SECOND-METACYCLIC FINITE 2-GROUPS

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Abstract. Second-metacyclic finite 2-groups are finite 2-groups with some non-metacyclic maximal subgroup and with all second-maximal subgroups being metacyclic. According to a known result there are only four non-metacyclic finite 2-groups with all maximal subgroups being metacyclic. The groups pointed in the title should contain some of these groups as a subgroup of index 2. There are seventeen second-maximal finite 2-groups, four among them being of order 16, ten of order 32 and three of order 64.

1. Introduction

A group $G$ is called metacyclic if there exists a cyclic normal subgroup $N$ of $G$ with cyclic factorgroup $G/N$. A group with some non-metacyclic maximal subgroup and with all second-maximal subgroups being metacyclic we call a second-metacyclic group. The aim of this article is to determine all second-metacyclic finite 2-groups.

The starting point is the following result of N. Blackburn.

Theorem 1.1. (see Janko [1, Th. 7.1]) Let $G$ be a minimal non-metacyclic 2-group. Then $G$ is one of the following groups:

(a) The elementary abelian group $E_8$ of order 8,
(b) The direct product $Q_8 \times Z_2$,
(c) The central product $Q_8 \ast Z_4$ of order $2^4$,
(d) $G = \langle a, b, c \mid a^4 = b^4 = [a, b] = 1, c^2 = a^2b^2, a^c = a^{-1}, b^c = a^2b^3 \rangle$.

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where \( G \) is special of order \( 2^5 \) with \( \exp(G) = 4 \), \( \Omega_1(G) = G' = Z(G) = \Phi(G) = \langle a^2, b^2 \rangle \cong E_4 \) and \( M = \langle a \rangle \times \langle b \rangle \cong Z_4 \times Z_4 \) is the unique abelian maximal subgroup of \( G \).

For brevity, we denote the second-metacyclic groups as \( MC(2) \)-groups. It is clear that their non-metacyclic maximal subgroups are minimal non-metacyclic groups and thus each \( MC(2) \)-group contains some group from Th. 1.1 as maximal subgroup. Especially, an \( MC(2) \)-group \( G \) is of order 16, 32 or 64. Our main result is stated in the following theorem.

**Theorem 1.2.** Let \( G \) be a second-metacyclic group. Then \( G \) is one of the following 17 groups:

(a) four groups of order 16:
\[ E_{16}, Z_4 \times E_4, D_8 \times Z_2, \text{ or the semidirect product } E_4 \cdot Z_4; \]

(b) ten groups of order 32:

(b1) \( G \) contains a subgroup \( H \) isomorphic with \( Q_8 \times Z_2 \) so that
\[
H = \langle a, b, c | a^4 = 1, b^2 = a^2, c^2 = 1, a^b = a^{-1}, a^c = a, b^c = b \rangle \\
\quad = \langle a, b \rangle \times \langle c \rangle
\]
\[
G_1 = \langle H, d | d^2 = 1, [a, d] = [b, d] = 1, c^d = a^2 c \rangle \\
\quad = \langle a, b \rangle * \langle c, d \rangle \cong Q_8 * D_8,
\]
\[
G_2 = \langle H, d | d^2 = 1, a^d = a, b^d = abc, c^d = a^2 c \rangle,
\]
\[
G_3 = \langle H, d | d^2 = a^2, a^d = a^{-1}, b^d = ab, c^d = c \rangle,
\]
\[
G_4 = \langle H, d | d^2 = a^2, a^d = a^{-1}, b^d = bc, c^d = c \rangle,
\]
\[
G_5 = \langle H, d | d^2 = c, a^d = a^{-1}, b^d = ab, c^d = c \rangle,
\]
\[
G_6 = \langle H, d | d^2 = c, [a, d] = [b, d] = [c, d] = 1 \rangle \\
\quad = \langle a, b \rangle \times \langle d \rangle \cong Q_8 \times Z_4,
\]
\[
G_7 = \langle H, d | d^2 = a, a^d = a, b^d = bc, c^d = a^2 c \rangle;
\]

(b2) \( G \) contains a subgroup \( H \) isomorphic with \( Q_8 \times Z_2 \) so that
\[
H = \langle a, b, c | a^4 = 1, b^2 = c^2 = a^2, a^b = a^{-1}, a^c = a, b^c = b \rangle \\
\quad = \langle a, b \rangle * \langle c \rangle
\]
and if \( L \leq G \), then \( L \not\cong Q_8 \times Z_2 \):
\[
G_8 = \langle H, d | d^2 = c, [a, d] = [b, d] = [c, d] = 1 \rangle,
\]
\[
G_9 = \langle H, d | d^2 = 1, a^d = a^{-1}, b^d = ab, c^d = c \rangle,
\]
\[
G_{10} = \langle H, d | d^2 = ac, a^d = a, b^d = ab, c^d = c \rangle; \]
(c) three groups of order 64:
\[ G > H = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2b^2, ab = a, a^{-1}, b^c = a^2b^3 \rangle : \]
\[ G_1 = \langle H, d \mid d^2 = a^2, a^d = a^3b^2, b^d = b^{-1}, c^d = c \rangle, \]
\[ G_2 = \langle H, d \mid d^2 = b^2, a^d = a^{-1}, b^d = a^2b^3, c^d = ac \rangle, \]
\[ G_3 = \langle H, d \mid d^2 = b, a^d = ab^2, b^d = b, c^d = ac \rangle. \]

We proceed by recalling some useful known results.

**Lemma 1.3.** Let \( E \) be an elementary abelian subgroup of a 2-group, and \( g \in G, \ g^2 \in E \). Then \( |C_E(g)|^2 \geq |E| \).

**Proof.** Because of \( g^2 \in E \) and \( E \) abelian, \( x^g \neq x \) for any \( x \in G \). Thus \( (xx^g)^2 = x^g x^3 = x^3 x = xx^g \), for all \( x \in G \), and so \( xx^g \in C_E(g) \). Now, for \( y \in G \), we have \( xx^g = yy^g \) if and only if \( xy \in C_E(g) \), which is equivalent with \( C_E(g) x = C_E(g) y \). Therefore \( xx^g \neq yy^g \) if and only if \( C_E(g) x \neq C_E(g) y \), and so \( |C_E(g)| \geq |E : C_E(g)| \). It follows \( |C_E(g)|^2 \geq |E| \). \( \square \)

**Theorem 1.4.** (see Janko [1, Proposition 1.9]) Let \( G \) be a p-group with a non-abelian subgroup \( P \) of order \( p^3 \). If \( C_G(P) \leq P \), then \( G \) is of maximal class. Especially, if \( p = 2 \), \( G \) is metacyclic.

**Theorem 1.5.** (see Janko [1, Proposition 1.14]) A 2-group \( G \) is metacyclic if and only if \( G \) and all its subgroups are generated by two elements.

**Theorem 1.6.** Let \( G \) be a group of order 16. Then \( G \) is isomorphic to some of the following groups:

\[ \text{(a) abelian groups: } Z_{16}, Z_8 \times Z_2, Z_4 \times Z_4, Z_4 \times E_4, E_{16}; \]
\[ \text{(b) non-abelian groups containing a cyclic maximal subgroup: } \]
\[ G = \langle a, b \mid a^8 = b^2 = 1, a^b = a^{-1} \rangle \cong D_{16} - \text{dihedral group}, \]
\[ G = \langle a, b \mid a^8 = b^4 = 1, a^b = a^{-1} \rangle \cong Q_{16} - \text{quaternion group}, \]
\[ G = \langle a, b \mid a^8 = b^2 = 1, a^b = a^{3} \rangle \cong SD_{16} - \text{semidihedral group}, \]
\[ G = \langle a, b \mid a^8 = b^2 = 1, a^b = a^{3} \rangle \cong M_{16} \]
\[ - \text{M-group, which is minimal non-abelian}; \]
\[ \text{(c) non-abelian groups with } \exp(G) = 4 : \]
\[ \text{(c1) } G \text{ containing a subgroup isomorphic with } E_8 : \]
\[ G \cong D_8 \times Z_2, \]
\[ G \cong E_4 \cdot Z_4, \text{ the semidirect product of } E_4 \text{ by } Z_4; \]
\[ \text{(c2) } G \text{ being } E_8 \text{-free. Then } G \text{ contains } H \cong Z_4 \times Z_2. \text{ If } \Omega_1(G) \not\leq H, \text{ then } G \cong Q_8 \times Z_4. \text{ If } \Omega_1(G) \leq H, \text{ then either } G \cong Q_8 \times Z_2 \text{ or } G \cong Z_4 \cdot Z_4, \text{ the semidirect product of } Z_4 \text{ by } Z_4, \text{ which is minimal non-abelian.} \]
2. Proof of the Theorem 1.2

(a) Groups of order 16.
As group $E_8$ is the only (minimal) nonmetacyclic group of order 8, the second metacyclic groups of order 16 are those among them which contain $E_8$. According to Th. 1.6, we have four such groups: $Z_4 \times E_4$, $E_16$, $D_8 \times Z_2$ and $E_8 \cdot Z_4$, the semidirect product of $E_4$ by $Z_4$.

(b) Groups of order 32.
According to Th. 1.1, such a group contains a subgroup isomorphic to $Q_8 \times Z_2$, or to $Q_8 \cdot Z_4$, the central product of $Q_8$ by $Z_4$.

(1) $G$ contains a subgroup $H$ isomorphic with $Q_8 \times Z_2$:
Let $H = \langle a, b, c \mid a^4 = 1, b^2 = a^2, c^2 = 1, a^b = a^{-1}, [a, c] = [b, c] = 1 \rangle = \langle a, b \rangle \cong Q_8 \times Z_2$. Now, $|G : H| = 2$, $G = \langle H, d \rangle$, $d^2 \in H$. Since $\Phi(H) = \Omega_1(H) = \langle x^2 \mid x \in H \rangle = \langle a^2 \rangle$ and $\Omega_1(H) = \langle x \in H \mid x^2 = 1 \rangle = \langle a^2 \rangle$, thus $(a^2)^1, (a^2, c) < H$ and $(a^2, c) < G$. The maximal subgroups of $H$ are the following ones: $\langle a, b \rangle \cong \langle ac, b \rangle \cong \langle ac, bc \rangle \cong Q_8$, and $\langle a, c \rangle \cong \langle b, c \rangle \cong \langle ab, c \rangle \cong Z_4 \times Z_2$.

We use the bar convention for subgroups and elements of factor groups.

For $\overline{G} = G/\langle a^2 \rangle$, we have $\overline{H} = H/\langle a^2 \rangle = \langle \overline{a}, \overline{b}, \overline{c} \rangle \cong E_8$, and $\overline{G} = \langle \overline{H}, \overline{d} \rangle$, $\overline{d}^2 \in \overline{H}$. By Lemma 1.3, it is $|C_{\overline{H}}(\overline{d})|^2 \geq |H| = 8$, and so $|C_{\overline{H}}(\overline{d})| \geq 4$. On the other hand, $(a^2, c) < G$ implies $(\overline{a}, \overline{c}) < \overline{G}$ and $\overline{c} \in C_{\overline{H}}(\overline{d})$. As $|C_{\overline{H}}(\overline{d}) \cap C_{\overline{H}}(\overline{d})| \geq 2$, some of the elements $\overline{a}, \overline{b}$, or $\overline{ab}$ is contained in $C_{\overline{H}}(\overline{d})$, and we can assume without loss that $\overline{c} \in C_{\overline{H}}(\overline{d})$. Now, $\overline{H} \setminus \langle \overline{a}, \overline{c} \rangle = \langle \overline{a}, \overline{c} \rangle \cdot \overline{b}$, and so: $G = \langle H, d \mid d^2 \in H, a^d = az_1, b^d = a^z b^2 z_2, c^d = cz_3 \rangle$, $\varepsilon, \eta \in \{0, 1\}$, $z_1, z_2, z_3 \in \{1, a^2\}$.

There are 3 cases with respect to the element $d$.

1) $\exists d \in G \setminus H$, s.th. $d^2 = 1$. Since $(a^2, c, d) \not\cong E_8$ it must be $c^d \neq c$, and so $c^d = a^2 c$. If $a^d = a^3$, then $(ac)^d = a^d c^d = a^3 a^2 c = ac$, and replacing $a$ with $ac$, we have without loss $a^d = a$, $b^d = a^2 c$. Now, $b^d = b^1 = b = (b^d)^d = (a^2 c b^2 z_2)^d = a^6 a^2 q c a^2 c b^2 z_2 z_2 = a^{2(x+y)} b$. Therefore $\varepsilon = \eta = 0$ or $\varepsilon = \eta = 1$. If $\varepsilon = \eta = 0$, then $b^d = b z_2$. For $z_2 = a^2$ it is $(bc)^d = b a^2 a^2 c = bc$, and replacing $bc$ with $b$, we have without loss $b^d = b$. Thus

\[ G_1 = \langle H, d \mid d^2 = 1, [a, d] = [b, d] = 1, c^d = a^2 c \rangle \cong \langle a, b \rangle \ast \langle c, d \rangle \cong Q_8 \ast D_8. \]

If $\varepsilon = \eta = 1$, then $b^d = ab z_2 = az_2 b c$. Now, replacing $a$ with $az_2$, we get without loss $b^d = abc$, and the group:

\[ G_2 = \langle H, d \mid d^2 = 1, a^d = a, b^d = abc, c^d = a^2 c \rangle. \]

2) $x \in G \setminus H \Rightarrow x^2 \neq 1$, $\exists d \in G \setminus H$ s.th. $d^4 = 1$.
Now $d^2 \in H$ and $d^2$ is an involution, $d^2 = \{a^2, c, a^2 c\}$. As $a^2 c$ and $c$ are interchangeable, we may assume that $d^2 \in \{a^2, c\}$. If $d^2 = c$, then $c^d = (d^2)^d = d^2 = c$. If $d^2 = a^2$, then $(cd)^d = ca^2 c d = ca^2 c z_2 = a^2 z_3 \neq 1$, by our assumption. Thus $z_3 = 1$, and so $c^d = c$ in both cases.
2a) Case $d^2 = a^2$.

Now, $(ad)^2 = ad^2a^d = a^3a_2z_1 = z_1 \neq 1$, by our assumption. Thus $z_1 = a^2$ and so $a^d = a^3$, $c^d = c$, $d^2 = a^2$. For $\varepsilon = 1$, $b^d = ac^\varepsilon b z_2$. Replacing $d$ with $z_2$, we may assume $b^d = ac^\eta b$. If $\eta = 1$, $b^d = abc$, and replacing $a$ with $ac$, we get $b^d = ab$, as in the case $\eta = 0$, and so

$$G_3 = \langle H, d \mid d^2 = a^2, a^d = a^{-1}, b^d = ab, c^d = c \rangle.$$ 

For $\varepsilon = 0$, it is $b^d = c^\eta b z_2$. If $z_2 = a^2$, then $b^d = (a^2 b)^d = a^2 c^\eta b a^2 = c^\eta b$. Replacing $d$ with $ad$, we get $b^d = c^\eta b$. For $\eta = 0$, $b^d = b$ and $(bd)^2 = bd^2 b^d = ba b^2 z_2 = 1$, a contradiction. Thus, $\eta = 1$, $b^d = bc$, and we have:

$$G_4 = \langle H, d \mid d^2 = a^2, a^d = a^{-1}, b^d = bc, c^d = c \rangle.$$ 

2b) Case $d^2 = c$.

For $\varepsilon = 1$, $b^d = ac^\varepsilon b z_2$, and replacing $a$ with $a z_2$, $b^d = ac^\eta b$. Again if $\eta = 1$, replacing $a$ with $ac$, we get $b^d = ab$. Now, $b^d = b = (b^d)^d = (ab)^d = a^4 b^d = a z_1 a b \Rightarrow z_1 = a^2$, and so:

$$G_5 = \langle H, d \mid d^2 = c, a^d = a^{-1}, b^d = ab, c^d = c \rangle.$$ 

For $\varepsilon = 0$, $b^d = c^\eta b z_2$. If $\eta = 1$, then $b^d = cbz_2$, and $(bd)^2 = bd^2 b^d = bc b z_2 = b^2 z_2 = a^2 z_2 \neq 1$, by our assumption. Thus $z_2 = 1$ and $(bd)^2 = a^2$, which leads to the case 2a). Thus we may assume that $\eta = 0$, and so $b^d = bz_2$, $a^d = a z_1$, $c^d = c$. Thus $(a^d, b^d) \in \{(a, b), (a, b^3), (a^3, b), (a^3, b^3)\}$. As $a, b$ and $ab$ are interchangeable here, and for $a^d = a^3$, $b^d = b^3 \Rightarrow (ab)^d = a^3 b^3 = ab$, there remain, without loss, only two cases: $a^d = a$, $b^d = b$ and $a^d = a$, $b^d = b^3$.

In the latter case $(ad)^2 = ad^2 a d = ac a = a^2 c$, $a^d = a^d = a$, $b^d = (b^3)^d = b^9 = b$, $(a^2 c)^d = (a^2 c)^d = a^2 c$, and replacing $c$ with $a^2 c$, and $d$ with $ad$, we get without loss, that $a^d = a$, $b^d = b$, $c^d = c$, and thus

$$G_6 = \langle H, d \mid d^2 = c, [a, d] = [b, d] = [c, d] = 1 \rangle.$$ 

3) $d \in G \setminus H \Rightarrow |d| = 8$.

Now, $d^2 \in H$ and $|d^2| = 4$. As all elements of order 4 in $H$ are interchangeable, we may assume that $d^2 = a$, and so $a^d = a$. Now, $d^2 = a$, $a^d = a$, $b^d = a^2 c^\varepsilon b z_2$, $c^d = c z^3$. If $\varepsilon = 1$, then $(bd)^2 = bd^2 b^d = ba c a b z_2 = c^\eta z_2$, an involution, against our assumption. Therefore $\varepsilon = 0$, and $b^d = c^\eta b z_2$. Now, $b^d = b^\varepsilon = b^3 = (b^d)^d = (c^\varepsilon b z_2)^d = c^\varepsilon z^3 c^\varepsilon b z_2 z_2 = z^3 \Rightarrow z^3 = b^d = a^2 \Rightarrow z_3 = a^2$. $\eta = 1$. Thus $b^d = bc z_2$, $c^d = a^2 c$, $a^d = a$. If $z_2 = a^2$, replacing $c$ with $a^2 c$, we get $b^d = bc$ and finally:

$$G_7 = \langle H, d \mid d^2 = a, a^d = ab^d = bc, c^d = a^2 c \rangle.$$ 

(b2) $G$ contains a subgroup $H$ isomorphic with $Q_8 \ast Z_4$ and if $L \leq G$, then $L \not\cong Q_8 \times Z_2$.

Let $H = \langle a, b, c \mid a^4 = 1, b^2 = c^2 = a^2, a^b = a^{-1}, [a, c] = [b, c] = 1 \rangle$. Again, $G = \langle H, d \rangle$, $d^2 \in H$. Now $G_1(H) = \Phi(H) = \langle a^2 \rangle$, $Z(H) = \langle c \rangle$. There are 8 elements of order 4: $a$, $a^3$, $b$, $a^2 b$, $ab$, $a^3 b$, $c$, $a^2 c$, and 7 involutions:
$a^2$, $ac$, $a^3c$, $bc$, $a^2bc$, $abc$, $a^3bc$. The maximal subgroups of $H$ are: $\langle a, b \rangle \cong Q_8$, $\langle a, c \rangle \cong \langle b, c \rangle \cong \langle ab, c \rangle \cong Z_4 \times Z_2$ and $\langle a, bc \rangle \cong \langle b, ac \rangle \cong \langle ab, ac \rangle \cong D_8$. Obviously, $\langle c, (a, b \rangle \not< H$ and so $\langle c, (a, b \rangle \not< G$. Again, for $G = \langle a^2 \rangle$, we have $H/(a^2) \cong \langle \bar{x}, \bar{y}, \bar{z} \rangle \cong E_8$, and $|C_{\bar{H}}(\bar{d})| \geq 4$ by Lemma 1.3. We may assume, without loss, that $C_{\bar{H}}(\bar{d}) \geq \langle \bar{a}, \bar{y} \rangle$, which implies $\bar{d} \in \langle \bar{a}, \bar{y} \rangle$. Returning to the original, we get two cases:

1) $G = \langle H, d \mid d^2 \in H, a^2 = a,z_1, b^2 = b z_2, c^2 = c z_3 \rangle$,
2) $G = \langle H, d \mid d^2 \in H, a^2 = a z_1, b^2 = a b z_2, c^2 = c z_3 \rangle$,

where $z_1, z_2, z_3 \in \langle a^2 \rangle$.

Case 1)

1a) $\exists d \in G \setminus H, |d| = 2$.

Now, $d^2$ is an involution on $H$. We may assume, without loss, that $d^2 = a^2$ or $d^2 = ac$. If $d^2 = a^2$, then $(ad)^2 = ad^2 a^d = a a^2 a z_1 = z_1 \neq 1$, $(bd)^2 = b z_2 \neq 1$, and $(abd)^2 = z_1 z_2 \neq 1$, a contradiction. If $d^2 = ac$, then $b^d = b c a = b^3 \neq (b^d)^3 = (b z_2)^2 = b z_2 a = b$, a contradiction again.

1c) $x \in G \setminus H \Rightarrow |x| = 8$, $d \in G \setminus H$, $d^2 \in H$.

We may assume, without loss, that $d^2 = a$, or $d^2 = c$. If $d^2 = a, b d^2 = (b z_2)^2 = b = b a = b^3$, a contradiction. Thus $d^2 = c, a d = a z_1, b^d = b z_2, c^d = c$. Now, $(a d, b^d) \in \{(a, b), (a, b^3), (a^3, b), (a^3, b^3)\}$. In the latter case $(a d)^2 = a^3 b^3 = a b$, and since $a$, $b$ and $a b$ may be replaced with each other, we may assume that $a d = a, b d = b$ or $a d = a, b d = b^3$. In the latter case $(a d)^2 = a^2 c$, $b a d = b$, and thus replacing $c$ with $a^2 c$ and $d$ with $a d$, the second case is reduced to the first, and we get

$G_8 = \langle H, d \mid d^2 = c, [a, d] = [b, d] = [c, d] = 1 \rangle = \langle a, b \rangle \times \langle c, d \rangle \cong Q_8 \times Z_2$.

Case 2)

Replacing $a$ with $a z_2$, we may assume that $b d = a b$.

2a) $\exists d \in G \setminus H, |d| = 2$.

$\langle a d, b^d \rangle \not\cong E_8 \Rightarrow (ad)^2 = ac z_1 z_3 \neq ac \Rightarrow z_3 \neq z_1$. If $a d = a$, then $b d^2 = b = (b^d)^2 = (ad)^2 = a a b = b^3$, a contradiction. Therefore $a d = a^3, c^d = c$, and

$G_9 = \langle H, d \mid d^2 = 1, a d = a^{-1}, b d = a, c^d = c \rangle$.

2b) $x \in G \setminus H \Rightarrow x^2 \neq 1, \exists d \in G \setminus H, |d| = 4$.

Now, without loss $d^2 = a^2$ or $d^2 = ac$ or $d^2 = bc$. If $d^2 = a^2$, then $(ad)^2 = z_1 \neq 1$ and $(cd)^2 = z_3 \neq 1$, thus $z_1 = z_3 = a^2$. So $G = \langle H, d \mid d^2 = a^2, a d = a^3, b d = a b, c^d = c^3 \rangle$. Here $(ac)^d = a^3 c^3 = ac$, and $G > \langle a, d, ac \rangle$.
\langle a, d \rangle \times \langle ac \rangle \cong Q_8 \times Z_2$, against the assumption. Actually, $G \cong G_2$. If $d^2 = ac$, then $b^d = (ab)^d = a\gamma_1ab = b^{ac} = b^3$, and so $z_1 = 1$. Now, $(ac)^d = ac = a^{d,e}_{d} = acz_3$, thus also $z_3 = 1$. Therefore:

\[ G_{10} = \langle H, d \mid d^2 = ac, a^d = a, b^d = ab, c^d = c \rangle. \]

If $d^2 = bc$, then $(bc)^d = bc = abc_{z_3} = a z_3 bc$, implying $a z_3 = 1$, a contradiction.

2c) $x \in G \setminus H \Rightarrow |x| = 8, \ d \in G \setminus H, \ d^2 \in H$.

We may assume, without loss, that $d^2 = a$ or $d^2 = b$ or $d^2 = c$. For $d^2 = a$, we get $(bd)^2 = bd^2 = baab = 1$, a contradiction. As $b^d = ab$, it cannot be $d^2 = b$. If $d^2 = c$, then $(bd)^2 = bcab = b^2 ca^b = a^2 ca^{-1} = ac$, and so $|bd| = 4$, a contradiction again.

(c) Groups of order 64.

According a previous remark and by Th. 1.1(d), such a group $G$ contains a subgroup

\[ H = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2 b^2, a^b = a, a^c = a^{-1}, b^c = a^3 b^3 \rangle, \]

where $\Omega_4(H) = \langle x \in H \mid x^2 = 1 \rangle = Z(H) = \Phi(H) = \langle a^2, b^2 \rangle \cong K \cong E_4$. One can easily check that there are only 4 square roots for $a^2$ (that is, such $x \in H$, that $x^2 = a^2$), and 12 square roots for $b^2$ and $a^2 b^2$ each. Thus $A = \langle a^2 \rangle \triangleleft H$.

The square roots of $a^2$ generate the subgroup $N = \langle a, a^2 \rangle \cong Z_4 \times Z_2$. The group $L = \langle a, b \rangle$ is the unique subgroup of $H$ isomorphic to $Z_4 \times Z_4$. Thus, $A, K, N, L$ are all characteristic in $H$ and consequently normal in $G$, as $H \triangleleft G$.

It can easily be seen that

\[ \begin{align*}
\text{Aut} H &= \Phi \cup \Psi, \text{ where} \\
\Phi &= \{ \varphi \mid \varphi : a \mapsto a \zeta_1, b \mapsto b \zeta_2, c \mapsto a^\alpha b^{2 \beta} c \}, \\
\Psi &= \{ \psi \mid \psi : a \mapsto a \zeta_1, b \mapsto ab \zeta_2, c \mapsto a^\alpha b^{1 + 2 \beta} c \},
\end{align*} \]

and $\zeta_1, \zeta_2 \in K, \ \alpha \in \{0, 1, 2, 3\}, \ \beta \in \{0, 1\}$.

As $A \triangleleft K \triangleleft N \triangleleft L \triangleleft H \triangleleft G$ is a normal chain $G$, we have:

\[ G = \langle H, d \mid \langle H \rangle, \ d^2, a^d = a z_1, b^d = a^7 b z_2, c^d = a^7 b^5 z_3 c \rangle, \]

where $z_1, z_2, z_3 \in K$ and $\gamma, \delta, \varepsilon \in \{0, 1\}$ and $(H)$ denotes the relations of $H$.

We split our proof into several steps.

(i) If $T < G$, $|T| = 8$, then $T$ is abelian and $T \not\cong E_8$.

Let $T$ be a nonabelian subgroup of order 8 in $G$, thus $T \cong D_8$ or $T \cong Q_8$ and $|Z(T)| = 2$. If $C_G(T) \leq T$, then $G$ is metacyclic by Th. 1.4, a contradiction. Hence $C_G(T) \not\leq T$ and take in $C_G(T)$ a subgroup $U$ of order 4 containing $Z(T)$. Now $\langle T, U \rangle = T \ast U$, the central product of $T$ and $U$, is isomorphic to some of the groups $D_8 \times Z_2$, $Q_8 \times Z_2$, or $Q_8 \ast Z_4 \cong D_8 \ast Z_4$. Thus $(T, U)$ would be a non-metacyclic subgroup of order 16 in $G$, a contradiction. Therefore every subgroup $T$ of order 8 in $G$ is abelian and, being metacyclic, it must be $T \not\cong E_8$. 

(ii) If \( g \in G \setminus H \), then \( g^2 \neq 1 \).

If \( g^2 = 1 \), then \( T = (K, g) = (a^2, b^2, g) \) is abelian, by (i), and isomorphic to \( E_8 \), a contradiction.

(iii) \( G/L \cong E_4 \).

Else \( G/L \cong Z_4 \) and \( G = \langle L, d \mid d^2 \in H \setminus L = Lc \rangle \). By (i) we see that we can assume without loss that \( a^2 = c \), and so \( d^4 = c^2 = a^2b^2 \). Now \( a^d = a^c = a^3 = (a^d)^d = (a z_1)^d = a z_1 z_1^d \) implying \( z_1^d = a^2 z_1 \) and thus \( z_1 \in \{ b^2, a^2 b^2 \} \) and \( (b^2)^d = a^2 b^2 = c^2 = (c^2)^d \), and so \( b^2 = c^2 \), a contradiction.

(iv) If there exists some \( d \in G \setminus H \), s.th. \( |d| = 4 \), then either \( G \cong G_1 = \langle a, b, c, d \mid (H), d^2 = a^2, a^d = a^3b^2, b^d = b^3, c^d = c \rangle \), or \( G \cong G_2 = \langle a, b, c, d \mid (H), d^2 = b^2, a^d = a^3, b^d = a^2 b^3, c^d = ac \rangle \). As \( d^2 \in L \) and \( d^2 \) is an involution, we have \( d^2 \in \Omega_1(L)^2 = a^2, b^2, a^2 b^2 = K^2 \). By (i), \( \langle a^2, b^2, d \rangle \) is abelian, and so \( b^d = b z_2 \). Now, \( (c^2)^d = c^2 = (c^d)^2 = (a^2 b^2 z_2 c)^2 = a^2 b^2 z_2 c a^3 b^3 z_3 = a^2 b^2 c \), implying \( d = 0 \) and \( c^d = a^2 z_3 c \). From \( c^d = (a^c)^d = a^2 c^d, b^d = (a^b)^d = a^2 b^2 c, \) and \( e^d (a^c)^d = (a^2 a^2 b^2 c)^d = b^2 c^d \), and, by (ii), \( (a d)^2 = a^2 d^2 a^d = a^2 d^2 z_1 \neq 1 \), \( (b d)^2 = b^2 d^2 z_2 \neq 1 \) and \( (a b d)^2 = a^2 b^2 d^2 z_1 z_2 \neq 1 \), we conclude that

\[
(2.1) \quad z_1 \neq a^2 d^2, \quad z_2 \neq b^2 d^2, \quad z_1 z_2 \neq a^2 b^2 d^2,
\]

and replacing, if needed, \( d \) by \( a d \) or \( b d \) or \( a b d \), we can assume, without loss, that \( c^d = a^2 c \), and so:

\[
(2.2) \quad c^d = c \ \text{ or } \ c^d = ac.
\]

Case 1) \( c^d = c \).

Now \( (a d)^2 = c^2 d^2 = a^2 b^2 d^2 \neq 1 \), by (ii), and so \( d^2 \neq a^2 b^2 \). Therefore \( d^2 \in \{ a^2, b^2 \} \). If \( d^2 = b^2 \), then \( (a d)^2 = c^2 d^2 = a^2 \). Thus, replacing \( d \) by \( a d \), we may assume without loss that \( d^2 = a^2 \). Besides of (1), we also have now the following conditions on \( z_1, z_2 \): \( (a c)^2 = a c d^2 a z_1 c = ac^2 d^2 a^c z_1 = c^2 d^2 z_1 \neq 1 \), and similarly \( (b c d)^2 = b^2 d^2 z_2 \neq 1 \), \( (a b c d)^2 = abcd^2 a z_1 b z_2 c = b^2 d^2 z_1 z_2 \neq 1 \), that is:

\[
(2.3) \quad z_1 \neq c^2 d^2, \quad z_2 \neq b^2 d^2, \quad z_1 z_2 \neq b^2 d^2.
\]

From (2.1) and (2.3) we get: \( z_1 \neq 1, b^2, z_2 \neq a^2 b^2, z_1 z_2 \neq b^2, a^2 b^2 \). If \( z_1 = a^2 \), then \( a^d = a^3 \), and \( (a, d \mid a^d = 1, d^2 = a^2, a^d = a^3) \cong Q_8 \), a contradiction, by (i). Thus \( z_1 = a^2 b^2, z_2 = a^2 b^2, a^2 b^2 z_2 = b^2, a^2 b^2 \), and so \( z_2 = b^2 \), giving: \( G_1 = \langle a, b, c, d \mid (H), d^2 = a^2, a^d = a^3 b^2, b^d = b^3, c^d = c \rangle \).

Case 2) \( c^d = ac \).

We already know that \( a^d = a z_1, b^d = b z_2 \) and \( d^2 \in \{ a^2, b^2, a^2 b^2 \} \). Now \( c = c^d = (c^d)^d = (a^d)^d = a z_1 c, \) so \( z_1 = a^2 \) and \( a^d = a^3 \). If \( d^2 = a^2 \), then \( (a, d \mid a^d = 1, d^2 = a^2, a^d = a^3) \cong Q_8 \), against (i). Therefore \( d^2 \in \{ b^2, a^2 b^2 \} \). Since \( (b d)^2 = b a^2 d^2 = b^2 z_2 d^2 \), we have \( z_2 \neq b_2 d_2 \).
Case 2.1) \( d^2 = b^2 \).

From \( z_2 \neq b^2d^2 \) it follows \( z_2 \in \{a^2, b^2, a^2b^2\} \). If \( z_2 = a^2 \), then \( b^2 = a^2b \) and \( \langle a, bd | a^4 = 1, (bd)^2 = a^2, abd = a^3 \rangle \cong Q_8 \), against (i). Similarly, for \( z_2 = b^2 \) we have \( \langle b, d | b^4 = 1, d^2 = b^2, b^d = b^3 \rangle \cong Q_8 \) again. It remains as the only possibility \( z_2 = a^2b^2 \), and we obtain the group

\[ G_2 = \langle a, b, c, d | (H), d^2 = b^2, a^d = a^3, b^d = a^2b^3, c^d = ac \rangle. \]

Case 2.2) \( d^2 = a^2b^2 \).

Because of \( z_2 \neq b^2d^2 = a^2 \), we have now \( z_2 \in \{1, b^2, a^2b^2\} \), that is \( b^d \in \{b, b^3, a^2b^3\} \). If \( b^d = b \), then \( (bd)^2 = bd^2d = ba^2b^2b = a^2 \), and \( \langle a, bd \rangle \cong Q_8 \), against (i). If \( b^d = b^3 \), then \( (ab)^d = (ab)^3 \) and \( \langle ab, d \rangle \cong Q_8 \), again the same contradiction. Therefore \( b^d = a^2b^3 \). Replacing \( a \) by \( a' = a^3b^2 \), \( b \) by \( b' = ab \), \( c \) by \( c' = bc \), we get:

\[ d^2 = b'^2, \quad a'^d = (a^3b^2)^d = ab^2 = a^3, \]
\[ b'^d = (ab)^d = a^3a^2b^3 = ab^3 = (a^3b^2)^2 \cdot (ab)^3 = a^2b^3, \]
\[ c'^d = (bc)^d = a^2b^3ac = a^3b^2bc = a' \cdot c', \]

that is the relations of \( G_2 \). Thus, this group is isomorphic to \( G_2 \).

(v) If all elements in \( G \setminus H \) are of order 8, then

\[ G = G_3 = \langle a, b, c, d | (H), d^2 = b, a^d = ab^2, b^d = b, c^d = ac \rangle. \]

As \( G/L \cong E_4 \), \( G = \langle H, d \rangle, |d| = 8 \), it follows that \( d^2 \) is an element of order 4 in \( L \). According to (\( * \)), all such elements are replaceable by \( a \) or \( b \), and so we may assume without loss that

\[ G = \langle H, d | (H), d^2 \in \{a, b\}, a^d = az_1, b^d = az_2b, c^d = a^\gamma z_3c \rangle, \]

where \( z_1, z_2, z_3 \in K, \gamma, \delta, \varepsilon \in \{0, 1\} \). Now \( (c^2)^d = (c^d)^2 = (a^\gamma z_3c)^2 = a^\gamma b^\delta z_3c^2a^\gamma a^\varepsilon z_3 = a^2c^2 \).

Case 1) \( d^2 = a \).

Now \( a^d = a \). If \( \varepsilon = 0 \) then \( z^d = z \) for \( z \in K \), and from \( (c^2)^d = c^2 = a^2 = c^2 \) it follows \( \delta = 0, c^d = a^\gamma z_3c \). Similarly, from \( c^d = c^a = a^2c = (a^\gamma z_3c)^d = a^\gamma z_3a^\gamma z_3c = a^2c \), we get \( \gamma = 1 \) and \( c^d = az_3c \). But now \( (cd)^2 = cd^2c^d \) is \( a^2c^2z_3 \in K \), and \( |d| = 4 \), against our assumption. If \( \varepsilon = 1 \), then \( b^d = abz_2 \), \( (b^d)^d = c^2 \) and \( c^d = b^2 = a^2c^2 \). Thus \( \delta = 1 \), and \( c^d = a^\gamma z_3c \).

Therefore, \( c^d = a^2c = (abz_2)^d = a^\gamma a^\varepsilon z_3a^\gamma z_3 = az_3c \), for some \( z \in K \). This implies \( z = a \), a contradiction because \( a \) is not in \( K \).

Case 2) \( d^2 = b \).

Now \( b^d = b \), and \( z^d = z \) for \( z \in K \). From \( (c^2)^d = c^2 = a^2c^2 \) it follows \( \delta = 0, c^d = a^\gamma z_3c \). Since \( [b, c] = b^{-1}bc = a^2b^2 = [c, b] \), we have \( c^d = c^d = a^\gamma z_3c = a^\gamma z_3a^\gamma z_3a^\gamma z_3 \), implying \( a^2z_3^\gamma = a^2b^2 \). Thus \( \gamma = 1, z_1 = b^2 \) and so \( a^d = ab^2, c^d = az_3c \). Replacing \( a \) by \( az_3 \), we get the group \( G_3 \) as stated above.
Remark 2.1. It is of some interest to check the maximal subgroups of second-metacyclic groups. We present their distribution in each of these groups in the following table:

(i) \(-|G|=-32:\)

<table>
<thead>
<tr>
<th>(g)</th>
<th>(G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(5 \cdot (Q_8 \times Z_2), 10 \cdot (Q_8 * Z_4));</td>
</tr>
<tr>
<td>(2)</td>
<td>(Q_8 \times Z_2, Q_8 * Z_4, 2 \cdot \text{SD}<em>{16}, 2 \cdot Q</em>{16}, M_{16});</td>
</tr>
<tr>
<td>(3)</td>
<td>(2 \cdot (Q_8 \times Z_2), Z_8 \times Z_2, 4 \cdot Q_{16});</td>
</tr>
<tr>
<td>(4)</td>
<td>(2 \cdot (Q_8 \times Z_2), Z_4 \times Z_4, 4 \cdot (Z_4 \cdot Z_4));</td>
</tr>
<tr>
<td>(5)</td>
<td>(Q_8 \times Z_2, Z_4 \cdot Z_4, Z_8 \times Z_2)</td>
</tr>
<tr>
<td>(6)</td>
<td>(Q_8 \times Z_2, 3 \cdot (Z_4 \times Z_4), 3 \cdot (Z_4 \cdot Z_4));</td>
</tr>
<tr>
<td>(7)</td>
<td>(Q_8 \times Z_2, 2 \cdot M_{16});</td>
</tr>
<tr>
<td>(8)</td>
<td>(Q_8 * Z_4, 3 \cdot (Z_8 \times Z_2), 3 \cdot M_{16});</td>
</tr>
<tr>
<td>(9)</td>
<td>(2 \cdot (Q_8 * Z_4), Z_8 \times Z_2, 2 \cdot \text{SD}<em>{16}, Q</em>{16}, D_{16});</td>
</tr>
<tr>
<td>(10)</td>
<td>(Q_8 * Z_4, Z_4 \times Z_4, M_{16}).</td>
</tr>
</tbody>
</table>

(ii) \(|G| = 64:\)

Denoting \(H_r = \langle a, b \mid a^8 = b^4 = 1, a^b = a^r \rangle\), we have:

<table>
<thead>
<tr>
<th>(g)</th>
<th>(G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(15 \cdot H);</td>
</tr>
<tr>
<td>(2)</td>
<td>(2 \cdot H, 2 \cdot H_3, 2 \cdot H_7, Z_8 \times Z_4);</td>
</tr>
<tr>
<td>(3)</td>
<td>(H, 2 \cdot H_5).</td>
</tr>
</tbody>
</table>

The factor groups \(G_i = G_i / \langle a^2 \rangle\), \(i = 1, 2, 3\) are isomorphic to the groups \(G_1, G_4\) and \(G_7\) of order 32 from (i), respectively.

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