ON A SYMMETRIC DESIGN (133,33,8) AND THE GROUP $E_8 \cdot F_{21}$ AS ITS AUTOMORPHISM GROUP

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ABSTRACT. This article presents the examination of the possibility that group $E_8 \cdot F_{21}$ operates on a symmetric design with parameters (133,33,8) as its automorphism group. The method based on coset enumeration in group is used.

1. INTRODUCTION

So far, only one symmetric design with parameters (133,33,8) is known [1, p. 625], on which operates Singer group of order 133. Its full automorphism group is $F_{18,19} \cdot Z_7$ of order 2394.

In this paper we examine the possibility that group $E_8 \cdot F_{21}$ would operate on a design with these parameters as its automorphism group. Namely, using the operation of this group, already several designs have been obtained [2], [5]. We use well known method based on coset enumeration in groups; see [2], [4]. By additional conditions on group operating the research is directed towards cases where the existence of design would be more likely, at least according to update knowledge and experience.

2. OPERATION OF THE GROUP

Group $G = E_8 \cdot F_{21}$ of order $168 = 2^3 \cdot 3 \cdot 7$ is a faithful extension of elementary abelian group of order 8 with Frobenius group $F_{21}$. In terms of generators and relations it is given as follows:

$$G = \langle a, b, c, d, e | a^7 = 1, b^3 = 1, c^2 = d^2 = e^2 = 1, (cd)^2 = (ce)^2 = (de)^2 = 1, b^{-1}ab = a^2, a^{-1}ca = d, a^{-1}da = e, a^{-1}ea = cd, b^{-1}cb = c, b^{-1}db = e, b^{-1}eb = de \rangle.$$

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We presume that $G$ is an automorphism group of a symmetric design $D$ with parameters $(133,33,8)$, that $Z_7 = \langle a \rangle$ operates fixed point free and that $Z_3 = \langle b \rangle$ stabilizes all $G$-orbits on design $D$. In that case $7$ is a divisor of all $G$-orbit lengths on $D$, while $3$ can be divisor of none. This leads us to a conclusion that lengths of $G$-orbits on $D$ must be indices of the subgroups of $G$ listed in Table 1.

<table>
<thead>
<tr>
<th>Subgroup</th>
<th>Degree of repres. (Index)</th>
<th>Number of fixed points of $Z_2$</th>
<th>Number of fixed points of $Z_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle b, c, d, e \rangle \approx E_8 \cdot Z_3$</td>
<td>7</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>$\langle b, d, e \rangle \approx A_4$</td>
<td>14</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>$\langle b, c \rangle \approx Z_6$</td>
<td>28</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>$\langle b \rangle \approx Z_3$</td>
<td>56</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Thus we shall need only the permutation representation of $G$-generators of degrees $7, 14, 28$ and $56$ provided by the corresponding computer program (Hrabe de Angelis). From the permutation representation of the generators we also determine the number of fixed points of prime-order automorphisms for all the necessary degrees (Table 1).

Let $f(Z_2)$ and $f(Z_3)$ denote respectively the number of fixed points of automorphisms of order 2 and 3 on design $D$. Using their well known upper and lower bounds

$$f(Z_2) \geq 1 + \frac{k - 1}{\lambda}, \quad f(Z_{2,3}) \leq k + \sqrt{k - \lambda},$$

as well as the fact $f(Z_2) \equiv 1 \pmod{2}$ and $f(Z_3) \equiv 1 \pmod{3}$, we obtain $f(Z_2) \in \{5, 7, 9, 11, \ldots, 37\}$ and $f(Z_3) \in \{1, 4, 7, \ldots, 37\}$.

Our additional assumption is that $Z_3$ has at most 7 fixed points on $D$. A motivation for this is the manner of acting of the automorphism of order 3 on Hall's design [3], the only so far known one with these parameters. Considering all this, we finally get possible lengths of $G$-orbits of points (and blocks) on design $D$ as given in Table 2.
ON A SYMMETRIC DESIGN (133,33,8)...

Table 2

<table>
<thead>
<tr>
<th>Fixed points schedule</th>
<th>Orbit lengths</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(Z_2)$</td>
<td>$f(Z_3)$</td>
</tr>
<tr>
<td>13</td>
<td>7</td>
</tr>
<tr>
<td>21</td>
<td>7</td>
</tr>
<tr>
<td>29</td>
<td>7</td>
</tr>
<tr>
<td>37</td>
<td>7</td>
</tr>
</tbody>
</table>

The task of finding orbit structures is performed by computer. They are obtained in three cases only, marked "*" in Table 2. After bringing certain number of them to contradiction, it's left to do the indexing of orbit structures given in Fig. 1 by means of computer. Here we make use of the algorithm presented in the next section.
Let $B_1, B_2, \ldots, B_t$ be $G$-orbits of blocks on design $D$. To index an orbit structure, that is to index a representative of each block orbit, means to find all points from point orbit $I$, $I \in \{1, 2, \ldots, t\}$, which lie on that block.

As a representative of the block orbit $B_i$, $i \in \{1, 2, \ldots, t\}$, we take block $p$ stabilized by the subgroup $H_i < G$ for which $[G : H_i] = |B_i|$. Such a block can contain only $H_i$-orbits, on $G$-orbits of points, in whole. One possible choice of points of block $p$, regarding this criterion, will be denoted $\text{compose\_block\_p}$ hereafter.

For the composed block $p$ we have to check the number of points that are common to it and each block of its own orbit $B_i$, that is, we check upon the accuracy of the relation $|p \cap p^a| = \lambda = 8$ for all $a \in G$. In fact, it is enough to check the intersection of $p$ and the representatives of $H_i$-orbits on $B_i$ because $H_i$ stabilizes $p$. The procedure of these consecutive checkings we shall call $\text{check\_inprod\_p}$.

Next, $p$ must be submitted to checking upon its intersection with the blocks from all the other orbits. Let block $q$ belong to $B_j$, $j \in \{1, \ldots, t\}$, $j \neq i$. Checking the criterion $|p \cap q^a| = \lambda$ for all $a \in G$ will be called
check_{outprod}. (p, q). Blocks $p_1 \in B_1, p_2 \in B_2, \cdots, p_t \in B_t$, that would satisfy all the cited conditions, completely determine design we are searching for. 

\[ \{p_1^\alpha, p_2^\alpha, \cdots, p_t^\alpha | \alpha \in G \} \] is the set of all blocks of $D$.

The procedure of indexing, which we accomplish iteratively in $t$ steps, is presented by algorithm in pseudocode, Fig. 2.

Algorithm for step 1:

```plaintext
while input matrices permit do
    compose_block_p
    check_inprod_p
    if p satisfactory then
        save p to 'RES 1'
    end
end
```

Algorithm for step $i, i = 2, 3, \ldots, t$:

```plaintext
while input matrices permit do
    begin
        compose_block_p
        check_inprod_p
        if p satisfactory then
            while not eof 'RES i-1' do
                begin
                    read (p_1, p_2, \ldots, p_{i-1}) from 'RES i-1'
                    w = 1
                    repeat
                        check_{outprod}. (p, p_w)
                        if p satisfactory then
                            w = w + 1
                        until p not satisfactory or w = i
                        if w = i then
                            save (p_1, \ldots, p_{i-1}, p) to 'RES i'
                        end
                    end
                end
            end
    end
end
```

Fig. 2

Statistics of the number of solutions obtained by indexing cited orbit structures, step by step, is given in Table 3.
Table 3

<table>
<thead>
<tr>
<th>Orbit structure</th>
<th>step 1</th>
<th>step 2</th>
<th>step 3</th>
<th>step 4</th>
<th>step 5</th>
<th>step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>O4</td>
<td>2</td>
<td>16</td>
<td>96</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(O5)₁</td>
<td>4</td>
<td>16</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(O5)₂</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(O5)₃</td>
<td>4</td>
<td>16</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(O5)₄</td>
<td>4</td>
<td>16</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(O6)₁</td>
<td>4</td>
<td>32</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(O6)₂</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(O6)₃</td>
<td>4</td>
<td>32</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

That result proves the following conclusion.

**Theorem.** There is no symmetric design with parameters (133,33,8) on which group \( G = E_8 \cdot F_{21} \) would operate so that \( Z_7 \) acts fixed point free and \( Z_3 \) stabilizes \( G \)-orbits having at most 7 fixed points.

**References**


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