ON INVERSE LIMITS OF COMPACT SPACES. 
CORRECTION OF A PROOF

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Abstract. For a compact Hausdorff space $X$ and an ANR for metrizable spaces $M$, one considers the space $M^X$ of all mappings from $X$ to $M$, endowed with the compact-open topology. Since a mapping $f: X' \to X$ induces a natural mapping $M^f: M^X \to M^{X'}$, an inverse system of compact Hausdorff spaces $X$ determines a direct system $M^X$ of spaces as well as the corresponding direct system of singular homology groups $H_\ast(M^X; G)$. There is a natural isomorphism between the direct limit $\text{dir lim} H_\ast(M^X; G)$ and the singular homology group $H_\ast(M^X; G)$, where $X = \text{inv lim} X$. This continuity theorem, used by some authors, was published more than 50 years ago. Unfortunately, the author discovered a serious error in the proofs of two lemmas on which the result depended. The present paper gives new correct proofs of these lemmas.

1. Introduction

A compact Hausdorff space $X$ and a metrizable space $M$ determine the space $M^X$ of all (continuous) mappings $\phi: X \to M$, endowed with the compact-open topology. Every mapping $f: X' \to X$ between compact Hausdorff spaces induces a mapping $M^f: M^X \to M^{X'}$, which assigns to $\phi \in M^X$ the composition $M^f(\phi) = \phi f$. Therefore, if one has an inverse system $X = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of compact Hausdorff spaces and a metrizable space $M$, the connecting mappings $p_{\lambda\lambda'}: X_{\lambda'} \to X_\lambda$, $\lambda \leq \lambda'$, induce mappings $P_{\lambda\lambda'} = M^f(p_{\lambda\lambda'}): M^{X_\lambda} \to M^{X_{\lambda'}}$ and one obtains a direct system $M^X = (M^{X_\lambda}, P_{\lambda\lambda'}, \Lambda)$. If $X$ is the limit of $X$ and $p_\lambda: X \to X_\lambda, \lambda \in \Lambda$, are the corresponding canonical projections, then they induce mappings $P^X = \text{dir lim} M^X = \text{inv lim} X$.

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$M^p: M^{X_\lambda} \to M^X$ such that $P_{X_\lambda} = P_{X_{\lambda'}}$. Denote by $H_n(\_; G)$ the functor of singular $n$-dimensional homology group with coefficients in an abelian group $G$. Denote by $P_{X_{\lambda'}}: H_n(M^{X_{\lambda'}}; G) \to H_n(M^{X_\lambda}; G)$ the homomorphism induced by $P_{X_{\lambda'}}: M^{X_{\lambda'}} \to M^{X_\lambda}$. Then, $M^X$ determines the direct system $H_n(M^X; G) = (H_n(M^{X_\lambda}; G), P_{X_{\lambda'}})$ of homology groups. Moreover, the mappings $P_{X_\lambda}$ induce homomorphisms $P_{\lambda n}: H_n(M^{X_\lambda}; G) \to H_n(M^{X}; G)$ such that $P_{X_{\lambda'}} P_{X_{\lambda'} n} = P_{X_\lambda n}$. Clearly, these homomorphisms induce a homomorphism $P_{X_\lambda n}$ from the direct limit $\limdir H_n(M^{X_\lambda}; G)$ to the homology group $H_n(M^X; G)$.

A straightforward verification shows that $P_{X_\lambda n}: \limdir H_n(M^{X_\lambda}; G) \to H_n(M^X; G)$ is a natural homomorphism, i.e., if $X' = (X_{\lambda'}, p'_{X_{\lambda'}}; \Lambda)$ is another system of compact Hausdorff spaces and $g = (g_{\lambda}, \Lambda): X' \to X$ is a level-preserving mapping of inverse systems, then the following diagram commutes.

\[
\begin{array}{ccc}
\limdir H_n(M^{X}; G) & \xrightarrow{P_{X_\lambda n}} & H_n(M^{X}; G) \\
\downarrow & & \downarrow \\
\limdir H_n(M^{X'}; G) & \xrightarrow{P_{X_{\lambda'}} n} & H_n(M^{X'}; G),
\end{array}
\]

where $H_n(M^{X}; G): H_n(M^{X}; G) \to H_n(M^{X'}; G)$ is the homomorphism induced by $g = \limdir g: X' \to X$ and $\limdir H_n(M^{X}; G): \limdir H_n(M^{X}; G) \to \limdir H_n(M^{X'}; G)$ is the homomorphism induced by $M^{X} = M^{X'} \to M^{X'}$.

The main result of the present paper is the following theorem.

**Theorem 1.1.** If $X = (X_\lambda, p_{X_{\lambda'}}; \Lambda)$ is an inverse system of compact Hausdorff spaces with limit $X = \limdir X$ and $M$ is an ANR for metrizable spaces, then the natural homomorphism $P_{X_\lambda n}: \limdir H_n(M^{X_\lambda}; G) \to H_n(M^{X}; G)$ is an isomorphism.

The special case when $X$ is an inverse sequence of metrizable compacta and $M$ is a compact ANR was proved already in [11] as Theorem 13 on page 200. The generalization to inverse systems of compact Hausdorff spaces and arbitrary ANRs appears in [12] as Theorem 6 on page 254. It follows the proof of the theorem in the special case, given in [11]. However, two lemmas used in that proof, i.e., a lemma due to M. Abe and Lemma 8, stated on page 199 of [12], had to be generalized as follows.

**Lemma 1.2.** Let $X = (X_\lambda, p_{X_{\lambda'}}; \Lambda)$ be an inverse system of compact Hausdorff spaces with limit $X = \limdir X$ and canonical projections $p_{X_{\lambda}}: X \to X_{\lambda}, \lambda \in \Lambda$. If $M$ is an ANR for metrizable spaces, then every mapping $f: X \to M$ admits a $\lambda \in \Lambda$ such that, for every $\mu \geq \lambda$, there exists a mapping $f_{\mu}: X \to M$, having the property that $f_{\mu} p_{\lambda} = f$. Moreover, if $\nu \geq \mu \geq \lambda$, then $f_{\nu} p_{\mu \nu} = f_{\nu}$.
**Lemma 1.3.** Let \((Y,X) = ((Y_\lambda, X_\lambda), q_{\lambda\lambda'}, \Lambda)\) be an inverse system of compact Hausdorff pairs, \(X_\lambda \subseteq Y_\lambda\), with limit \((Y,X) = \lim (Y,X)\) and canonical projections \(q_\lambda: (Y,X) \rightarrow (Y_\lambda, X_\lambda), \lambda \in \Lambda\). Let \(p_{\lambda\lambda'}: X_{\lambda'} \rightarrow X_\lambda\) and \(p_\lambda: X \rightarrow X_\lambda\) be the restrictions of the mappings \(q_{\lambda\lambda'}\) and \(q_\lambda\) to \(X_{\lambda'}\) and \(X\), respectively. If \(M\) is an ANR for metrizable spaces and for a given \(\lambda \in \Lambda\), \(f_\lambda: X_\lambda \rightarrow M\) is a mapping such that \(f_\lambda p_\lambda: X \rightarrow M\) admits an extension \(g: Y \rightarrow M\) to all of \(Y\), then there is a \(\mu \geq \lambda\) such that \(f_\lambda p_\mu: X_\mu \rightarrow M\) admits an extension \(g_\mu: Y_\mu \rightarrow M\) to all of \(Y_\mu\).

By an extension we always mean a continuous extension. Lemmas 1.2 and 1.3 appear in [12] as Theorems 4 and 5. Unfortunately, the proofs given in that paper are not correct. Indeed, they are based on the following theorem of R. Arens (see [3, Theorem 4.1]).

Let \(C\) be a closed convex subset of a Banach space \(L\). Every mapping \(f: A \rightarrow C\) of a closed subset \(A\) of a (Hausdorff) paracompact space \(X\) to \(C\) admits an extension \(g: X \rightarrow C\) to the whole space \(X\).

The assumption that \(C\) is closed in \(L\) is not fulfilled in the application made in [12]. In fact, there the Arens’ theorem was misstated, because “closed convex” was replaced by “convex”.

In the present paper we give correct proofs of Lemmas 1.2 and 1.3 and thus, we obtain a correct proof of Theorem 1.1. The new proof of Lemma 1.3 uses the following proposition.

**Proposition 1.4.** Every ANR for metrizable spaces is an ANE for compact Hausdorff spaces.

Yu.T. Lisica proved the following theorem (see [10, Theorem 1]).

Every convex subset of a Banach space is an AE for paracompact \(p\)-spaces.

By the well-known Kuratowski-Wojdulski embedding theorem, every metrizable space embeds in a Banach space as a closed subset of its convex hull ([17], also see [14, I.3.1, Theorem 2]). Therefore, the following proposition holds.

**Proposition 1.5.** Every ANR for metrizable spaces is an ANE for paracompact \(p\)-spaces.

\(p\)-spaces were introduced by A.V. Arhangelskiǐ ([4]), who proved that paracompact \(p\)-spaces coincide with spaces which admit perfect mappings to metrizable spaces ([4, Theorem 16]). It is clear from this characterization, that compact Hausdorff spaces are paracompact \(p\)-spaces. Consequently, Proposition 1.4 is an immediate consequence of Proposition 1.5. In fact, Proposition 1.4 is an immediate consequence of the Kuratowski-Wojdulski embedding theorem and the following special case of the theorem of Lisica.
Proposition 1.6. Every convex subset C of a Banach space is an AE for compact Hausdorff spaces.

Although the paper [12] has been published more than 50 years ago, to the author it appears justified to correct the faulty proofs, because the paper has been cited at least in the papers [1, 2, 5, 8, 9, 15] and [16] and Theorem 1.1 was used in an essential way in [15] and [16].

2. Proofs of Lemmas 1.2 and 1.3

In the proofs we will use the fact that the limit $p = (p_{x}, \Lambda): X \rightarrow X$ of an inverse system of compact Hausdorff spaces $X = (X_{\lambda}, p_{x\lambda}, \Lambda)$ is a resolution ([13, Theorem 6.20]) and a homotopy expansion ([13, Corollary 7.8]). This allows us to use property (R1) (see [13, page 104 and Lemma 6.3]), property (B2) (see [13, page 107, Theorem 6.7 and Remark 6.13]) and Morita’s property (M1) ([13, page 129]).

Proof of Lemma 1.2. Since $p: X \rightarrow X$ is a homotopy expansion, it has Morita’s property (M1). Therefore, for a mapping $f: X \rightarrow M$ into an ANR for metrizable spaces $M$, there exist a $\lambda \in \Lambda$ and a mapping $f_{\lambda}: X_{\lambda} \rightarrow M$ such that $f_{\lambda}p_{x} \simeq f$. For $\mu \geq \lambda$, put $f_{\mu} = f_{\lambda}p_{x\mu}$. Note that $f_{\mu}p_{\mu} = f_{\lambda}p_{x\mu}p_{\mu} = f_{\lambda}p_{x} \simeq f$. If $\nu \geq \mu$, then $\nu \geq \lambda$ and thus, $f_{\nu} = f_{\lambda}p_{x\nu} = f_{\lambda}p_{x\mu}p_{\mu} = f_{\mu}p_{\mu}$.

Proof of Lemma 1.3. For $\lambda \in \Lambda$, let $f_{\lambda}: X_{\lambda} \rightarrow M$ and $g: Y \rightarrow M$ be mappings such that $g|X = f_{\lambda}|X$. Since $M$ is an ANR for metrizable spaces and $X_{\lambda}$ is a closed subset of the compact Hausdorff space $Y_{\lambda}$, Proposition 1.4 yields an open neighborhood $U_{X}$ of $X_{\lambda}$ in $Y_{\lambda}$ and an extension $f_{\lambda}: U_{X} \rightarrow M$ of $f_{\lambda}$. Since $M$ is an ANR for metrizable spaces, there exists an open covering $V$ of $M$ such that any two $V$-near mappings into $M$ are homotopic ([14, I.3.2, Corollary 1]). Since $q$ is a resolution, by property (R1) for $q$, there exist a $\lambda' \in \Lambda$ and a mapping $g_{\lambda'}: Y_{\lambda'} \rightarrow M$ such that the mappings $g_{\lambda'}q_{\lambda'}, g: Y \rightarrow M$ are $\nu$-near. There is no loss of generality in assuming that $\lambda' \geq \lambda$.

Let us show that $g_{\lambda'}p_{\lambda'}(X)$ and $f_{\lambda}p_{\lambda'}|p_{\lambda'}(X)$ are $\nu$-near mappings. Indeed, if $x_{\lambda} \in p_{\lambda'}(X)$, there is a point $x \in X$ such that $x_{\lambda} = p_{\lambda}(x)$. Note that $g_{\lambda}(x_{\lambda}) = g_{\lambda}p_{\lambda}(x) = g_{\lambda}q_{\lambda}(x)$. Therefore, there is a $V \in \nu$ such that $g_{\lambda}(x_{\lambda}), g(x) \in V$. The assertion follows, because $g(x) = f_{\lambda}p_{x} = f_{\lambda}p_{\lambda}p_{x}(x) = f_{\lambda}p_{\lambda}(x_{\lambda}) = g_{\lambda}(x_{\lambda})$ and thus, $g_{\lambda}(x_{\lambda}), f_{\lambda}p_{\lambda}(x_{\lambda}) \in V$.

Now put $U_{\lambda'} = (q_{\lambda'})^{-1}(U_{\lambda}) \subseteq Y_{\lambda'}$ and note that $p_{\lambda'}(X) \subseteq U_{\lambda'}$, because $q_{\lambda'}p_{\lambda'}(X) = p_{\lambda'}(X) = p_{\lambda}(X) \subseteq X_{\lambda} \subseteq U_{\lambda}$. Also note that for $y_{\lambda'} \in U_{\lambda'}$, one has $q_{\lambda'}(y_{\lambda'}) \in U_{\lambda}$ and therefore, $f_{\lambda}q_{\lambda'}: U_{\lambda'} \rightarrow M$ is well defined. Moreover, since the restrictions $g_{\lambda'}|p_{\lambda'}(X)$ and $f_{\lambda}q_{\lambda'}|p_{\lambda'}(X) = f_{\lambda}p_{\lambda'}|p_{\lambda'}(X) = f_{\lambda}p_{\lambda'}|p_{\lambda'}(X)$ are $\nu$-near mappings, there exists an open neighborhood $U_{\lambda'} \subseteq U_{\lambda}$ of $p_{\lambda'}(X)$ in $Y_{\lambda}$ such that the mappings $g_{\lambda'}|U_{\lambda'}$, and $f_{\lambda}q_{\lambda'}|U_{\lambda'}$ are also $\nu$-near mappings.
Since $U'_X \cap X_M$ is an open neighborhood of $p_M(X)$ in $X_M$, property
(B2) of the inverse limit $p = (p_M, A^*): X \to X$ yields a $\mu \geq X$ such that
$p_{X_M}(X_M) \subseteq U'_X \cap X_M \subseteq U'_X$. We now define a mapping $g_\mu: Y_\mu \to M$ by
the formula $g_\mu = g_{\lambda'q_{\lambda',\mu}}$. Let us show that $g_\mu|X_\mu$ and $f_\lambda p_{\lambda\mu}$
are $\mathcal{V}$-near mappings and therefore,

$$g_\mu|X_\mu \simeq f_\lambda p_{\lambda\mu}.$$

Indeed, if $x_\mu \in X_\mu$, then $p_{X_M}(x_\mu) \in U'_X$ and therefore, there is a $V \in \mathcal{V}$ such that
the points $g_{\lambda'q_{\lambda',\mu}}(x_\mu), T_{\lambda'}g_{\lambda'q_{\lambda',\mu}}(x_\mu) \in V$. However,
$g_{\lambda'q_{\lambda',\mu}}(x_\mu) = g_\mu(x_\mu)$ and $T_{\lambda'}g_{\lambda'q_{\lambda',\mu}}(x_\mu) = f_\lambda p_{\lambda\mu}(x_\mu)$, because
$p_{X_M}(x_\mu) \in X_M$ and thus, $g_{\lambda'q_{\lambda',\mu}}(x_\mu) = p_{\lambda\mu}(x_\mu)$.

Clearly, the mapping $g_\mu|X_\mu: X_\mu \to M$ admits an extension to all of $Y_\mu$, because $g_\mu: Y_\mu \to M$ is such a mapping. Now the homotopy extension
property (HEP) shows that the mapping $f_\lambda p_{\lambda\mu}$, being homotopic to $g_\mu|X_\mu$, also admits an extension to all of $Y_\mu$. 

**Remark 2.1.** Recall that a pair of spaces $(Z, B)$, where $B$ is a closed
subset of $Z$, is said to have the HEP with respect to a space $M$, provided every
mapping $F: (B \times I) \cup (Z \times 0) \to M$ admits an extension $H: (Z \times I) \to M$. The
well-known Dowker lemma asserts that this is the case, whenever $Z$ is normal
and there is a neighborhood $U$ of $B$ in $Z$ such that $F$ admits an extension
$G: (U \times I) \cup (Z \times 0) \to M$ ([7, Lemma IV.2.1]). Consequently, if $C$ is a class
of normal spaces, having the property that $Z \in C$ implies $Z \times I \in C$, it follows
that every pair $(Z, B)$, where $Z \in C$, has the HEP with respect to any space $M$,
which is an absolute neighborhood extensor for the class $C$. In particular,
Proposition 1.4 shows that this is the case when $C$ is the class of compact
Hausdorff spaces and $M$ is an ANR for metrizable spaces.

3. A more elementary proof of Proposition 1.6

**Proof.** Let $(X, A)$ be a pair of compact Hausdorff spaces and let $f: A \to C$
be a mapping to a convex subset $C$ of a Banach space $L$. We must exhibit
an extension $g: X \to C$ of $f$ to all of $X$. First note that there is an extension
$F: X \to L$ of $f: A \to C \subseteq L$. It suffices to apply the Arens theorem to $L$,
viewed as a closed convex subset of $L$. Now note that $(L, f(A))$ is a pair
of metrizable spaces; $f(A)$ is closed in $L$, because it is compact. Consider
the inclusion mapping $i: f(A) \to C$. Applying the well-known Dugundji
extension theorem ([6], also see [14, I.3.1, Theorem 3]) to $(L, f(A))$ and $C \subseteq L$,
one obtains a mapping $h: L \to C$ such that $h|f(A) = i$. Clearly, $g = hF: X \to C$
is a mapping which extends $g$, because $g(a) = hF(a) = hF(a) = f(a)$, for
$a \in A$.

**References**


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