ON THE POWER PROPERTY OF THE DENSITY TOPOLOGY IN THE PLANE

Pavel Pyrih

ABSTRACT. We prove that the density topology in the plane does not have the power property.

1. Introduction.

Any continuous transformation on a topological space $X$ will be called a mapping. We say that $f : A \rightarrow X$ is light on $A \subset X$ if $f$ is non-constant on any non-degenerate continuum in $A$.

We say that a topology on $X$ has the power property if for any open set $U \subset X$, any open and light mapping $f : U \rightarrow X$ and any point $x \in U$ there exist a number $n \in \mathbb{N}$ and a neighborhood $V \subset U$ of $x$ such that for each $y \in f(V) \setminus f(\{x\})$ the set $f^{-1}(\{y\})$ has cardinality $n$. In other words the power property of a topology means that each open and light mapping on any open set is locally (punctured neighborhoods) $n$ to one.

We recall the power property of the Euclidean topology in the plane originally due to Stoïlow. (see [3], [4], [5, Chap. VII, 5.1, p. 88]):

**Theorem 1.1.** Let $A$ and $B$ be 2-manifolds and $f(A) = B$ be light and open. For any ordinary point $q \in B$ and any $p \in f^{-1}(q)$, there exists a closed 2-cell neighborhood $E$ of $p$ and an integer $k$ such that $f \upharpoonright E$ is topologically equivalent to $w = z^k$ on $|z| \leq 1$.

This implies a simple observation on the holomorphic functions (see [5, Chap. VII, Theorem 5.3, p. 88]):

* Research supported by the grant No. GAUK 186/96 of Charles University.
1991 Mathematics Subject Classification. 54A10, 54C10.
Key words and phrases. Power property, density topology.
Corollary 1.2. The mapping generated by a non-constant differentiable function \( w = f(z) \) in a region \( R \subset \mathbb{C} \) is locally equivalent to a power mapping.

A similar result closely related to the power property was obtained by B. Fuglede in [2, Proposition 4.3, p. 292] for the case of the fine topology in potential theory. He proved that a finely holomorphic function is finely locally equivalent to a power mapping. It would be of interest to know whether one can obtain a power property for even finer topology than for the fine topology from potential theory.

In connection with this remark let us formulate two problems.

Problem 1.3. Which topological spaces have the power property?

Problem 1.4. Does the fine topology from potential theory have the power property?

We show that the density topology on \( \mathbb{C} \) (the density being measured using the Lebesgue measure \( \lambda \) and the discs centered at a point) does not have the power property.

2. Density topology example.

We start with a simple observation

Family lemma 2.1. For any \( t \in (0,1) \) we define the 'family' \( H_t = \{ t/2^k : k \geq 0 \} \). Given the 'prison' set \( M \subset (0,1) \) of the Lebesgue measure \( \lambda(M) = 1/2^n \) there exists \( t \in (0,1) \) with a 'free family' \( F_t = H_t \setminus M \) of cardinality at least \( n \).

Proof. The assertion is obviously fulfilled with \( M = (0,1/2^n) \). We can try to avoid the free family of cardinality at least \( n + 1 \) with a prison of measure \( 1/2^n \). For each \( t \in (1/2,1) \) there must be at most \( n \) members of the family \( H_t = \{ t/2^k : k \geq 0 \} \) free (i.e. outside the prison \( M \)). The most efficient way is to build the prison \( M = (0,1/2^n) \). \( \square \)

Proposition 2.2. The density topology in the plane does not have the power property.

Proof. We set \( D = \{ r(\cos \pi t + i \sin \pi t) \in \mathbb{C} : 0 \leq r < 1, 0 < t < 2 \} \) and define a 'corkscrew'-type mapping \( f : D \to \mathbb{C} \) by the formula

\[
f(r(\cos \pi t + i \sin \pi t)) = r(\cos 2\pi \varphi(t) + i \sin 2\pi \varphi(t))
\]

where \( \varphi(t) = t - 1 \) for \( t \in (1,2] \), \( \varphi(t) = \varphi(2^{n+1} t) \) for \( t \in (1/2^{n+1}, 1/2^n] \), \( n \geq 0 \). (With \( t \) decreasing from 2 to 1 the 'lower half' of \( D \) is mapped onto \( D \) counter-clockwise, then the 'speed' of rotation increases in such a way that)

\[
f(\{ r(\cos \pi t + i \sin \pi t) \in \mathbb{C} : 0 \leq r < 1, t \in (1/2^{n+1}, 1/2^n) \}) = D
\]
for $n \geq 0$.)

We consider the mapping $f : D \to \mathbb{C}$ with the density topology on both $D$ and $\mathbb{C}$. We see that

(i) $D$ is a density open set (the missing segment has the Lebesgue measure zero);
(ii) $f$ is density continuous at $D \setminus \{0\}$ ($f$ is piecewisely a rotation in $D \setminus \{0\}$);
(iii) $f$ is density continuous at 0;

Proof of (iii) : For any density open set $V$ containing 0, the density of a set $f^{-1}(V)$ at 0 can be calculated using the 'radial segments'

$\{r(\cos \pi t + i \sin \pi t) \in \mathbb{C} : 0 \leq r < 1, t \in (1/2^{n+1}, 1/2^n]\}$

of $D$ defined by the segments $t \in (1/2^{n+1}, 1/2^n]$, the density of $V$ at 0 gives the density of $f^{-1}(V)$ at 0. ◇

(iv) $f$ is density open at $D \setminus \{0\}$ ($D$ is piecewisely a rotation in $D \setminus \{0\}$);
(v) $f$ is density open at $\{0\}$;

Proof of (v) : The density of $U$ at 0 gives the estimate of the Lebesgue measure of

$U \cap \{r(\cos \pi t + i \sin \pi t) \in \mathbb{C} : 0 \leq r < 1, t \in (1, 2)\}$

and we obtain the estimate of the density of $f(U)$ at 0. ◇

Moreover, $f$ is light on $D$. Hence we summarize :

(vi) $f$ is a light and open mapping on an open set $D$ on a topological space $\mathbb{C}$ with the density topology;

Finally we prove that

(vii) for any density open set $V \subset D$ containing 0 and given $n \in \mathbb{N}$ there exists $y \in f(V)$ such that the set $V \cap f^{-1}(\{y\})$ has cardinality at least $n$.

Proof of (vii) : There is a density open set $U \subset V$ containing 0 and a Euclidean open set $G$ containing the density closed set $\mathbb{C} \setminus V$ such that $G$ and $U$ are disjoint - see [1], the Lusin-Menchoff property of the density topology. When $U$ reaches the density $1 - 1/2^n$ at 0 for some $R \in (0, 1)$, i.e.

$$\lambda(\{r(\cos \pi t + i \sin \pi t) \in \mathbb{C} : 0 \leq r < R, t \in (0, 2)\}) > (1 - 1/2^n)\pi R^2,$$

we can using the polar coordinates obtain $r \in (0, R)$ such that the set

$$M = \{t \in (0, 2) : r(\cos \pi t + i \sin \pi t) \in G\}$$
is Euclidean open in $(0,2)$ (since $G$ is Euclidean open in $\mathbb{C}$) with $\lambda(M) \leq 1/2^n$. Using Family lemma 2.1 with $M$ as the prison set we conclude that there exists $t \in (0,1)$ with the 'free family' set $F_t = \{t_1, \ldots, t_n\} \subset (0,1)$ disjoint with $M$, being of cardinality at least $n$. Then $y = f(r(\cos \pi t_1 + i \sin \pi t_1)) = \cdots = f(r(\cos \pi t_n + i \sin \pi t_n)) \in f(V)$ due to the definitions of $F_t$ and $f$, and consequently $f^{-1}(\{y\})$ has cardinality at least $n$. ☐

The mapping $f : D \to \mathbb{C}$ shows that the density topology does not have the power property. □

REFERENCES


(Received: 9.3.1998.)