Snails in Hyperbolic Plane

ABSTRACT
The properties of the limaçon of Pascal in the Euclidean plane are well known. The aim of this paper is to obtain the curves in the hyperbolic plane having the similar properties. That curves are named hyperbolic snails and defined as the circle pedal curves. It is shown that all of them are circular quartics, while some of them are entirely circular.

Key words: limaçon of Pascal, hyperbolic plane, entirely circular 4-order curves

MSC 2010: 51M09, 51M10, 51M15

1 Introduction

The properties of the limaçon of Pascal in the Euclidean plane are well known. It is a bicircular curve of fourth order, that can be obtained as a circle pedal curve, ([5], pp. 133–134). The pedal point (pole) $P$ is a node, cusp or isolated double point of this pedal curve depending on whether it is outside, on or inside the circle. The limaçon has cusps at the absolute points and a singular focus that coincides with the midpoint of the segment $OP$, where $O$ is the center of the circle. There are the limaçons of Pascal that have singular foci lying on them.

The limaçon of Pascal can also be obtained by the inversion of a conic if its focus coincides with the pole of the inversion, ([5], p. 122). The limaçon of Pascal possesses a node, cusp or isolated double point depending on whether the generating conic is a hyperbola, parabola or ellipse, respectively.

The limaçon of Pascal possesses an axis of symmetry. One can ask himself if there is a curve in the hyperbolic plane having the similar properties.

Let $a$ be the absolute conic for the Cayley-Klein model of the hyperbolic plane (H-plane) represented by a circle of classical Euclidean plane. The interior points of the absolute conic are called real points, exterior points are ideal points and points of the absolute conic are called absolute points, [4].

A perpendicularity in the H-plane is defined by the absolute polarity. This means that two lines are perpendicular iff one passes through the absolute pole of the other. The pedal of a given curve with respect to a point $P$ is the locus of the foot of the perpendicular from the point $P$ to the tangent line to the given curve, [4].

A curve in the H-plane is circular if it touches the absolute conic at least at one point. If a curve possesses a common tangent with the absolute conic (isotropic asymptote) at each intersection point, it is entirely circular, [6].

A curve having two touching points with the absolute conic, possess a singular focus defined as an intersection of isotropic tangent lines at the absolute points, [3].

2 Hyperbolic snail

Definition 1 A hyperbolic snail (H-snail) is a circle pedal curve.

There are three classes of circles in the H-plane. Therefore, different types of H-snail can be expected. The circles are classified into the hypercycles, cycles and horocycles depending on whether they touch the absolute conic at two different real points, at a pair of imaginary points, or whether their four absolute points coincide, respectively, [4].
Theorem 1 \textit{H-snail is a fourth order curve touching the absolute conic at two points.}

\textbf{Proof.} Let us construct the pedal curve $k^4$ of a circle $c_2$. Let the given circle $c_2$ be e.g. a hypercycle with the center $O$ and absolute touching points $O_1$ and $O_2$ and let $P$ be the pole of the pedal transformation, Figure 1. The construction should be made in the following way:

The connecting line $PT$, where $T$ is the absolute pole of the tangent line $t$ of the circle $c_2$, intersects $t$ in a point $T_N$ lying on the required curve.

Absolute touching points $O_1, O_2$ obviously lie on the curve $k^4$ since they are the feet of the perpendiculars from the point $P$ to the isotropic asymptotes $OO_1$ and $OO_2$.

Through each of the intersection points $A_1, A_2$ of the absolute conic $a$ and the polar line $p$ of the pole $P$ with respect to $a$, pass two tangent lines to the hypercycle. Consequently, $A_1, A_2$ are double points of the required curve. According to Chasles correspondence principal, $k^4$ is fourth order curve as it is the result of $(1, 2)$-correspondence between the first order pencil of lines $(P)$ and the second order pencil $(c_2)$ of the tangent lines of the conic $c_2$.

Two tangent lines to the hypercycle $c_2$ pass through the point $P$. Their poles are located on the polar line $p$. The connecting lines of those points with the point $P$ are the tangent lines of $k^4$ at its double point $P$.

The constructed curve $k^4$ is the fourth order curve touching the absolute conic at $O_1, O_2$. It has three double points $O, A_1, A_2$, and a quadruple focus $O$.

If the given circle $c_2$ is cycle or horocycle, it is similar, Figure 2. The cycle touches the absolute conic at the pair of imaginary points, and the same holds for its pedal curve $k^4$. The horocycle pedal curve hyperosculates the absolute conic $a$ at the touching point $O = O_1 = O_2$ with the intersection multiplicity 4. \hfill \Box

Corollary 1 \textit{If the pole $P$ of the pedal transformation is an absolute point, H-snail is an entirely circular fourth order curve.}

\textbf{Proof.} If the pole $P$ lies on the absolute conic, three double points $P, A_1, A_2$ of H-snail coincide with the point $P$ at which both tangents coincide with the line $p$, Figure 2.

Generally, an entirely circular quartic in the H-plane possesses six quadruple foci. The isotropic tangent lines $OO_1$, $OO_2$ of $k^4$ intersect the twice counted isotropic tangent line $p$ at the points $F_1, F_2$, respectively. Therefore, entirely circular H-snail $k^4$ possesses two quadruple foci $O, P$, and two eightfold foci $F_1, F_2$. \hfill \Box

Remark 1 \textit{If the pedal point $P$ is a real point, $k^4$ has two imaginary double points on the absolute conic. These are imaginary contact points of the absolute conic and tangents passing through the point $P$.}

The pole $P$ is a node, cusp or isolated double point depending on whether it is outside, on or inside the circle $c_2$, respectively.
Remark 2 Let us denote by $c_2$ the reciprocal curve of the circle $c_2$ in the absolute polarity, Figure 1. It’s easy to see that the pedal curve $k_4$ of the circle $c_2$ is the inverse curve of the circle $c_2$ with respect to the same pole $P$, [8]. This fact simplifies derivation of some constructive and synthetic conclusions about H-snails.

The limaçon of Pascal possesses an axis of symmetry. The analogous statement holds in the hyperbolic plane.

Theorem 2 Let the H-snail $k_4$ be the pedal curve of the circle $c_2$ with the center $O$ with respect to the pedal point $P$. The line $s = OP$ is an axis of symmetry of the H-snail.

Proof. The fact that every circle of the H-plane is a collinear image of the absolute conic $a$, [3], will be applied in the proof. Let $T_N$ be a point on $k_4$, obtained as the foot of the perpendicular from the pole $P$ to the tangent line $t$, Figure 4. Let $q$ be the line through $T_N$ perpendicular to $s$. The connecting line $z = ST$ intersects the circle $c_2$ in the point $T'$, being the pole of the tangent line $t'$ and inverse of the point $T_N$. By $A$ and $B$ the intersection points of line $s$ with lines $z$ and $q$ are denoted. A circle is a symmetric curve with respect to every diameter. In other words, the points $T$ and $T'$ are equally distanced from the line $s$ and equality $(SA, TT') = -1$ holds. After connecting these four points with the pole $P$ a harmonic quadruple of lines is obtained, [1]. Accordingly, $(SB, T_N T'_N) = -1$. This finishes the proof. □

For further studies of the entirely circular quartics in the hyperbolic plane the papers [2], [7] and [8] can be consulted.

References