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SYMMETRIC (36,15,6) DESIGN HAVING U(3,3) AS AN AUTOMORPHISM GROUP

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ABSTRACT. Up to isomorphism there are four symmetric (36,15,6) designs with automorphisms of order 7. Full automorphism group of one of them is the Chevalley group $G(2,2) \cong U(3,3) : Z_2$ of order 12096. Unitary group $U(3,3)$ acts transitively on that design.

1. INTRODUCTION AND PRELIMINARIES

A symmetric (v,k,λ) design is a finite incidence structure $(\mathcal{P},\mathcal{B},I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P}\times\mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = |\mathcal{B}| = v$,
2. Every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. Every pair of elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

Let $\mathcal{D} = (P, B, I)$ be a symmetric (v,k,λ) design and $G \leq Aut\mathcal{D}$. Group G has the same number of point and block orbits. Let us denote the number of $G-$orbits by t, point orbits by $\mathcal{P}_1,\dots,\mathcal{P}_t$, block orbits by $\mathcal{B}_1,\dots,\mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$, $|\mathcal{B}_i| = \Omega_i$. We shall denote points of the orbit $\mathcal{P}_r$ by $\mathcal{P}_r = \{r_1,\dots,r_{\omega_r-1}\}$. Further, denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with the representative of the block orbit $\mathcal{B}_i$. For those numbers the following equalities hold:

$$\sum_{r=1}^{t} \gamma_{ir} = k, \tag{1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r}\gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij}\cdot(k-\lambda). \tag{2}$$

Definition 1. *The $(t\times t)$-matrix (γ_{ir}) with entries satisfying properties* (1) *and* (2) *is called the orbit structure for parameters (v,k,λ) and orbit distribution $(\omega_1,\dots,\omega_t)$, $(\Omega_1,\dots,\Omega_t)$.*

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Definition 2. *The set of indices of points of the orbit $\mathcal{P}_r$ indicating which points of $\mathcal{P}_r$ are incident with the representative of the block orbit $\mathcal{B}_i$ is called the index set for the position (i,r) of the orbit structure.*

2. Construction of the design

Let ρ be an automorphism of a symmetric design. We shall denote by $F(\rho)$ the number of points fixed by ρ. In that case, the number of blocks fixed by ρ is also $F(\rho)$.

Lemma 1. *Let ρ be an automorphism of a symmetric $(36,15,6)$ design. If $|\rho| = 7$, then $F(\rho) = 1$.*

Proof It is known that $F(\rho) < k+\sqrt{n}$ and $F(\rho) \equiv v(mod\,|\rho|)$. Therefore, $F(\rho) \in \{1,8,15\}$. If $F(\rho) = 8$, then the fixed structure must be a symmetric $(8,8,6)$ design. Such a design doesn't exist, therefore $F(\rho) \neq 8$. The case $F(\rho) = 15$ can be eliminated in the similar way. □

Lemma 2. *Up to isomorphism there are exactly two orbit structures for cyclic automorphism group of order 7 and a symmetric $(36,15,6)$ design. Those structures are:*

OS1	1	7	7	7	7	7
1	1	7	7	0	0	0
7	1	4	1	3	3	3
7	1	1	4	3	3	3
7	0	3	3	5	2	2
7	0	3	3	2	5	2
7	0	3	3	2	2	5

OS2	1	7	7	7	7	7
1	1	7	7	0	0	0
7	1	4	1	3	3	3
7	1	1	4	3	3	3
7	0	3	3	4	4	1
7	0	3	3	1	4	4
7	0	3	3	4	1	4

Proof Solving equations (1) and (2). □

Theorem 3. *Up to isomorphism there are four symmetric $(36,15,6)$ designs with automorphism of order 7. Let us denote them by $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$. Full automorphism groups of those designs are: $Aut\mathcal{D}_1 \cong Aut\mathcal{D}_2 \cong Frob_{21}$, $Aut\mathcal{D}_3 \cong G(2,2)$, $Aut\mathcal{D}_4 \cong Frob_{21} \times Z_2$.*

Proof Indexing of the column and row correponding to the fixed point and block is trivial. Therefore, we shall take into consideration only right-lower (5×5) submatrices of orbit structures. Indexing of the structure $OS1$ leads to designs $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$. Orbit structure $OS2$ leads to the design $\mathcal{D}_4$. Index sets which could occure in the case of $OS1$ are:

$$0 = \{0\},\ \dots\ ,\ 6 = \{6\},\ 7 = \{0,1\},\ \dots\ ,\ 27 = \{5,6\}$$

$$28 = \{0,1,2\},\ \dots\ ,\ 62 = \{4,5,6\},$$

$$63 = \{0,1,2,3\},\ \dots\ ,\ 97 = \{3,4,5,6\}$$

$$98 = \{0,1,2,3,4\},\ \ldots\ ,\ 118 = \{2,3,4,5,6\}.$$

Design $\mathcal{D}_3$ is presented in terms of index sets as follows:

64	0	31	31	31
0	81	36	44	54
31	36	108	24	13
31	44	24	98	11
31	54	13	11	109

With the help of the computer program by V. Tonchev, we got followig orders of full automorphism groups: $|Aut\mathcal{D}_1| = |Aut\mathcal{D}_2| = 21$, $|Aut\mathcal{D}_3| = 12096$, $|Aut\mathcal{D}_4| = 42$. Using the GAP [5] we have determine that $Aut\mathcal{D}_3 \cong G(2,2)$ and $Aut\mathcal{D}_4 \cong Frob_{21} \times Z_2$. $\square$

Derived Chevalley group $G(2,2)'$ is isomorphic to the unitary group $U(3,3)$ of order 6048. Simple group $U(3,3)$ acts transitively on the design $\mathcal{D}_3$.

We have also found out that automorphism groups $Frob_{21}$ and $Frob_{14}$ acts on the design $\mathcal{D}_3$ with orbit distributions $(1,7,7,21)$ and $(1,7,7,7,14)$ respectively. It is interesting that $U(3,3)$ doesn't contain subgroup isomorphic to $Frob_{14}$.

It is obvious that the design $\mathcal{D}_3$ have null polarity. Therefore, it is possible to construct strongly regular graph corresponding to that design.

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