c \cdot \frac{\chi(1)}{d} \) is rational. Since \( m = -i\psi(w) \) is an algebraic integer, it follows that \( m \) is a rational integer so \( m \) divides \( \chi(1) \).

This completes the proof of our theorem.

Now we are ready to offer another proof of the following

**Theorem 3** (Isaacs-Zisser [IZ]). Let \( G > \{1\} \) be a group and suppose that there is a faithful \( \chi \in \text{Irr}(G) \) such that

\[ \chi^2 = a\psi + b\tilde{\psi}, \]

where \( a, b \) are positive integers and \( \psi, \tilde{\psi} \in \text{Irr}(G) \). Then \( G \) is a direct product of a cyclic 2-group of order not exceeding 4 and a group of odd order.

**Proof.** Since \( \psi_1 = \psi \) and \( \psi_2 = \tilde{\psi} \) are algebraically conjugate, one can apply Theorem 2 to our group \( G \). By that theorem, \( P \) is cyclic. It remains to prove that \( |P| \leq 4 \) and \( P \) is normal in \( G \). We claim that \( |\text{Irr}(\chi^{(2)})| = 1 \) (recall that \( \chi^{(2)} : x \mapsto \chi(x^2) \)). Otherwise, \( \chi^{(2)} = b_1\psi_1 + b_2\psi_2 \), where \( b_1, b_2 \) are nonzero rational integers. It follows that

\[ (\chi^{(2)})_N = b_1(\psi_1)_N + b_2(\psi_2)_N \]

(recall that \( (\chi^{(2)})_N, (\psi_1)_N, (\psi_2)_N \) are irreducible, by Theorem 2(h,i)). We have \( (\chi^{(2)})_N = (\psi_s)_N \) for some \( s \in \{1, 2\} \) (see the proof of Theorem 2(g)). Using this and the equality \( b_1 + b_2 = 1 \), we get \( (\psi_1)_N = (\psi_2)_N \). It follows that \( (\psi_1)_N = (\psi_1)_N \), i.e., character \( (\psi_1)_N \) is real, contrary to Theorem 2(i). Thus,
one of numbers \(b_1, b_2\) equals 0 so our claim is proven, i.e., \(\chi^{(2)} \in \{\psi_1, \psi_2\}\). Assume, for definiteness, that \(\chi^{(2)} = \psi_1\). Thus,
\[
\chi^2 = a_1\psi_1 + a_2\psi_2 \quad \text{and} \quad \chi^{(2)} = \psi_1.
\]
As above, set \(\psi = \psi_1\). Assume that \(|P| > 4\). Take in \(P\) an element \(w\) of order 8. Set \(v = w^2\). Then, by (10),
\[
\psi(w) = \chi^{(2)}(w) = \chi(v) = ci\chi(1), \quad \text{where} \ c = \pm 1.
\]
Since \(\chi\) and \(\psi\) have the same degree, we get \(\psi(w) = ci\psi(1)\) so \(|\psi(w)| = \psi(1)|\) whence \(w \in Z(\psi)\). Let \(H \in \text{Syl}_2(Z(\psi))\); then \(H\) is normal in \(G\) so \(HN = H \times N\) and \(w\) centralizes \(N\). Since \(w\) centralizes \(P \in \text{Syl}_2(G)\), by Theorem 2(e), we see that \(w \in Z(G)\) since \(G = P \cdot N\), and so \(|\chi(w)| = \chi(1)|\). Then equalities
\[
\psi_2(w) = \psi_1(w) = -\psi(w) \quad \text{and} \quad |\psi(w)| = \chi(1)
\]
imply
\[
\chi(1)^2 = |\chi(w)|^2 = |a_1\psi(w) + a_2\psi_2(w)| = |(a_1 - a_2)\psi(w)| = |a_1 - a_2||\psi(w)| = |a_1 - a_2|\chi(1).
\]
Hence \(\chi(1) = |a_1 - a_2|\). On the other hand, \(\chi(1) = a = a_1 + a_2\). Thus \(a_1 + a_2 = |a_1 - a_2|\), a contradiction since \(a_1, a_2 > 0\). Thus \(|P| \leq 4\).

Let us prove that the subgroup \(P\) is normal in \(G\). In view of Theorem 2(b), one may assume that \(|P| = 4\) and \(P = \langle \psi \rangle\). Then, by (9), \(\psi(v) = -\psi(1)|\) so \(v \in Z(\psi)\) and hence \(P \leq Z(\psi)\). Since \(P\) is characteristic in \(Z(\psi)\) (indeed, \(P\) is a Sylow 2-subgroup of the abelian group \(Z(\psi)\); see Theorem 2(d)), it follows that \(P\) is normal in \(G\), and the proof is complete.

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References

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