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Common fixed point and invariant approximations for subcompatible mappings in convex metric spaces

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Abstract. A common fixed point theorem for subcompatible mappings satisfying a generalized contractive condition (in the framework of a convex metric space) is proved and also utilized to derive some invariant approximation results.

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1. Introduction

The interplay between the geometry of Banach spaces and fixed point theory is not only natural but also proving very fruitful. In particular, geometric properties play a key role in metric fixed point problems (see [8] and references cited therein). There exist numerous results banking heavily on geometric properties of Banach spaces which mark the beginning of a new mathematical field wherein the metric fixed point theorems are proved with the aid of geometric properties of Banach spaces. The utility of approximation theory is enormous. By now approximation theory intersects with almost every other branch of analysis and plays a very fruitful role in the applied sciences and engineering. Broadly speaking, approximation theory is concerned with the approximation of a continuous function by a polynomial carried out in several concrete ways these days. In fixed point theory also the approximation of a fixed point is carried out and one of the most applied such result of fixed point approximation is due to Scarf [16]. In the recent past, fixed point theorems have been extensively applied to best approximation theory and in the course of a last four decades several interesting results have been established. One may recall that Meinardus [14] was the first to notice such possibility by using Schauder Fixed Point Theorem to best approximation theory. Thereafter, Brosowski [4] obtained celebrated results and generalized the Meinardus's result. Several authors (e.g. [10, 23, 25]) have further improved the results of Brosowski [4] in several ways. In the year 1988, Sahab et al. [15] extended the result of Hicks and Humpheries [10] and

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Singh [23] by considering a pair of mappings wherein one is linear and the other one is nonexpansive.

In 1970, Takahashi [26] introduced the notion of convex metric spaces and proved some fixed point theorems for nonexpansive mappings in such spaces. Afterwards, many authors have discussed the existence of a fixed point as well as the convergence of iterative processes for nonexpansive mappings in such spaces (see [5, 6, 11]). Recently, Beg et al. [3] employed convex metric spaces to prove results on the existence of the common fixed point and utilize the same to prove the existence of the best approximant for relatively contractive commuting mappings which also generalize the core result of Sahab et al. [15] which has witnessed intense research activity around their result in the last several years.

In this paper, we establish an existence result on the common fixed point for a contractive subcompatible pair of mappings in the setting of the convex metric space which is further utilized to prove some results in invariant approximation. In the process, results due to Beg et al. [3], Al-Thagafi [2], Brosowski [4], Meinardus [14], Singh [23, 24] and Sahab et al. [15] are also generalized and improved by considering relatively general classes of noncommuting mappings satisfying a Gregus [7] type contraction condition in the setting of convex metric spaces.

2. Preliminaries

For the material to be presented here, the following definitions are required:

Definition 1 (see [26]). *Let $(\mathcal{X}, d)$ be a metric space. A continuous mapping $\mathcal{W}$: $\mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$ is said to be a convex structure on $\mathcal{X}$, if for all $x, y \in \mathcal{X}$ and $\lambda \in [0,1]$ the following condition is satisfied:*

$$d(u, \mathcal{W}(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y),$$

for all $u \in \mathcal{X}$ wherein obviously $\mathcal{W}(x, x, \lambda) = x$.

A metric space $\mathcal{X}$ equipped with a convex structure is called a convex metric space. Obviously, Banach space and each of its convex subsets are simple examples of convex metric spaces with $\mathcal{W}(x, y, \lambda) = \lambda x + (1 - \lambda) y$, but a *Fréchet* space need not be a convex metric space. There are many examples of convex metric spaces which cannot be embedded in any Banach space. For substantiation, the following two examples can be recalled:

Example 1. *Let I be the unit interval $[0,1]$ and $\mathcal{X}$ the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i, b_i], I_j = [a_j, b_j]$ and $\lambda (0 \leq \lambda \leq 1)$, we define a mapping $\mathcal{W}$ by $\mathcal{W}(I_i, I_j; \lambda) = [\lambda a_i + (1 - \lambda) a_j, \lambda b_i + (1 - \lambda) b_j]$ and define a metric d in $\mathcal{X}$ by the Hausdorff distance, i.e.*

$$d(I_i, I_j) = \sup_{a \in I} \{ | \inf_{b \in I_i} \{ |a - b| \} - \inf_{c \in I_j} \{ |a - c| \} | \}.$$

Example 2. *A linear space $\mathcal{L}$ equipped with the following two properties is a natural convex metric space:*

(1) For $x, y \in \mathcal{L}$, $d(x,y) = d(x-y,0)$;

(2) For $x, y \in \mathcal{L}$ *and* $\lambda (0 \leq \lambda \leq 1)$,

$$d(\lambda x + (1-\lambda)y, 0) \leq \lambda d(x,0) + (1-\lambda)d(y,0).$$

Definition 2. *A subset* $\mathcal{K}$ *of a convex metric space* $(\mathcal{X}, d)$ *is said to be convex, if* $\mathcal{W}(x,y,\lambda) \in \mathcal{K}$ *for all* $x, y \in \mathcal{K}$ *and* $\lambda \in [0,1]$. *The set* $\mathcal{K}$ *is said to be q-starshaped if there exists* $q \in \mathcal{K}$ *such that* $\mathcal{W}(x,q,\lambda) \in \mathcal{K}$ *for all* $x \in \mathcal{K}$ *and* $\lambda \in [0,1]$. *Clearly, q-starshaped subsets of* $\mathcal{X}$ *contain all convex subsets of* $\mathcal{X}$ *as a proper subclass.*

Definition 3. *A convex metric space* $(\mathcal{X}, d)$ *is said to satisfy the Property* (I), *if for all* $x, y, z \in \mathcal{X}$ *and* $\lambda \in [0,1]$,

$$d(\mathcal{W}(x,z,\lambda), \mathcal{W}(y,z,\lambda)) \leq \lambda d(x,y).$$

For motivation and further details in respect of Property (I), *one can be referred to Guay et al. [9] (e.g. Definition 3.2).*

Definition 4 (see [11, 26])**.** *A continuous function* $\mathcal{S}$ *from a closed convex subset* $\mathcal{K}$ *of a convex metric space* $(\mathcal{X}, d)$ *into itself is said to be* $\mathcal{W}$*-affine if* $\mathcal{S}(\mathcal{W}(x,y,\lambda)) = \mathcal{W}(\mathcal{S}x, \mathcal{S}y, \lambda)$ *whenever* $\lambda \in [0,1] \cap \mathcal{Q}$ *and* $x, y \in \mathcal{K}$, *where* $\mathcal{Q}$ *stands for the set of rational numbers.*

Definition 5 (see [23])**.** *Let* $\mathcal{K}$ *be a subset of a metric space* $(\mathcal{X}, d)$. *Let* $x_0 \in \mathcal{X}$. *An element* $y \in \mathcal{K}$ *is called a best approximant to* $x_0 \in \mathcal{X}$, *if*

$$d(x_0, y) = \inf\{d(x_0, z) : z \in \mathcal{K}\}.$$

Let $\mathcal{P}_{\mathcal{K}}(x_0)$ *be the set of best* $\mathcal{K}-$*approximants to* x_0 *and so*

$$\mathcal{P}_{\mathcal{K}}(x_0) = \{z \in \mathcal{K} : d(x_0, z) = d(x_0, \mathcal{K})\}.$$

Definition 6. *Let* $\mathcal{T}$ *be a self-map defined on a subset* $\mathcal{K}$ *of a metric space* $(\mathcal{X}, d)$. *A best approximant* y *in* $\mathcal{K}$ *to an element* x_0 *in* $\mathcal{X}$ *with* $\mathcal{T}x_0 = x_0$ *is an invariant approximation in* $\mathcal{X}$ *to* y *if* $\mathcal{T}y = y$.

Example 3 (see [24])**.** *Let* $\mathcal{X} = \mathbb{R}$ *and* $\mathcal{K} = [0, \frac{1}{2}]$. *Define* $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ *as follows:*

$$\mathcal{T}x = \begin{cases} x-1, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{x+1}{2}, & \text{if } x > \frac{1}{2}. \end{cases} \tag{1}$$

Clearly, $\mathcal{T}(\mathcal{K}) = \mathcal{K}$ *and* $\mathcal{T}(1) = 1$ *(i.e.*$x_0 = 1$*). Also*

$$\mathcal{P}_{\mathcal{K}}(x_0) = \left\{\frac{1}{2}\right\}.$$

Hence, $\mathcal{T}$ *has a fixed point in* $\mathcal{P}_{\mathcal{K}}(x_0)$ *which is a best approximation to* x_0 *in* $\mathcal{K}$. *Thus,* $\frac{1}{2}$ *is an invariant approximation.*

Definition 7 (see [12]). *A pair* $(\mathcal{T}, \mathcal{S})$ *of self-mappings of a metric space* $\mathcal{X}$ *is said to be compatible, if* $d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}\mathcal{T}x_n) \to 0$, *whenever* $\{x_n\}$ *is a sequence in* $\mathcal{X}$ *such that* $\mathcal{T}x_n, \mathcal{S}x_n \to t \in \mathcal{X}$.

Every commuting pair of mappings is compatible but the converse implication is not true in general.

Definition 8 (see [19, 20]). *Let* $\mathcal{K}$ *be a q-starshaped subset of convex metric space* $(\mathcal{X}, d)$ *such that* q *remains fixed under* $\mathcal{S}$ *and invariant under both* $\mathcal{T}$ *and* $\mathcal{S}$. *Then* $\mathcal{T}$ *and* $\mathcal{S}$ *are called* $\mathcal{R}$*-subcommuting on* $\mathcal{K}$, *if for all* $x \in \mathcal{K}$ *there exists a real number* $\mathcal{R} > 0$ *such that* $d(\mathcal{S}\mathcal{T}x, \mathcal{T}\mathcal{S}x) \leq (\frac{\mathcal{R}}{k})d(seg[\mathcal{T}x, q], \mathcal{S}x)$ *for each* $k \in (0, 1]$. *If* $\mathcal{R} = 1$, *then the maps are called* 1*-subcommuting.* $\mathcal{S}$ *and* $\mathcal{T}$ *are called* $\mathcal{R}$*-subweakly commuting on* $\mathcal{K}$, *if for all* $x \in \mathcal{K}$ *there exists a real number* $\mathcal{R} > 0$ *such that* $d(\mathcal{S}\mathcal{T}x, \mathcal{T}\mathcal{S}x) \leq \mathcal{R}d(\mathcal{S}x, seg[\mathcal{T}x, q])$, *where* $seg[x, q] = \mathcal{W}(x, q, k), 0 \leq k \leq 1$.

Remark 1.

(i) Notice that commutativity implies $\mathcal{R}$*-subcommutativity which in turn implies* $\mathcal{R}$*-weak commutativity (see [17, 18, 19]).*

(ii) Also, it is straightforward to notice that commuting maps are $\mathcal{R}$*-subweakly commuting maps whereas* $\mathcal{R}$*-subweakly commuting maps are* $\mathcal{R}$*-weakly commuting but the converse implications are not true in general (see [20]).*

To justify some of the above remarks, we cite the following examples (see [19, 20]):

Example 4. *Let* $\mathcal{X} = \mathbb{R}$ *with norm* $\|x\| = |x|$ *and* $\mathcal{M} = [1, \infty)$. *Let* $\mathcal{T}, \mathcal{S} : \mathcal{M} \to \mathcal{M}$ *be defined by*

$$\mathcal{T}x = x^2 \text{ and } \mathcal{S}x = 2x - 1$$

for all $x \in \mathcal{M}$. *Then* $\mathcal{T}$ *and* $\mathcal{S}$ *are* $\mathcal{R}$*-weakly commuting with* $\mathcal{R} = 2$. *However, they are not* $\mathcal{R}$*-subcommuting because*

$$|\mathcal{T}\mathcal{S}x - \mathcal{S}\mathcal{T}x| \leq (\frac{\mathcal{R}}{k})|(k\mathcal{T}x + (1-k)q) - \mathcal{S}x|$$

does not hold for $x = 2$ *and* $k = \frac{2}{3}$, *where* $q = 1 \in Fix(\mathcal{S})$.

Example 5. *Let* $\mathcal{X} = \mathbb{R}$ *with norm* $\|x\| = |x|$ *and* $\mathcal{M} = [1, \infty)$. *Let* $\mathcal{T}, \mathcal{S} : \mathcal{M} \to \mathcal{M}$ *be defined by*

$$\mathcal{T}x = 4x - 3 \text{ and } \mathcal{S}x = 2x^2 - 1$$

for all $x \in \mathcal{M}$. *Then* $\mathcal{M}$ *is* 1*-starshaped with* $1 \in Fix(\mathcal{S})$ *and both* $\mathcal{T}$ *and* $\mathcal{S}$*-invariant. Also,* $|\mathcal{T}\mathcal{S}x - \mathcal{S}\mathcal{T}x| = 24(x-1)^2$. *Further,*

$$|\mathcal{T}\mathcal{S}x - \mathcal{S}\mathcal{T}x| \leq (\frac{\mathcal{R}}{k})|(k\mathcal{T}x + (1-k)q) - \mathcal{S}x|$$

for all $x \in \mathcal{M}$, *where* $\mathcal{R} = 12$ *and* $q = 1 \in Fix(\mathcal{S})$. *Thus,* $\mathcal{T}$ *and* $\mathcal{S}$ *are* $\mathcal{R}$*-subcommuting on* $\mathcal{M}$ *but not commuting on* $\mathcal{M}$.

Example 6. *Let $\mathcal{X} = \mathbb{R}^2$ with norm $\|(x,y)\| = \max\{|x|,|y|\}$, and let $\mathcal{T}$ and $\mathcal{S}$ be defined by*

$$\mathcal{T}(x,y) = (2x-1, y^3) \text{ and } \mathcal{S}(x,y) = (x^2, y^2)$$

for all $(x,y) \in \mathcal{X}$. Then $\mathcal{T}$ and $\mathcal{S}$ are $\mathcal{R}$-subweakly commuting on $\mathcal{K} = \{(x,y) : x \geq 1, y \geq 1\}$ but not commuting on $\mathcal{K}$.

Definition 9 (see [1, 13, 21, 22]). *Suppose that $\mathcal{K}$ is a q-starshaped subset of a metric space $\mathcal{X}$. For the selfmaps $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{K}$ with $q \in Fix(\mathcal{S})$, define $\bigwedge_q(\mathcal{S},\mathcal{T}) = \cap\{\bigwedge(\mathcal{S},\mathcal{T}_k) : 0 \leq k \leq 1\}$, where $\mathcal{T}_k x = seg[\mathcal{T}x, q]$ and $\bigwedge(\mathcal{S},\mathcal{T}_k) = \{\{x_n\} \subset \mathcal{K} : lim_n \mathcal{S}x_n = lim_n \mathcal{T}_k x_n = t \in \mathcal{K}\}$. Then $\mathcal{S}$ and $\mathcal{T}$ are called subcompatible, if $lim_n d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) = 0$ for all sequences $x_n \in \bigwedge_q(\mathcal{S},\mathcal{T})$.*

Obviously, subcompatible maps are compatible but the converse statement does not hold in general as substantiated by the following example.

Example 7 (see [1, 13]). *Let $\mathcal{X} = \mathbb{R}$ with usual metric and $\mathcal{K} = [1,\infty)$. Let $\mathcal{S}(x) = 2x-1$ and $\mathcal{T}(x) = x^2$, for all $x \in \mathcal{K}$. Let $q = 1$. Then $\mathcal{K}$ is q-starshaped with $\mathcal{S}q = q$. Note that $\mathcal{S}$ and $\mathcal{T}$ are compatible. For any sequence $\{x_n\}$ in $\mathcal{K}$ with $lim_n x_n = 2$, we have $lim_n \mathcal{S}x_n = lim_n \mathcal{T}_{\frac{2}{3}} x_n = 3 \in \mathcal{K}$. However, $lim_n d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) \neq 0$. Thus $\mathcal{S}$ and $\mathcal{T}$ are not subcompatible maps.*

Notice that $\mathcal{R}$-subweakly commuting and $\mathcal{R}$-subcommuting maps are subcompatible. The following simple example reveals that the converse statement is not true in general.

Example 8 (see [1, 13]). *Let $\mathcal{X} = \mathbb{R}$ with usual metric and $\mathcal{K} = [0,\infty)$. Let $\mathcal{S}(x) = \frac{x}{2}$ if $0 \leq x < 1$ and $\mathcal{S}x = x$ if $x \geq 1$, and $\mathcal{T}(x) = \frac{1}{2}$ if $0 \leq x < 1$ and $\mathcal{T}x = x^2$ if $x \geq 1$. Then $\mathcal{K}$ is 1-starshaped with $\mathcal{S}1 = 1$ and $\bigwedge_q(\mathcal{S},\mathcal{T}) = \{\{x_n\} : 1 \leq x_n < \infty\}$. Note that $\mathcal{S}$ and $\mathcal{T}$ are subcompatible but not $\mathcal{R}$-weakly commuting for all $\mathcal{R} > 0$. Thus $\mathcal{S}$ and $\mathcal{T}$ are neither $\mathcal{R}$-subweakly commuting nor $\mathcal{R}$-subcommuting maps.*

The following result is also needed in the sequel.

Theorem 1 (see [11], Corollary 3.2). *Let $\mathcal{T}$ and $\mathcal{S}$ be a pair of compatible self-maps on a closed convex subset $\mathcal{K}$ of the convex metric space $(\mathcal{X}, d)$, satisfying:*

$$d(\mathcal{T}x, \mathcal{T}y) \leq ad(\mathcal{S}x, \mathcal{S}y) + (1-a)\max\{d(\mathcal{T}x, \mathcal{S}x), d(\mathcal{T}y, \mathcal{S}y)\}, \tag{2}$$

for $x, y \in \mathcal{K}$, where $0 < a < 1$ is a constant. If $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$ and $\mathcal{S}$ is $\mathcal{W}$-affine and continuous, then $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point $z \in \mathcal{K}$ and $\mathcal{T}$ is continuous at z.

3. Main Result

The following result is a fixed point theorem for a relatively general class of noncommuting mappings in the framework of a convex metric space.

Theorem 2. *Let $\mathcal{K}$ be a nonempty convex subset of a convex metric space $(\mathcal{X}, d)$ satisfying the property (I). Let $\mathcal{T}$ and $\mathcal{S}$ be a pair of self-maps defined on $\mathcal{K}$ which is subcompatible. Assume that $\mathcal{S}(\mathcal{K}) = \mathcal{K}$, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is $\mathcal{W}$-affine and also continuous. If $\mathcal{T}$ and $\mathcal{S}$ satisfy*

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(\mathcal{S}x, \mathcal{S}y) + \tfrac{(1-k)}{k} \max\{d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y)\}, \tag{3}$$

for all $x, y \in \mathcal{K}$ where $0 < k < 1$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point provided $\mathcal{K}$ is compact and $\mathcal{T}$ is continuous.

Proof. Choose a sequence $\{k_n\} \subset (0, 1)$ with $k_n \to 1$ as $n \to \infty$. For each $n \in \mathbb{N}$, define $\mathcal{T}_n : \mathcal{K} \to \mathcal{K}$ as follows:

$$\mathcal{T}_n x = \mathcal{W}(\mathcal{T}x, q, k_n) \tag{4}$$

for some $q \in \mathcal{K}$. Obviously, for each n, $\mathcal{T}_n$ maps $\mathcal{K}$ into itself as $\mathcal{K}$ is convex. Subcompatibility of the pair $(\mathcal{S}, \mathcal{T})$, $\mathcal{W}$-affinity of $\mathcal{S}$, $q = \mathcal{S}q$ and the property (I) (in respect of any $\{x_m\} \subset \mathcal{K}$ with $\lim_m \mathcal{T}_n x_m = \lim_m \mathcal{S}x_m = t \in \mathcal{K}$) together imply that $\lim_m \mathcal{T}_n x_m = \lim_m \mathcal{W}(\mathcal{T}x_m, q, k_n) = \lim_m \mathcal{T}_{k_n} x_m = t$ by the assumption of $\mathcal{T}_\lambda$ and subcompatibility of $\mathcal{T}$ and $\mathcal{S}$.

$$\begin{aligned}
0 &\leq \lim_m d(\mathcal{T}_n \mathcal{S}x_m, \mathcal{S}\mathcal{T}_n x_m) \\
&= \lim_m d(\mathcal{W}(\mathcal{T}\mathcal{S}x_m, q, k_n), \mathcal{S}\mathcal{W}(\mathcal{T}x_m, q, k_n)) \\
&= \lim_m d(\mathcal{W}(\mathcal{T}\mathcal{S}x_m, q, k_n), \mathcal{W}(\mathcal{S}\mathcal{T}x_m, \mathcal{S}q, k_n)) \\
&= \lim_m d(\mathcal{W}(\mathcal{T}\mathcal{S}x_m, q, k_n), \mathcal{W}(\mathcal{S}\mathcal{T}x_m, q, k_n)) \\
&\leq k_n \lim_m d(\mathcal{T}\mathcal{S}x_m, \mathcal{S}\mathcal{T}x_m).
\end{aligned}$$

Hence $\{\mathcal{T}_n\}$ and $\mathcal{S}$ are compatible for each n and $x_m \in \mathcal{K}$ whereas $\mathcal{T}_n(\mathcal{K}) \subseteq \mathcal{K} = \mathcal{S}(\mathcal{K})$, $\mathcal{S}$ is affine and $q \in Fix(\mathcal{S})$. Also, for all $x, y \in \mathcal{K}$, one can write (in view of (3), (4) and the property (I)) that

$$\begin{aligned}
d(\mathcal{T}_n x, \mathcal{T}_n y) &= d(\mathcal{W}(\mathcal{T}x, q, k_n), \mathcal{W}(\mathcal{T}y, q, k_n)) \\
&\leq k_n d(\mathcal{T}x, \mathcal{T}y) \\
&\leq k_n \{d(\mathcal{S}x, \mathcal{S}y) + (\frac{1-k_n}{k_n}) \max\{d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y)\}\},
\end{aligned}$$

i.e.,

$$d(\mathcal{T}_n x, \mathcal{T}_n y) \leq k_n d(\mathcal{S}x, \mathcal{S}y) + (1 - k_n) \max\{d(\mathcal{T}_n x, \mathcal{S}x), d(\mathcal{T}_n y, \mathcal{S}y)\}, \tag{5}$$

for all $x, y \in \mathcal{K}$ and $0 < k_n < 1$. Since $\mathcal{K}$ is compact, therefore using Theorem 1, for every $n \in \mathbb{N}$, $\mathcal{T}_n$ and $\mathcal{S}$ have a common fixed point x_n in $\mathcal{K}$, i.e.,

$$x_n = \mathcal{T}_n x_n = \mathcal{S}x_n. \tag{6}$$

As $\mathcal{K}$ is compact and $\{x_n\}$ is a sequence in $\mathcal{K}$, so $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $x_m \to y \in \mathcal{K}$. As $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{W}$ are continuous and

$$x_m = \mathcal{S}x_m = \mathcal{T}_m x_m = \mathcal{W}(\mathcal{T}x_m, q, \lambda_m),$$

so it follows that $y = \mathcal{T}y = \mathcal{S}y$. □

We demonstrate Theorem 2 with the help of the following example.

Example 9. *Consider the real vector space* $\mathcal{X} = \mathbb{R}^2$ *equipped with natural metric wherein* $\mathcal{W}(x, y, \lambda) = \lambda x + (1-\lambda)y$. *Define self-maps* $\mathcal{T}$ *and* $\mathcal{S}$ *on the convex metric space* $(\mathcal{X}, d)$ *as follows (for arbitrary* (x, y) *in* $\mathcal{X} = \mathbb{R}^2$*):*

$$\mathcal{S}(x, y) = (x, y) \quad \text{and} \quad \mathcal{T}(x, y) = (-x, -y).$$

If we take $\mathcal{K} = \{(x, y) : x^2 + y^2 \leq 1\}$, *then* $\mathcal{K}$ *satisfies the property* (I) *and* $\mathcal{K}$ *is compact together with* $q = (0, 1) \in Fix(\mathcal{S})$. *Also,* $\mathcal{S}$ *is continuous,* $\mathcal{W}$*-affine and* $\mathcal{S}(\mathcal{K}) = \mathcal{K}$. *Further, the pair* $(\mathcal{T}, \mathcal{S})$ *is commuting and hence subcompatible besides* $\mathcal{T}(\mathcal{K}) = \mathcal{K}$ *and* $\mathcal{T}$ *is continuous. For the verification of condition (3) (for arbitrary* $x = (x_1, y_1)$ *and* $y = (x_2, y_2)$ *in* $\mathcal{K}$*), we have:*

$$\begin{aligned} d(\mathcal{T}(x_1, y_1), \mathcal{T}(x_2, y_2)) &= d((-x_1, -y_1), (-x_2, -y_2)) \\ &= d(((x_2 - x_1), (y_2 - y_1)), (0, 0)) \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d(\mathcal{S}x, \mathcal{S}y) \\ &\leq d(\mathcal{S}x, \mathcal{S}y) + (\frac{1-k}{k}) \\ &\quad \times \max\{d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y)\}. \end{aligned}$$

Thus all the conditions of Theorem 2 are satisfied. Notice that $(0, 0)$ *remains fixed under* $\mathcal{T}$ *and* $\mathcal{S}$ *both which, in all, substantiates Theorem 2.*

The following relatively simple example also illustrates Theorem 2.

Example 10. *Consider* $\mathcal{X} = \mathbb{R}$ *endowed with usual metric and* $\mathcal{K} = [0, 1]$ *is a subset of* $\mathbb{R}$ *which is indeed convex. Define* $\mathcal{T}x = 0$ *(for all* $x \in \mathcal{K}$*) and* $\mathcal{S}x = x$ *(for all* $x \in \mathcal{K}$*). Using routine calculations, one can easily show that* $\mathcal{T}$ *and* $\mathcal{S}$ *satisfy condition (3) together with all other conditions of Theorem 2. Notice that* $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) = \{0\}$.

Our next example exhibits that conditions of convexity of $\mathcal{K}$ are necessary in Theorem 2.

Example 11. *Let* $\mathcal{X} = \mathbb{R}$ *be endowed with usual metric and* $\mathcal{K} = \{0, 1, 1 - \frac{1}{n+1} : n \in \mathbb{N}\}$. *Define* $\mathcal{T}0 = \frac{1}{2}$ *and* $\mathcal{T}1 = \mathcal{T}(1 - \frac{1}{n+1}) = 0$ *for all* $n \in \mathbb{N}$. *Clearly,* $\mathcal{K}$ *is not convex. Define* $\mathcal{S}x = x$ *(for all* $x \in \mathcal{K}$*). Now* $\mathcal{T}$ *and* $\mathcal{S}$ *satisfy (3) together with all other conditions of Theorem 2 except the convexity of* $\mathcal{K}$. *Notice that* $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) = \emptyset$.

As an immediate consequence of Theorem 2, we can have the following.

Corollary 1. *Let $\mathcal{K}$ be a nonempty convex subset of a convex metric space $(\mathcal{X}, d)$ satisfying the property (I). Let $\mathcal{T}$ and $\mathcal{S}$ be a pair of self-maps of $\mathcal{K}$ which is $\mathcal{R}$-subweakly commuting. Suppose that $\mathcal{S}(\mathcal{K}) = \mathcal{K}$, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is $\mathcal{W}$-affine and also continuous. If $\mathcal{T}$ and $\mathcal{S}$ satisfy*

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(\mathcal{S}x, \mathcal{S}y) + (\tfrac{1-k}{k}) \max\{d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y)\}, \qquad (7)$$

for all $x, y \in \mathcal{K}$, where $0 < k < 1$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point, provided $\mathcal{K}$ is compact and $\mathcal{T}$ is continuous.

As an application of Theorem 2, we derive a more general result in invariant approximation theory for subcompatible pairs (a generalized class of noncommuting pairs) in the framework of the convex metric space.

Theorem 3. *Let $\mathcal{T}$ and $\mathcal{S}$ be self-maps of a convex metric space $(\mathcal{X}, d)$ and $\mathcal{K}$ a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{K}) \subseteq \mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$, where $x_0 \in \mathcal{X}$. Suppose $\mathcal{D} = \mathcal{P}_{\mathcal{K}}(x_0)$ is nonempty convex such that $\mathcal{S}(\mathcal{D}) = \mathcal{D}$, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is continuous as well as $\mathcal{W}$-affine and the pair $(\mathcal{T}, \mathcal{S})$ is subcompatible on $\mathcal{D}$. If $\mathcal{T}$ and $\mathcal{S}$ satisfy (for all $x, y \in \mathcal{D}' = \mathcal{D} \cup \{x_0\}$)*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \begin{cases} d(\mathcal{S}x, \mathcal{S}x_0), & \text{if } y = x_0; \\ d(\mathcal{S}x, \mathcal{S}y) + (\frac{1-k}{k}) & \\ \quad \times \max\{d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y)\}, & \text{if } y \in \mathcal{D} \end{cases} \qquad (8)$$

where $0 < k < 1$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point in $\mathcal{D}$, provided $\mathcal{D}$ is compact and $\mathcal{T}$ is continuous.

Proof. Firstly, we show that $\mathcal{T}$ is a self-map on $\mathcal{D}$, i.e. $\mathcal{T} : \mathcal{D} \to \mathcal{D}$. To do this, let $y \in \mathcal{D}$, then $\mathcal{S}y \in \mathcal{D}$ as $\mathcal{S}(\mathcal{D}) = \mathcal{D}$. In case $y \in \partial\mathcal{K}$, then $\mathcal{T}y \in \mathcal{K}$ as $\mathcal{T}(\partial\mathcal{K}) \subseteq \mathcal{K}$. Owing to the fact that $\mathcal{T}x_0 = x_0 = \mathcal{S}x_0$, from (8) one may have

$$d(\mathcal{T}y, x_0) = d(\mathcal{T}y, \mathcal{T}x_0) \leq d(\mathcal{S}y, \mathcal{S}x_0) = d(\mathcal{S}y, x_0) = d(x_0, \mathcal{K}),$$

which shows that $\mathcal{T}y \in \mathcal{D}$, and in all $\mathcal{T}$ and $\mathcal{S}$ are self-maps on $\mathcal{D}$. Thus all the conditions of Theorem 2 are satisfied and hence there exists a $z \in \mathcal{D}$ such that $\mathcal{T}z = z = \mathcal{S}z$. $\square$

We furnish the following example to demonstrate the validity of the hypotheses of Theorem 3:

Example 12. *Consider the real vector space $\mathcal{X} = \mathbb{R}^2$ equipped with metric*

$$d((x_1, y_1), (x_2, y_2)) = | x_1 - x_2 | + | y_1 - y_2 |$$

wherein $\mathcal{W}(x, y, \lambda) = \lambda x + (1 - \lambda)y$. Define self-maps $\mathcal{T}$ and $\mathcal{S}$ on the convex metric space $(\mathcal{X}, d)$ as follows:

$$\mathcal{S}(x, y) = (x, y) \text{ and } \mathcal{T}(x, y) = \begin{cases} (x, y), & \text{if } y \leq x, \\ (x, x), & \text{if } y \geq x. \end{cases}$$

Take $\mathcal{K} = \{(x,x) : x \in \mathbb{R}\}$ *and* $x_0 = (1,-1)$. *Then* $(1,-1) \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$ *and* $\mathcal{D} = \mathcal{P}_{\mathcal{K}}(x_0)$ *is the line segment joining the points* $(-1,-1)$ *and* $(1,1)$ *which is indeed nonempty and convex. Also* $\mathcal{T}$ *and* $\mathcal{S}$ *are continuous,* $\mathcal{W}$*-affine,* $\mathcal{S}(\mathcal{D}) = \mathcal{D}$ *and* $(0,0) \in Fix(\mathcal{S})$. *Also, the pair* $(\mathcal{T},\mathcal{S})$ *is commuting and hence subcompatible besides* $\mathcal{T}(\partial\mathcal{K}) = \mathcal{K} \subseteq \mathcal{K}$. *For the verification of condition (8), for arbitrary* (x,x) *and* (y,y)*, in* $\mathcal{K}$*, we distinguish the following two cases:*
Case I: If

$$y = x_0 = (1,-1),$$

then

$$d(\mathcal{T}(x,x),\mathcal{T}(1,-1)) = d((x,x),(1,-1)) =\mid x-1 \mid + \mid x+1 \mid= d(\mathcal{S}x,\mathcal{S}x_0)$$

Case II:

$$\begin{aligned} d(\mathcal{T}(x,x),\mathcal{T}(y,y)) &= d((x,x),(y,y)) =\mid x-y \mid + \mid x-y \mid= d(\mathcal{S}x,\mathcal{S}y) \\ &\leq d(\mathcal{S}x,\mathcal{S}y) + (\frac{1-k}{k})\max\{d(seq[\mathcal{T}x,q],\mathcal{S}x), d(seq[\mathcal{T}y,q],\mathcal{S}y)\}. \end{aligned}$$

Thus all the conditions of Theorem 3 are satisfied. Notice that the segment joining $(-1,-1)$ *and* $(1,1)$ *remains fixed under* $\mathcal{T}$ *and* $\mathcal{S}$ *both which in all substantiates Theorem 3. Notice that* $\mathcal{K}$ *is not compact in this example and hence Theorem 2 cannot be used in the context of this example. However, the set* $\mathcal{D}$ *is compact as needed.*

The following corollary is an immediate consequence of Theorem 3.

Corollary 2. *Let* $\mathcal{T}$ *and* $\mathcal{S}$ *be self-maps of a convex metric space* $(\mathcal{X},d)$ *and* $\mathcal{K}$ *a subset of* $\mathcal{X}$ *such that* $\mathcal{T}(\partial\mathcal{K}) \subseteq \mathcal{K}$*, where* $\partial\mathcal{K}$ *stands for the boundary of* $\mathcal{K}$ *and* $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$*, where* $x_0 \in \mathcal{X}$*, where* $x_0 \in \mathcal{X}$*. Suppose* $\mathcal{D} = \mathcal{P}_{\mathcal{K}}(x_0)$ *is nonempty convex such that* $\mathcal{S}(\mathcal{D}) = \mathcal{D}$*,* $q \in Fix(\mathcal{S})$*,* $\mathcal{S}$ *is continuous as well as* $\mathcal{W}$*-affine and the pair* $(\mathcal{T},\mathcal{S})$ *is* $\mathcal{R}$*-subweakly commuting on* $\mathcal{D}$*. If* $\mathcal{T}$ *and* $\mathcal{S}$ *satisfy (for all* $x,y \in \mathcal{D}' = \mathcal{D} \cup \{x_0\}$*)*

$$d(\mathcal{T}x,\mathcal{T}y) \leq \begin{cases} d(\mathcal{S}x,\mathcal{S}x_0), & \text{if } y = x_0; \\ d(\mathcal{S}x,\mathcal{S}y) + (\frac{1-k}{k}) & \\ \ \times \max\{d(seq[\mathcal{T}x,q],\mathcal{S}x), d(seq[\mathcal{T}y,q],\mathcal{S}y)\}, & \text{if } y \in \mathcal{D} \end{cases} \tag{9}$$

where $0 < k < 1$*, then* $\mathcal{T}$ *and* $\mathcal{S}$ *have a common fixed point in* $\mathcal{D}$*, provided* $\mathcal{D}$ *is compact and* $\mathcal{T}$ *is continuous.*

Before stating our next theorem, we need to define

$$\mathcal{D}^* = \mathcal{P}_{\mathcal{K}}(x_0) \cap \mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0),$$

where

$$\mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0) = \{x \in \mathcal{K} : \mathcal{S}x \in \mathcal{P}_{\mathcal{K}}(x_0)\}.$$

Theorem 4. *Let $\mathcal{T}$ and $\mathcal{S}$ be self-maps of a convex metric space $(\mathcal{X}, d)$ and $\mathcal{K}$ a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) \subseteq \mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$, where $x_0 \in \mathcal{X}$. Suppose $\mathcal{D}^*$ is nonempty convex such that $\mathcal{S}(\mathcal{D}^*) = \mathcal{D}^*$, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is nonexpansive and $\mathcal{W}$-affine on $\mathcal{P}_{\mathcal{K}} \cup \{x_0\}$, and the pair $(\mathcal{T}, \mathcal{S})$ is subcompatible on $\mathcal{D}^*$. If $\mathcal{T}$ and $\mathcal{S}$ satisfy (for all $x, y \in \mathcal{D}^* \cup \{x_0\}$)*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \begin{cases} d(\mathcal{S}x, \mathcal{S}x_0), & \text{if } y = x_0; \\ d(\mathcal{S}x, \mathcal{S}y) + (\frac{1-k}{k}) & \\ \quad \times \max\{d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y)\}, & \text{if } y \in \mathcal{D}^* \end{cases} \tag{10}$$

where $0 < k < 1$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point in $\mathcal{P}_{\mathcal{K}}(x_0)$, provided $\mathcal{D}^*$ is compact and $\mathcal{T}$ is continuous.

Proof. Let $x \in \mathcal{D}^*$. Then, $x \in \mathcal{P}_{\mathcal{K}}(x_0)$ and hence $d(x, x_0) = d(x_0, \mathcal{K})$. Notice that for any $t \in (0, 1)$,

$$d(\mathcal{W}(x, x_0, t), x_0) = d(\mathcal{W}(x, x_0, t), \mathcal{W}(x_0, x_0, t)) \leq td(x, x_0) < d(x_0, \mathcal{K}).$$

Now, it follows that the segment $\{\mathcal{W}(x, x_0, t) : 0 < t < 1\}$ and the set $\mathcal{K}$ are disjoint. Thus x is not in the interior of $\mathcal{K}$ and so $x \in \partial\mathcal{K} \cap \mathcal{K}$. Since $\mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) \subset \mathcal{K}$, $\mathcal{T}x$ must be in $\mathcal{K}$. Now, proceeding on the lines of the proof of Theorem 3, we have $\mathcal{T}x \in \mathcal{P}_{\mathcal{K}}(x_0)$. As $\mathcal{S}$ is nonexpansive on $\mathcal{P}_{\mathcal{K}}(x_0) \cup \{x_0\}$, we have

$$d(\mathcal{S}\mathcal{T}x, x_0) \leq d(\mathcal{T}x, \mathcal{T}x_0) \leq d(\mathcal{S}x, \mathcal{S}x_0) = d(\mathcal{S}x, x_0) = d(x_0, \mathcal{K}).$$

Thus $\mathcal{S}\mathcal{T}x \in \mathcal{P}_{\mathcal{K}}(x_0)$ and so $\mathcal{T}x \in \mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0)$. Hence $\mathcal{T}x \in \mathcal{D}^*$. Consequently, $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{D}^* = \mathcal{S}(\mathcal{D}^*)$. Now, in view of Theorem 3, $\mathcal{P}_{\mathcal{K}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$. □

Remark 2. *It is straightforward to notice that Theorem 4 is trivial if $x_0 \in \mathcal{K}$. Otherwise the disjointness of $\mathcal{K}$ with the segment $\mathcal{W}(x, x_0, t)$ is no longer necessarily true if $x_0 \in \mathcal{K}$ (e.g. Example 12).*

In what follows, we observe that Example 12 can be utilized to demonstrate Theorem 4.

Example 13. *One can easily notice that the hypotheses of Theorem 4 can be demonstrated by Example 12 because $(\partial\mathcal{K} \cap \mathcal{K}) = \mathcal{K}$ and $\mathcal{T}\mathcal{K} = \mathcal{K}$ and henceforth $\mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) = \mathcal{K} \subset \mathcal{K}$. Also $\mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0) = \mathcal{K} = \mathcal{P}_{\mathcal{K}}(x_0)$ and $\mathcal{D}^* = \mathcal{P}_{\mathcal{K}}(x_0) \cap \mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0) = \mathcal{D}$. The detailed verification is already available in Example 12.*

Corollary 3. *Let $\mathcal{T}$ and $\mathcal{S}$ be self-maps of a convex metric space $(\mathcal{X}, d)$ and $\mathcal{K}$ a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) \subseteq \mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$, where $x_0 \in \mathcal{X}$. Suppose $\mathcal{D}^*$ is nonempty convex such that $\mathcal{S}(\mathcal{D}^*) = \mathcal{D}^*$, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is continuous and $\mathcal{W}$-affine, and the pair $(\mathcal{T}, \mathcal{S})$ is commuting on $\mathcal{D}^*$. If $\mathcal{T}$ and $\mathcal{S}$ satisfy (10) for all $x, y \in \mathcal{D}^* \cup \{x_0\}$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point in $\mathcal{P}_{\mathcal{K}}(x_0)$, provided $\mathcal{D}^*$ is compact and $\mathcal{T}$ is continuous.*

Proof. Let $x \in \mathcal{D}^*$, then proceeding as in the proof of Theorem 4, we obtain $\mathcal{T}x \in \mathcal{P}_{\mathcal{K}}(x_0)$. Moreover, since $\mathcal{T}$ commutes with $\mathcal{S}$ on $\mathcal{D}^*$, $\mathcal{T}$ and $\mathcal{S}$ satisfy (10), henceforth

$$d(\mathcal{S}\mathcal{T}x, x_0) = d(\mathcal{T}\mathcal{S}x, \mathcal{T}x_0) \leq d(\mathcal{S}^2x, \mathcal{S}x_0) = d(\mathcal{S}x, x_0) = d(x_0, \mathcal{K}).$$

Thus $\mathcal{S}\mathcal{T}x \in \mathcal{P}_{\mathcal{K}}(x_0)$ and so $\mathcal{T}x \in \mathcal{D}_{\mathcal{K}}^{\mathcal{S}}(x_0)$. Hence $\mathcal{T}x \in \mathcal{D}^*$. Consequently, $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{D}^* = \mathcal{S}(\mathcal{D}^*)$. Now, in view of Theorem 3, $\mathcal{P}_{\mathcal{K}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$. □

Corollary 4. *Let $\mathcal{T}$ and $\mathcal{S}$ be self-maps of a convex metric space $(\mathcal{X}, d)$and $\mathcal{K}$ a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) \subseteq \mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$, where $x_0 \in \mathcal{X}$. Suppose $\mathcal{D}^*$ is nonempty q-starshaped such that $\mathcal{S}(\mathcal{D}_{\mathcal{K}}^{\mathcal{S}}(x_0)) \cap \mathcal{D}^* \subset \mathcal{S}(\mathcal{D}^*) \subset \mathcal{D}^*$. Further, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is continuous and $\mathcal{W}$-affine, and the pair $(\mathcal{T}, \mathcal{S})$ is commuting on $\mathcal{D}^*$. If $\mathcal{T}$ and $\mathcal{S}$ satisfy (10) for all $x, y \in \mathcal{D}^* \cup \{x_0\}$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point in $\mathcal{P}_{\mathcal{K}}(x_0)$, provided $\mathcal{D}^*$ is compact and $\mathcal{T}$ is continuous.*

Proof. Let $x \in \mathcal{D}^*$. Proceeding on the lines of Theorem 4, we obtain $\mathcal{T}x \in \mathcal{D}^*$, i.e. $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{D}^*$, $x \in \partial\mathcal{K} \cap \mathcal{K}$ and so $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) \subset \mathcal{S}(\mathcal{K})$. Therefore, we can choose $y \in \mathcal{K}$ such that $\mathcal{T}x = \mathcal{S}y$. As $\mathcal{S}y = \mathcal{T}x \in \mathcal{P}_{\mathcal{K}}(x_0)$, it follows that $y \in \mathcal{D}_{\mathcal{K}}^{\mathcal{S}}(x_0)$. Consequently, $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{S}(\mathcal{D}_{\mathcal{K}}^{\mathcal{S}}(x_0)) \subset \mathcal{P}_{\mathcal{K}}(x_0)$. Therefore, $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{S}(\mathcal{D}_{\mathcal{K}}^{\mathcal{S}}) \cap \mathcal{D}^* \subset \mathcal{S}(\mathcal{D}^*) \subset \mathcal{D}^*$. Now, in view of Corollary 3, $\mathcal{P}_{\mathcal{K}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$. □

Remark 3. *It is straightforward to observe that $\mathcal{S}(\mathcal{P}_{\mathcal{K}}(x_0)) \subset \mathcal{P}_{\mathcal{K}}(x_0)$ implies $\mathcal{P}_{\mathcal{K}}(x_0) \subset \mathcal{D}_{\mathcal{K}}^{\mathcal{S}}(x_0)$ and henceforth $\mathcal{D}^* = \mathcal{P}_{\mathcal{K}}(x_0)$. Consequently, Theorem 4, and Corollaries 3 and 4 remain valid when $\mathcal{D}^* = \mathcal{P}_{\mathcal{K}}(x_0)$.*

Remark 4. *Theorem 3 as well as Corollary 4 improve Theorem* 6 *of Beg [3] owing to the fact that we have employed a relatively more generalized nonexpansive subcompatible pair of mappings as opposed to a relatively contractive commuting pair.*

Remark 5. *Theorem 3 together with Corollary 4 improves Theorem* 3.2 *of Al - Thagafi [2], Theorem* 3 *of Sahab et al. [15] and corresponding relevant results contained in Singh [23, 24] as we have utilized a relatively more generalized nonexpansive subcompatible pair of mappings in the setting of the convex metric space.*

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