MATHEMATICAL COMMUNICATIONS
Math. Commun. **16**(2011), 13–20.

Fixed point theorems in modular spaces

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Received November 26, 2008; accepted May 12, 2010

Abstract. In this paper we establish new fixed point theorems for modular spaces.

AMS subject classifications: 46A32

Key words: modular space, non-expansive mapping, Fatou property, Δ_2-condition, Musielak-Orlicz space

1. Introduction

The theory of modular space was initiated by Nakano [5] in connection with the theory of order spaces and redefined and generalized by Musielak ans Orlicz [4]. By defining a norm, particular Banach spaces of functions can be considered. Metric fixed theory for these Banach spaces of functions has been widely studied (see [1]). Another direction is based on considering an abstractly given functional which controls the growth of the functions. Eventhough a metric is not defined, many problems in fixed point theory for nonexpansive mappings can be formulated in modular spaces.
In this paper, a fixed point theorem for nonlinear contraction in the modular space is proved. In order to do this, for the sake of convenience, some definitions and notations are recalled from [2, 3, 4, 5] and [8].

Definition 1. *Let X be an arbitrary vector space over $K(=\mathbb{R}$ or $\mathbb{C})$. A functional $\rho : X \to [0, +\infty)$ is called modular if:*

1. $\rho(x) = 0$ *if and only if* $x = 0$.
2. $\rho(\alpha x) = \rho(x)$ *for* $\alpha \in K$ *with* $|\alpha| = 1, \forall x \in X$.
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *if* $\alpha, \beta \geq 0, \alpha + \beta = 1, \forall x, y \in X$.

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Definition 2. *If 3) in Definition 5 is replaced by:*

$$\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$$

for $\alpha, \beta \geq 0, \alpha^s + \beta^s = 1$ *with an* $s \in (0, 1]$, *then the modular* ρ *is called an* s*-convex modular; and if* $s = 1$, ρ *is called a convex modular.*

Definition 3. *A modular* ρ *defines a corresponding modular space, i.e. the space* X_ρ *is given by*

$$X_\rho = \{x \in X |\ \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Definition 4. *Let* X_ρ *be a modular space.*

1. *A sequence* $\{x_n\}_n$ *in* X_ρ *is said to be:*
   (a) ρ*-convergent to* x *if* $\rho(x_n - x) \to 0$ *as* $n \to +\infty$.
   (b) ρ*-Cauchy if* $\rho(x_n - x_m) \to 0$ *as* $n, m \to +\infty$.
2. X_ρ *is* ρ*-complete if any* ρ*-Cauchy sequence is* ρ*-convergent.*
3. *A subset* $B \subset X_\rho$ *is said to be* ρ*-closed if for any sequence* $\{x_n\}_n \subset B$ *with* $x_n \to x$, *then* $x \in B$. $\bar{B}^\rho$ *denotes the closure of* B *in the sense of* ρ.
4. *A subset* $B \subset X_\rho$ *is called* ρ*-bounded if:*
   $$\delta_\rho(B) = \sup_{x,y \in B} \rho(x - y) < +\infty,$$
   where $\delta_\rho(B)$ *is called the* ρ*-diameter of* B.
5. *We say that* ρ *has Fatou property if:*
   $$\rho(x - y) \leq \liminf \rho(x_n - y_n)$$
   whenever
   $$x_n \xrightarrow{\rho} x \text{ and } y_n \xrightarrow{\rho} y.$$
6. ρ *is said to satisfy the* Δ_2*-condition if :* $\rho(2x_n) \to 0$ *as* $n \to +\infty$ *whenever* $\rho(x_n) \to 0$ *as* $n \to +\infty$.

Example 1 (see [3])**.** *Let* (Ω, Σ, μ) *be a measure space. A real function* φ *defined on* $\Omega \times \mathbb{R}^+$ *will be said to belong to the class* Φ *if the following conditions are satisfied,*

(i) $\varphi(\omega, u)$ *is a nondecreasing continuous of* u *such that* $\varphi(\omega, 0) = 0$, $\varphi(\omega, u) > 0$ *for* $u > 0$ *and* $\varphi(\omega, u) \to \infty$ *as* $u \longrightarrow \infty$,

(ii) $\varphi(\omega, u)$ *is a* Σ*-measurable function of* ω *for all* $u \geq 0$,

(iii) $\varphi(\omega, u)$ *is a convex function of* u, *for all* $\omega \in \Omega$.

Moreover, consider X, *the set of all real-valued* Σ*-measurable and finite* μ*-almost everywhere functions on* Ω, *with equality* μ*-almost everywhere. Since* $\varphi(\omega, |x(\omega)|)$ *is a* Σ*-measurable function of* $\omega \in \Omega$ *for every* $x \in X$, *set*

$$\rho(x) = \int_\Omega \varphi(\omega, |x(\omega)|) d\mu(\omega). \tag{1}$$

It is easy to check that ρ *is a convex modular on* X. *The associated modular function space* X_ρ *is called Musielak-Orlicz space and it will be denoted by* L^φ.

2. Main Result

Definition 5 (see [6]). *Let T be a selfmapping on a modular space (X_ρ, ρ). A fixed point of T is said to be contractive if all the Picard iterates of T converge to this fixed point.*

Definition 6. *A selfmapping T on a modular space (X_ρ, ρ) is said to be*

- *contractive if $\rho(c(Tx - Ty)) < \rho(l(x - y))$ for all $x, y \in X_\rho$ with $x \neq y$ and for some c and l with $0 < l < c$;*
- *asymptotically regular if $\lim_{n\to\infty} \rho(T^{n+1}x - T^n x) = 0$ for each $x \in X$.*

Definition 7 (see [6]). *A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a gauge function of the class of*

(i) Φ_1 *if for any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < t < \delta$ implies $\varphi(t) \leq \varepsilon$;*

(ii) Φ_2 *if for any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon \leq t < \delta$ implies $\varphi(t) \leq \varepsilon$;*

(iii) Φ *if for any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varphi(\delta) \leq \varepsilon$.*

It can be easily seen that $\Phi \subset \Phi_2 \subset \Phi_1$.

Lemma 1 (see [6]). *Let T be a selfmapping of an orbitrary set X and let $E : X \to \mathbb{R}^+$ be a real-valued function defined on X. Suppose that the following conditions hold:*

(i) There exists a function $\varphi \in \Phi_1$ such that $E(Tx) \leq \varphi(E(x))$ for all $x \in X$;

(ii) $E(x) > 0$ implies $E(Tx) < E(x)$ and $E(x) = 0$ implies $E(Tx) = 0$.

Then $\lim_{n\to\infty} E(T^n x) = 0$ for each $x \in X$.

Lemma 2 (see [6]). *Suppose $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. Then $\varphi \in \Phi$ if and only if φ satisfies the following conditions:*

(i) φ is nondecreasing and $\varphi(t) \leq t$ for all $t \geq 0$;

(ii) if $\varphi(\varepsilon^+) = \varepsilon$ for some $\varepsilon > 0$, then there exists $\delta > \varepsilon$ such that $\varphi(\delta) = \varepsilon$.

Lemma 3 (see [6]). *For any function $\phi \in \Phi$ there exists a right continuous function $\varphi \in \Phi$ such that $\phi \leq \varphi$. Moreover, one can choose φ to satisfy also the condition:*

$$\varphi(t) > 0 \quad \forall\, t > 0.$$

Lemma 4 (see [6]). *Let T be a selfmapping on an arbitrary set x and let $E : X \to \mathbb{R}^+$ and $F : X \to \mathbb{R}^+$ be two real-valued functions defined on X. Suppose that*

$$E(Tx) \leq F(x) \quad \forall\, x \in X.$$

Then the following statements are equivalent:

(i) There is a function $\varphi \in \Phi_1$ such that $E(Tx) \leq \varphi(F(x))$ for any $x \in X$.

(ii) *For any* $\varepsilon > 0$ *there is* $\delta > \varepsilon$ *such that* $\varepsilon < F(x) < \delta$ *implies* $E(Tx) \leq \varepsilon$

Theorem 1. *Let* T *be a selfmapping on a modular space* (X_ρ, ρ) *and let* $F : X_\rho^2 \to \mathbb{R}^+$ *be a real-valued function defined on* X_ρ^2. *Suppose that*

$$\rho(Tx - Ty) \leq F(x, y) \quad \forall\ x, y \in X_\rho.$$

Then the following statements are equivalent:

(i) *There exists* $\varphi \in \Phi_1$ *such that* $\rho(Tx - Ty) \leq \varphi(F(x, y))$ *for any* $x, y \in X_\rho$.

(ii) *For any* $\varepsilon > 0$ *there exists* $\delta > \varepsilon$ *such that* $\varepsilon < F(x, y) < \delta$ *implies* $\rho(Tx - Ty) \leq \varepsilon$.

Proof. The proof is an immediate result of Lemma 3.1 in [6] by setting $E = \rho$. □

Lemma 5. *[6] Let* T *be a selfmapping on an arbitrary set* X *and let* $E : X \to \mathbb{R}^+$ *and* $F : X \to \mathbb{R}^+$ *be two real-valued functions defined on* X. *Suppose that*

$$F(x) = 0 \ \ \textit{implies} \ \ E(Tx) = 0.$$

Then the following statements are equivalent:

(i) *There exists a function* $\varphi \in \Phi_1$ *such that* $E(Tx) < \varphi(F(x))$ *for any* $x \in X$ *with* $F(x) > 0$.

(ii) *For any* $\varepsilon > 0$ *there is* $\delta > \varepsilon$ *such that* $\varepsilon \leq F(x) < \delta$ *implies* $E(Tx) < \varepsilon$.

Theorem 2. *Let* T *be a selfmapping on a modular space* (X_ρ, ρ) *and let* $F : X_\rho^2 \to \mathbb{R}^+$ *be a real-valued function defined on* X_ρ^2. *Suppose that*

$$F(x, y) = 0 \ \ \textit{implies} \ \ \rho(Tx - Ty) = 0. \tag{2}$$

Then the following statements are equivalent:

(i) *There exists* $\varphi \in \Phi_2$ *such that* $\rho(Tx - Ty) < \varphi(F(x, y))$ *for any* $x, y \in X_\rho$ *with* $F(x, y) > 0$.

(ii) *For any* $\varepsilon > 0$ *there exists* $\delta > \varepsilon$ *such that* $\varepsilon \leq F(x, y) < \delta$ *implies* $\rho(Tx - Ty) < \varepsilon$.

Proof. The theorem is a special case of Lemma 3.4 in [6]. It suffices to let $E = \rho$, the result is immediate. □

The following corollary is a special case of Theorem 2 by setting $F = \rho$.

Corollary 1. *Let* T *be a selfmapping on a modular space* (X_ρ, ρ). *Then the following statements are equivalent:*

(i) *There exists* $\varphi \in \Phi_2$ *such that* $\rho(Tx - Ty) < \varphi(\rho(x - y))$ *for any* $x, y \in X_\rho$ *with* $x \neq y$.

(ii) *For any* $\varepsilon > 0$ *there exists* $\delta > \varepsilon$ *such that* $\varepsilon \leq \rho(x - y) < \delta$ *implies* $\rho(Tx - Ty) < \varepsilon$.

In (i) one can choose φ to be also nondecreasing, right continuous and satisfying

$$\varphi(t) > 0 \quad \text{for all} \quad t > 0. \tag{3}$$

Theorem 3 (see [7])**.** *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Assume that $\psi : \mathbb{R} \to (0, \infty)$ is an increasing and upper semi continuous function satisfying*

$$\psi(t) < t, \quad \text{for all } t.$$

Let B be a ρ_closed subset of X_ρ and $T : B \to B$ a mapping such that there exist $c, l \in \mathbb{R}^+$ with $c > l$,

$$\rho(c(Tx - Ty)) \leq \psi(\rho(l(x - y)))$$

for all $x, y \in B$. Then T has a fixed point.

Using Theorem 3 and Corollary 1 we obtain the following result.

Corollary 2. *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition and let the following condition hold:*

For any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon \leq \rho(l(x - y)) < \delta$ implies $\rho(c(Tx - Ty)) < \varepsilon$, for fixed $c > l > 0$

Then T has a fixed point.

Proof. The function ψ in Theorem 3 is clearly a function in the set Φ_2, since for all t

$$\psi(t) < t.$$

By Corollary 1, the mapping T satisfies the condition of Theorem 3. Therefore, T has a fixed point. □

Theorem 4. *Let T be a continuous and asymptotically regular selfmapping on a ρ-complete modular space (X_ρ, ρ) which ρ satisfies the Δ_2-condition. And let the following conditions hold for $0 < l < c$:*

(i) there exists a $\varphi \in \Phi_1$ such that

$$\rho(c(Tx - Ty)) \leq \varphi\varrho(l(x - y))$$

for all $x, y \in X_\rho$.

(ii) $\rho(c(Tx - Ty)) < \varrho(l(x - y))$ for all $x, y \in X_\rho$ with $x \neq y$,

where $\varrho(x - y) = \rho(x - y) + \gamma[\rho(Tx - x) + \rho(Ty - y)]$ and $\gamma \geq 0$.

Then T has a unique contractive fixed point.

Proof. We prove that $(T^n x)$ for each $x \in X_\rho$ is a Cauchy sequence. Let $x_n = T^n x$, since T is asymptotically regular, then the sequence $(T^n x - T^{n-1}x)$ converges to zero and because $\varphi \in \Phi_1$ for $\varepsilon > 0$, there is a $\delta > \varepsilon$ such that for every $t \in \mathbb{R}^+$, $\varepsilon < t < \delta$ implies $\varphi(t) \leq \varepsilon$.

Without loss of generality, we can assume $\delta > 2\varepsilon$. Since $\rho(T^n x - T^{n-1}x) \to 0$ and ρ satisfies the Δ_2-condition, hence $\rho(\alpha_0 l(T^n x - T^{n-1}x)) \to 0$ where

$$\frac{l}{c} + \frac{1}{\alpha_0} = 1,$$

therefore, there exists $N \geq 1$ such that

$$\rho(\alpha_0 l(T^n x - T^{n-1}x)) < \frac{\delta - \varepsilon}{1 + 2\gamma} \quad \forall n \geq N \tag{4}$$

By induction we show that

$$\rho(l(x_m - x_n)) < \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma} \quad \forall m, n \in \mathbb{N}; m \geq n \geq N. \tag{5}$$

Suppose $n \geq N$ is fixed. Obviously, (5) holds for $m = n$. Now, let for $m \geq n$ (5) holds, we investigate that for $m + 1$. We have:

$$\begin{aligned} \rho(l(x_{m+1} - x_n)) &= \rho(l(x_{m+1} - x_{n+1} + x_{n+1} - x_n)) \\ &= \rho(\tfrac{cl}{c}(x_{m+1} - x_{n+1}) + \tfrac{\alpha_0 l}{\alpha_0}(x_{n+1} - x_n)) \\ &\leq \rho(c(x_{m+1} - x_{n+1})) + \rho(\alpha_0 l(x_{n+1} - x_n)) \end{aligned}$$

Claim:

$$\rho(c(Tx_m - Tx_n)) = \rho(c(x_{m+1} - x_{n+1})) \leq \varepsilon \tag{6}$$

We consider two cases.

Case 1: $\varrho(l(x_n - x_m)) \leq \varepsilon$.

From (ii) we get

$$\rho(c(Tx - Ty)) \leq \varrho(l(x - y));$$

therefore,

$$\rho(c(Tx_m - Tx_n)) \leq \varrho(l(x_n - x_m)) \leq \varepsilon.$$

So the claim is established.

Case 2: $\varrho(l(x_m - x_n)) > \varepsilon$.

By (i)

$$\rho(c(Tx_m - Tx_n)) \leq \varphi(\varrho(l(x_m - x_n))) \tag{7}$$

Then by the definition of $\varrho(x - y)$ we obtain

$$\varrho(l(x_m - x_n)) = \rho(l(x_m - x_n)) + \gamma[\rho(l(T^n x - T^{n-1}x)) + \rho(l(T^m x - T^{m-1}x))]$$

Now from (4) and (5) we get

$$\varrho(l(x_m - x_n)) < \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma} + 2\gamma\frac{\delta - \varepsilon}{1 + 2\delta} = \delta$$

We note that since $\alpha_0 > 1$ hence $\rho(l(T^n x - T^{n-1}x)) \leq \rho(\alpha_0 l(T^n x - T^{n-1}x))$. Therefore

$$\varepsilon < \varrho(l(x_m - x_n)) < \delta$$

and then

$$\varphi(\varrho(l(x_m - x_n))) \leq \varepsilon.$$

And (7) implies (6). Now,

$$\begin{aligned}\rho(l(x_{m+1}-x_n)) &\leq \rho(\alpha_0 l(T^n x - T^{n-1}x)) + \rho(c(Tx_m - Tx_n))\\ &\leq \rho(\alpha_0 l(T^n x - T^{n-1}x)) + \varepsilon\\ &< \tfrac{\delta-\varepsilon}{1+2\gamma} + \varepsilon = \tfrac{\delta+2\gamma\varepsilon}{1+2\gamma},\end{aligned}$$

and (5) is proved. Since $\delta < 2\varepsilon$, then (5) concludes that $\rho(l(x_m - x_n)) < 2\varepsilon$ for all $m, n \in \mathbb{N}$ that $m \geq n \geq N$. Consequently, (lx_n) and therefore (x_n) is a Cauchy sequence and because X_ρ is complete, there is a point $a \in X_\rho$ such that $T^n x \to a$ and because of the continuity of T, a is a fixed point. Now, if a and b are two different fixed points of T, then

$$\rho(c(a-b)) = \rho(c(Ta - Tb)) < \varrho(l(a-b)) = \rho(l(a-b)),$$

which is impossible because $l < c$, hence a is a unique fixed point of T and the proof is complete. □

Theorem 5. *Let T be a continuous and asymptotically regular selfmapping on a complete modular space (X_ρ, ρ) which ρ satisfies the Δ_2-condition. And let the following conditions hold for $0 < l < c$:*

(i) For any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < \varrho(l(x-y)) < \delta$ implies $\rho(c(Tx - Ty)) \leq \varepsilon$;

(ii) $\rho(c(Tx - Ty)) < \varrho(l(x-y))$ for all $x, y \in X_\rho$ with $x \neq y$.

Then T has a contractive fixed point.

Proof. By choosing $F(x,y) = \rho(l(x-y))$ in Theorem 1, condition (i) is equivalent to

$$\rho(c(Tx - Ty)) \leq \varphi(\rho(l(x-y)))$$

for a gauge function $\varphi \in \Phi_1$. Therefore, the condition of Theorem 4 holds. So T has a fixed point. □

Acknowledgement

The authors would like to thank the referees for their helpful suggestions.

References

[1] T. Domimguez Benavides, M. A. Khamsi, S. Samadi, *Uniformly Lipschitzian mappings in modular function spaces*, Nonlinear Anal. **46**(2001), 267–278.

[2] A. Hajji, E. Hanebaly, *Fixed point theorem and its application to perturbed integral equations in modular function spaces.*, Electron. J. Differential Equations **46**(2001), 267–278.

[3] M. A. Khamsi, *Nonlinear semigroups in modular function spaces*, Math. Japon. **37**(1992), 291–299.

[4] J. Musielak, W. Orlicz, *On modular spaces*, Studia Math. **18**(1959), 591–597.

[5] H Nakano, *Modular semi-ordered spaces*, Tokyo, 1959.

[6] P. D. Prinov, *Fixed point theorems in metric spaces*, Nonlinear Anal. **64** (2006), 546–557.

[7] A. Razani, E. Nabizadeh, M. Beyg Mohammadi, S. Homaei Pour, *Fixed point of nonlinear and asymptotic contractions in the modular space*, Abstr. Appl. Anal. **2007**(2007), Article ID 40575.

[8] A. Ait Taleb, E. Hanebaly, *A fixed point theorem and its application to integral equations in modular function Spaces*, Proc. Amer. Math. Soc. **128**(2000), 419–426.