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A generalization of some well-known distances and related isometries

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Abstract. In this paper, we define a family of distance functions in the real plane, m-distance function, which includes the taxicab, Chinese checker, maximum, and alpha distance functions as special cases, and we show that the m-distance function determines a metric. Then we give some properties of the m-distance, and determine isometries of the plane with respect to the m-distance. Finally, we extend the m-distance function to three- and n-dimensional spaces, and show that each extended distance function determines a metric. We also give some properties of the m-distance in three-dimensional space.

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Key words: metric, m-distance, taxicab distance, Chinese checker distance, maximum distance, alpha distance, minimum distance set, isometry

1. Introduction

The *taxicab metric* was given in a family of metrics of the real plane by Minkowski [11]. Later, Chen [2] developed the *Chinese checker metric*, and Tian [16] gave a family of metrics, *alpha metric*, which includes the taxicab and Chinese checker metrics as special cases. Metric geometries based on these metrics have been recently studied and developed in many directions. See [1, 3, 4, 5, 6, 7, 9, 10, 12, 13] and [14] for some of studies.

In this work, we define a new distance function in the real plane $\mathbb{R}^2$, d_m, which includes all of the distance functions mentioned above as special cases (see Remark 2 in the end of the article). We show that the m-distance function determines a metric in $\mathbb{R}^2$, that is, d_m is a function from $\mathbb{R}^2 \times \mathbb{R}^2$ to $[0, \infty)$ satisfying the following conditions: (1) $d_m(A, B) = 0$ if and only if $A = B$, (2) $d_m(A, B) = d_m(B, A)$, and (3) $d_m(A, B) \leq d_m(A, C) + d_m(C, B)$ for all A, B, and C in $\mathbb{R}^2$. Then we give some properties of the m-distance, and determine the isometries of the plane with respect to the m-distance. Finally, we extend the m-distance function to three- and n-dimensional spaces, and show that each extended distance function determines a metric. We also give some properties of the m-distance in a three-dimensional space.

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2. The m-distance function in $\mathbb{R}^2$

The m-distance function and the m-distance between two points in the Cartesian plane are defined as follows:

Definition 1. *Let $A = (x_1, x_2)$ and $B = (y_1, y_2)$ be two points in $\mathbb{R}^2$. For each real numbers a, b, and m such that $a \geq b \geq 0 \neq a$, the function $d_m : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined by*

$$d_m(A, B) = (a\Delta_{AB} + b\delta_{AB}) \diagup \sqrt{1 + m^2}, \tag{1}$$

where $\Delta_{AB} = \max\{|(x_1 - y_1) + m(x_2 - y_2)|, |m(x_1 - y_1) - (x_2 - y_2)|\}$ and $\delta_{AB} = \min\{|(x_1 - y_1) + m(x_2 - y_2)|, |m(x_1 - y_1) - (x_2 - y_2)|\}$, is called the m-distance function in $\mathbb{R}^2$, and the real number $d_m(A, B)$ is called the m-distance between points A and B.

Remark 1. *Clearly, there are infinitely many different distance functions in the family of distance functions defined above, depending on values a, b and m. One can find the definition not to be well-defined since the m-distance between two points can also change according to values a and b. To remove this confusion, we must use values a and b on behalf of the distance, just as we use m. This can be done easily; for example by using the notation $m(a, b)$ instead of m in phrases d_m and m-distance. But we keep on using m for the sake of shortness, supposing values a and b are initially determined and fixed unless otherwise stated.*

In all that follows, we use relations $a' = a \diagup \sqrt{1 + m^2}$ and $b' = b \diagup \sqrt{1 + m^2}$ to shorten phrases. The following proposition shows that the m-distance function defined above is a metric.

Proposition 1. *The m-distance function determines a metric in $\mathbb{R}^2$.*

Proof. Clearly, $d_m(A, B)=0$ if and only if $A=B$, and $d_m(A, B)=d_m(B, A)$ for all A, B in $\mathbb{R}^2$. So, we only will show that d_m satisfies the triangle inequality, that is, $d_m(A, B) \leq d_m(A, C) + d_m(C, B)$ for points $A=(x_1, x_2)$, $B=(y_1, y_2)$ and $C=(z_1, z_2)$ in $\mathbb{R}^2$.
Let

$$\begin{aligned} p_{1i} &= \left|(-1)^j (x_i - y_i) + m(x_j - y_j)\right|, \\ p_{2i} &= \left|(-1)^j (x_i - z_i) + m(x_j - z_j)\right|, \\ p_{3i} &= \left|(-1)^j (y_i - z_i) + m(y_j - z_j)\right| \end{aligned}$$

for $i, j \in \{1, 2\}, i \neq j$. Thus, $d_m(A, B) = a' \max\{p_{11}, p_{12}\} + b' \min\{p_{11}, p_{12}\}$, $d_m(A, C) = a' \max\{p_{21}, p_{22}\} + b' \min\{p_{21}, p_{22}\}$ and $d_m(C, B) = a' \max\{p_{31}, p_{32}\} + b' \min\{p_{31}, p_{32}\}$. It is also easy to see that $p_{1i} \leq p_{2i} + p_{3i}$ for $i \in \{1, 2\}$. Then we have

$$\begin{aligned} d_m(A, B) &= a' \max\{p_{11}, p_{12}\} + b' \min\{p_{11}, p_{12}\} \\ &\leq a' \max\{p_{21} + p_{31}, p_{22} + p_{32}\} + b' \min\{p_{21} + p_{31}, p_{22} + p_{32}\}. \end{aligned} \tag{2}$$

Using this inequality, one can easily see that d_m satisfies the triangle inequality by examining the following four cases:

Case I: If $p_{21} \geq p_{22}$ and $p_{31} \geq p_{32}$, then $(p_{21} + p_{31}) \geq (p_{22} + p_{32})$ and $d_m(A, B) \leq (a'p_{21} + p_{31}) + b'(p_{22} + p_{32}) = d_m(A, C) + d_m(C, B)$.

Case II: If $p_{21} \leq p_{22}$ and $p_{31} \leq p_{32}$, then $(p_{21} + p_{31}) \leq (p_{22} + p_{32})$ and $d_m(A, B) \leq a'(p_{22} + p_{32}) + b'(p_{21} + p_{31}) = d_m(A, C) + d_m(C, B)$.

Case III: If $p_{21} \geq p_{22}$ and $p_{31} \leq p_{32}$, then there are two subcases:

i) Let $(p_{21} + p_{31}) \geq (p_{22} + p_{32})$. Since $b' \leq a'$ and $p_{31} \leq p_{32}$, it is clear that $(a' - b')(p_{31} - p_{32}) \leq 0 \Leftrightarrow a'(p_{31} - p_{32} + p_{21} - p_{21}) + b'(p_{32} - p_{31} + p_{22} - p_{22}) \leq 0 \Leftrightarrow a'(p_{21} + p_{31}) + b'(p_{22} + p_{32}) \leq (a'p_{21} + b'p_{22}) + (a'p_{32} + b'p_{31})$. Therefore,

$$d_m(A, B) \leq d_m(A, C) + d_m(C, B).$$

ii) Let $(p_{21} + p_{31}) \leq (p_{22} + p_{32})$. Since $b' \leq a'$ and $p_{21} \geq p_{22}$, it is clear that $(a' - b')(p_{22} - p_{21}) \leq 0 \Leftrightarrow a'(p_{22} - p_{21} + p_{32} - p_{32}) + b'(p_{21} - p_{22} + p_{31} - p_{31}) \leq 0 \Leftrightarrow a'(p_{22} + p_{32}) + b'(p_{21} + p_{31}) \leq (a'p_{21} + b'p_{22}) + (a'p_{32} + b'p_{31})$. Therefore,

$$d_m(A, B) \leq d_m(A, C) + d_m(C, B).$$

Case IV: If $p_{21} \leq p_{22}$ and $p_{31} \geq p_{32}$, then there are two subcases:

i) Let $(p_{21} + p_{31}) \geq (p_{22} + p_{32})$. Since $b' \leq a'$ and $p_{21} \leq p_{22}$, it is clear that $(a' - b')(p_{21} - p_{22}) \leq 0 \Leftrightarrow a'(p_{21} - p_{22} + p_{31} - p_{31}) + b'(p_{22} - p_{21} + p_{32} - p_{32}) \leq 0 \Leftrightarrow a'(p_{31} + p_{21}) + b'(p_{32} + p_{22}) \leq (a'p_{31} + b'p_{32}) + (a'p_{22} + b'p_{21})$. Therefore,

$$d_m(A, B) \leq d_m(A, C) + d_m(C, B).$$

ii) Let $(p_{21} + p_{31}) \leq (p_{22} + p_{32})$. Since $b' \leq a'$ and $p_{31} \geq p_{32}$, it is clear that $(a' - b')(p_{32} - p_{31}) \leq 0 \Leftrightarrow a'(p_{32} - p_{31} + p_{22} - p_{22}) + b'(p_{31} - p_{32} + p_{21} - p_{21}) \leq 0 \Leftrightarrow a'(p_{32} + p_{22}) + b'(p_{31} + p_{21}) \leq (a'p_{31} + b'p_{32}) + (a'p_{22} + b'p_{21})$. Therefore,

$$d_m(A, B) \leq d_m(A, C) + d_m(C, B).$$

□

Let $A = (x_1, x_2)$ and $B = (y_1, y_2)$ be two points in the Cartesian coordinate plane. Let l_A be the line through A with slope m, and l_B be the line through B with slope $-1/m$ (We denote the slope of lines parallel to y-axis by ∞, and suppose that if $m = 0$ and $k \in \mathbb{R}$-$\{0\}$ then, $k/m = \infty$, throughout the paper to shorten phrases). Since the Euclidean distances from A to l_B and from B to l_A are $d_E(A, l_B) = |(x_1 - y_1) + m(x_2 - y_2)| \big/ \sqrt{1 + m^2}$ and $d_E(B, l_A) = |m(x_1 - y_1) - (x_2 - y_2)| \big/ \sqrt{1 + m^2}$, the m-distance between points A and B can be given by

$$d_m(A, B) = a \max\{d_E(A, l_B), d_E(B, l_A)\} + b \min\{d_E(A, l_B), d_E(B, l_A)\}. \quad (3)$$

According to this fact, the m-distance between points A and B is constant a multiple of the Euclidean length of one of the shortest paths from A to B composed of line segments, each parallel to one of lines with slope m, $-1/m$, $[m(a^2 - b^2) + 2ab] \big/ [(a^2 - b^2) - 2abm]$, or $[m(a^2 - b^2) - 2ab] \big/ [(a^2 - b^2) + 2abm]$ (see Figure 1). Although, there exist generally infinitely many shortest paths between points A and B, we prefer to use the ones in Figure 1, and call each of them a *basic way*. Also we call each of lines

$mx - y = 0$ and $x + my = 0$ an *axis of direction*, determined by the real number m. Notice that $d_m(A,B) = d_{-1/m}(A,B)$.

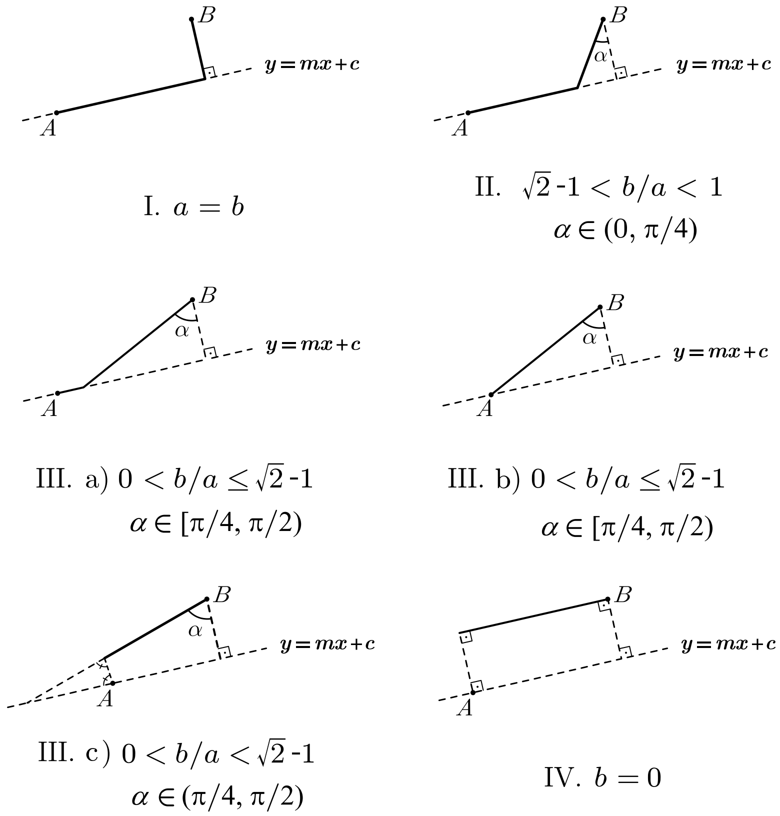

Figure 1: *The basic ways between points A and B with respect to the m-distance. In these figures, $b/a = \sec\alpha - \tan\alpha$, when $b \neq 0$.*

3. Some properties related to the m-distance

Let us denote the real plane endowed with the m-metric by $\mathbb{R}^2_m$. Then the unit m-circle in $\mathbb{R}^2_m$ is the set of points (x, y) in the plane which satisfy the equation

$$(a\max\{|x+my|\,, |mx-y|\} + b\min\{|x+my|\,, |mx-y|\})/\sqrt{1+m^2} = 1. \qquad (4)$$

One can see by calculation that if $0 < b/a < 1$, then the unit m-circle is an octagon with vertices $A_1 = \left(\frac{1}{ak}, \frac{m}{ak}\right)$, $A_2 = \left(\frac{1-m}{(a+b)k}, \frac{1+m}{(a+b)k}\right)$, $A_3 = \left(\frac{-m}{ak}, \frac{1}{ak}\right)$, $A_4 = \left(\frac{-1-m}{(a+b)k}, \frac{1-m}{(a+b)k}\right)$, $A_5 = \left(\frac{-1}{ak}, \frac{-m}{ak}\right)$, $A_6 = \left(\frac{m-1}{(a+b)k}, \frac{-1-m}{(a+b)k}\right)$, $A_7 = \left(\frac{m}{ak}, \frac{-1}{ak}\right)$, $A_8 = \left(\frac{1+m}{(a+b)k}, \frac{m-1}{(a+b)k}\right)$, where $k = \sqrt{1+m^2}$. If $a = b$ or $b = 0$, then the unit m-circle is a square with vertices A_1, A_3, A_5, A_7 or A_2, A_4, A_6, A_8, respectively (see Figure 2).

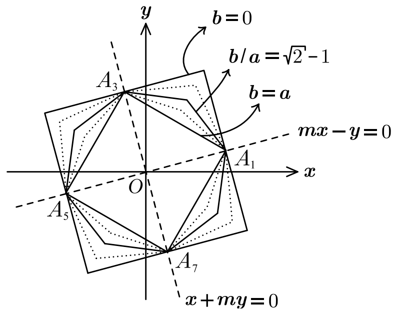

Figure 2: *Graph of unit m-circles*

The points A_1, A_3, A_5 and A_7 of the unit m-circle lie on the Euclidean circle $x^2 + y^2 = 1/a^2$. All vertices of unit m-circle lie on this Euclidean circle if and only if $a = b$ or $b/a = \sqrt{2} - 1$. In $\mathbb{R}^2_m$, it is easy to see that the ratio of the circumference of an m-circle to its diameter is $\pi_m = 4(a^2 + b^2)/(a^2 + ab)$.

Now, we examine the minimum distance set of points A and B in $\mathbb{R}^2_m$. The *minimum distance set* of A and B, $M(A, B)$, is defined by

$$M(A, B) = \{X : d_m(A, X) + d_m(X, B) = d_m(A, B)\}.$$

In the Euclidean plane, the minimum distance set of A and B is the line segment joining points A and B. It is not difficult to see that the minimum distance set of points A and B in $\mathbb{R}^2_m$ is generally a region whose boundary is a parallelogram with diagonal AB and with sides parallel to one of axes of direction or one of angle bisectors of axes of direction, as shown in Figure 3.

The following proposition gives an equation which relates the Euclidean distance to the m-distance between the points in the Cartesian coordinate plane:

Proposition 2. *For any two points A and B in $\mathbb{R}^2$ that do not lie on a vertical line, if n is the slope of the line through A and B, then*

$$d_E(A, B) = \rho(n) d_m(A, B) \tag{5}$$

where $\rho(n) = \sqrt{1 + n^2}/(a' \max\{|1 + mn|, |m - n|\} + b' \min\{|1 + mn|, |m - n|\})$. *If A and B lie on a vertical line, then*

$$d_E(A, B) = [1/(a' \max\{1, |m|\} + b' \min\{1, |m|\})] d_m(A, B). \tag{6}$$

Proof. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ with $x_1 \neq y_1$; then $n = (y_2 - y_1)/(x_2 - x_1)$. Clearly, $d_m(A, B) = |x_1 - x_2| (a' \max\{|1 + mn|, |m - n|\} + b' \min\{|1 + mn|, |m - n|\})$ and $d_E(A, B) = |x_1 - x_2| (1 + m^2)^{1/2}$, thus we have $d_E(A, B) = \rho(n) d_m(A, B)$, where $\rho(n) = (1 + n^2)^{1/2}/(a' \max\{|1 + mn|, |m - n|\} + b' \min\{|1 + mn|, |m - n|\})$. If $x_1 = y_1$, then $d_m(A, B) = |x_1 - x_2| (a' \max\{1, |m|\} + b' \min\{1, |m|\})$ and we have equation (6). □

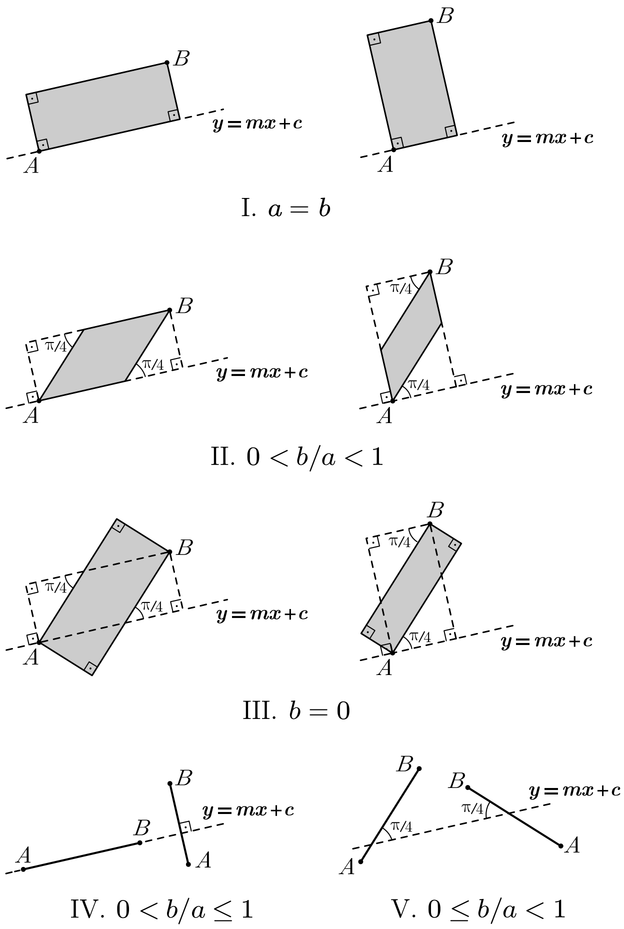

Figure 3: *Possible shapes of the minimum distance set of points A and B with respect to the m-distance*

The following two corollaries follow directly from Proposition 2:

Corollary 1. *Let A, B, C and D be four points in $\mathbb{R}^2$. If lines AB and CD are coincident, parallel or perpendicular to each other, then*

$$d_m(A,B) = d_m(C,D) \text{ if and only if } d_E(A,B) = d_E(C,D).$$

Corollary 2. *If A, B and X are three distinct collinear points in $\mathbb{R}^2$, then*

$$d_m(A,X) / d_m(X,B) = d_E(A,X) / d_E(X,B).$$

For two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in $\mathbb{R}^2$, by R_{AB} we denote a rectangular region (or a line segment) bounded by the lines through A or B and parallel to the axes of direction, which are $mx - y - mx_i + y_i = 0$ and $x + my - x_i -$

$my_i = 0$ for $i \in \{1, 2\}$. The following two propositions follow directly from equation (3), that is the geometric interpretation of the m-distance.

Proposition 3. *Let A, B and C be three points in $\mathbb{R}^2$ such that $C \in R_{AB}$. Then, $d_m(A,B) \geq d_m(A,C)$. In addition, if $b > 0$, then $d_m(A,B) = d_m(A,C)$ if and only if $B = C$. If $b = 0$, then $d_m(A,B) = d_m(A,C)$ if and only if C is on the line segment BD where D is a corner of R_{AB} such that $d_E(B,D) \leq d_E(A,D)$.*

Proposition 4. *Let A, B, C and D be four points in $\mathbb{R}^2$. If R_{AB} and R_{CD} are congruent, then $d_m(A,B) = d_m(C,D)$.*

4. Isometries of the plane with the m-distance

Notice that by Corollary 1, the m-distance between two points is invariant under all translations and rotations of $\pi/2$, π and $3\pi/2$ radians around a point. In addition to these transformations, one can easily see by Proposition 4 that the m-distance between two points is also invariant under the reflections about the lines parallel to an axis of direction or one of angle bisectors of axes of direction, which are the lines with slope m, $\frac{-1}{m}$, $\frac{m-1}{1+m}$ or $\frac{1+m}{1-m}$ (see Figure 4).

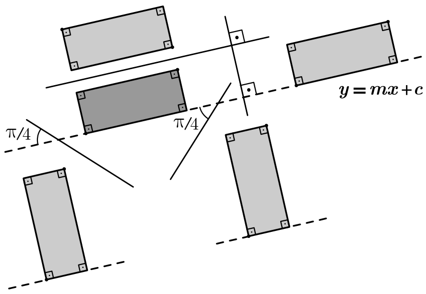

Figure 4: *The reflections about lines with slope m, $\frac{-1}{m}$, $\frac{m-1}{1+m}$ or $\frac{1+m}{1-m}$*

As a special case, if $b/a = \sqrt{2} - 1$ in d_m, since the rotation of $\pi/4$ and the reflection about a line making an angle of $\pi/8$ with the line $y = mx$ map the basic way of two points to the basic way of the images of the points, the m-distance between the points is preserved under these transformations (see Figure 5). Thus we can immediately state that the rotations of $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$ radians around a point, and the reflections about the lines making angles of $\pi/8$, $3\pi/8$, $5\pi/8$ or $7\pi/8$ radians with an axis of direction (or about the lines with slope

$$\frac{(1-\sqrt{2})m-1}{(1-\sqrt{2})+m}, \frac{(1+\sqrt{2})m-1}{(1+\sqrt{2})+m}, \frac{(1-\sqrt{2})m+1}{(1-\sqrt{2})-m} \quad or \quad \frac{(1+\sqrt{2})m+1}{(1+\sqrt{2})-m})$$

preserve the m-distance. Using the unit m-circle, one can easily see that there are no other rotations or reflections that preserve the m-distance.

Now, we state following two propositions as results of our observations above.

Proposition 5. *A reflection about the line $y = nx + c$ is an isometry of $\mathbb{R}^2_m$ if and only if*

$$n \in \left\{ m, -\frac{1}{m}, \frac{m-1}{1+m}, \frac{1+m}{1-m} \right\}$$

when $b/a \neq \sqrt{2} - 1$, and

$$n \in \left\{ m, -\frac{1}{m}, \frac{m-1}{1+m}, \frac{1+m}{1-m}, \frac{(1-\sqrt{2})m-1}{(1-\sqrt{2})+m}, \frac{(1+\sqrt{2})m-1}{(1+\sqrt{2})+m}, \frac{(1-\sqrt{2})m+1}{(1-\sqrt{2})-m}, \frac{(1+\sqrt{2})m+1}{(1+\sqrt{2})-m} \right\}$$

when $b/a = \sqrt{2} - 1$.

Proposition 6. *A rotation of θ around a point is an isometry of $\mathbb{R}^2_m$ if and only if $\theta \in \left\{ \frac{t\pi}{2} + 2k\pi : 0 \leq t \leq 3,\ k, t \in \mathbb{Z} \right\}$, when $b/a \neq \sqrt{2} - 1$, and $\theta \in \left\{ \frac{t\pi}{4} + 2k\pi : 0 \leq t \leq 7,\ k, t \in \mathbb{Z} \right\}$, when $b/a = \sqrt{2} - 1$.*

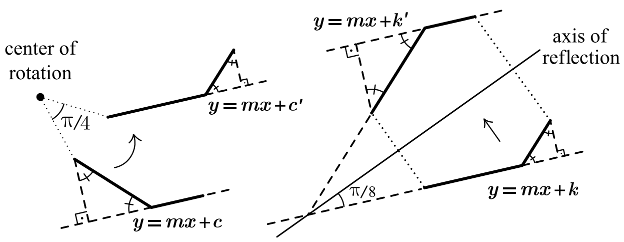

Figure 5: *The rotation of $\pi/4$ radians around a point, and the reflection about the line making angle of $\pi/8$ radians with the line $y = mx$*

Clearly, a composition of two isometries is also an isometry. Thus the Euclidean isometries that preserve m-distance are all translations, reflections given in Proposition 5, rotations given in Proposition 6, and their compositions. The question that must be answered now is: are there any other bijections of $\mathbb{R}^2$ onto $\mathbb{R}^2$ which preserve the m-distance? The following proposition and its corollary help us to give the answer of this question.

Proposition 7. *Let A and B be two distinct points in $\mathbb{R}^2_m$, and let $\tau : \mathbb{R}^2_m \to \mathbb{R}^2_m$ be an isometry. Then $\tau(M(A, B)) = M(\tau(A), \tau(B))$.*

Proof. Let $Y \in \tau(M(A, B))$. Then

$$\begin{aligned} Y \in \tau(M(A, B)) &\Leftrightarrow \exists X \in M(A, B) \text{ such that } Y = \tau(X) \\ &\Leftrightarrow d_m(A, X) + d_m(X, B) = d_m(A, B) \\ &\Leftrightarrow d_m(\tau(A), \tau(X)) + d_m(\tau(X), \tau(B)) = d_m(\tau(A), \tau(B)) \\ &\Leftrightarrow Y = \tau(X) \in M(\tau(A), \tau(B)). \end{aligned}$$

□

Corollary 3. *Let A and B be two distinct points in $\mathbb{R}^2_m$, and let $\tau : \mathbb{R}^2_m \to \mathbb{R}^2_m$ be an isometry. Then τ maps the vertices of $M(A,B)$ to the vertices of $M(\tau(A), \tau(B))$, and preserves the shape of $M(A,B)$.*

By the following proposition the answer to the question above is: no.

Theorem 1. *If $\tau : \mathbb{R}^2_m \to \mathbb{R}^2_m$ is an isometry, then it is a Euclidean isometry.*

Proof. Let A and B be two distinct points on a line not parallel to axes of direction and the angle bisectors of them, and let C be a corner of R_{AB} (distinct from A and B). Suppose that τ is a bijection of $\mathbb{R}^2$ onto itself that preserves the m-distance with $0 < b/a < 1$, and $\tau(A) = A'$, $\tau(B) = B'$ and $\tau(C) = C'$. By Proposition 7, τ maps $M(A,C)$ onto $M(A',C')$, $M(B,C)$ onto $M(B',C')$, and $M(A,B)$ onto $M(A',B')$. Since τ preserves the shapes of minimum distance sets of points by Corollary 3, $M(A',C')$ and $M(B',C')$ are line segments, and $M(A',B')$ is a parallelogram. Therefore, each pair of points A', C' and B', C' lies on a line parallel to one of axes of direction or one of angle bisectors of axes of direction, and points A', B' lie on a line not parallel to axes of direction and the angle bisectors of them. Now, one can see that the points A', B', C' form a right triangle (right angle at C') with legs parallel to axes of direction or with legs parallel to angle bisectors of axes of direction. If legs are parallel to axes of direction, then it is obvious by Corollary 1 that $d_E(A,C) = d_E(A',C')$ and $d_E(B,C) = d_E(B',C')$, so that triangle ABC is congruent to triangle $A'B'C'$ and $d_E(A,B) = d_E(A',B')$. If legs are parallel to angle bisectors of axes of direction, then it follows $b/a = \sqrt{2} - 1$, and therefore $d_E(A,C) = d_E(A',C')$ and $d_E(B,C) = d_E(B',C')$. Thus triangle ABC is congruent to triangle $A'B'C'$ again, and $d_E(A,B) = d_E(A',B')$. Hence, τ is a Euclidean isometry. The cases $b = 0$ and $b = 1$ are similar and left to the reader. □

In $\mathbb{R}^2_m$, let us denote by R_m the set of isometric rotations of θ, $\theta \in [0, 2\pi)$, around the origin, and by S_m the set of isometric reflections about the lines through the origin. Then we have the following corollary:

Corollary 4. *Let $\tau : \mathbb{R}^2_m \to \mathbb{R}^2_m$ be an isometry such that $\tau(O) = O$. Then $\tau \in R_m$ or $\tau \in S_m$.*

Consequently, if $b/a \neq \sqrt{2} - 1$, then we have the orthogonal group $O_m(2) = R_m \cup S_m$, consisting of four reflections and four rotations which give us the dihedral group D_4, that is the Euclidean symmetry group of the square. If $b/a = \sqrt{2} - 1$, then we have the orthogonal group $O_m(2) = R_m \cup S_m$, consisting of eight reflections and eight rotations which give us the dihedral group D_8, that is the Euclidean symmetry group of the regular octagon. The following proposition shows that all isometries of $\mathbb{R}^2_m$ are in $T(2)O_m(2)$, where $T(2)$ is the group of all translations of the plane.

Theorem 2. *Let $f : \mathbb{R}^2_m \to \mathbb{R}^2_m$ be an isometry. Then there exists a unique $T_A \in T(2)$ and $g \in O_m(2)$ such that $f = T_A \circ g$.*

Proof. Suppose that $f(O) = A$, where $A = (a_1, a_2)$. Define $g = T_{-A} \circ f$. It is clear that g is an isometry and $g(O) = O$. Thus, $g \in O_m(2)$ by Corollary 4, and $f = T_A \circ g$. The proof of uniqueness is trivial. □

5. The m-distance in $\mathbb{R}^3$

The m-distance between points A and B in $\mathbb{R}^3$ can be defined as follows:

Definition 2. *Let $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ be two points in $\mathbb{R}^3$. For each real numbers a, b, and m such that $a \geq b \geq 0 \neq a$, the function $d_m : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ defined by*

$$d_m(A, B) = (a\Delta_{AB} + b\delta_{AB}) \diagup \sqrt{1 + m^2}, \tag{7}$$

where $\Delta_{AB} = \max\{p_1, p_2, p_3\}$ and $\delta_{AB} = \min\{p_1 + p_2, p_1 + p_3, p_2 + p_3\}$ with $p_1 = |(x_1 - y_1) + m(x_2 - y_2)|$, $p_2 = |m(x_1 - y_1) - (x_2 - y_2)|$, $p_3 = |x_3 - y_3| \sqrt{1 + m^2}$, is called the m-distance function in $\mathbb{R}^3$, and the number $d_m(A, B)$ is called the m-distance between points A and B.

The following lemma helps us to show that the m-distance function defined above is a metric in $\mathbb{R}^3$.

Lemma 1. *Let $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ be two points in $\mathbb{R}^3$. If $\Delta_{AB} = p_j$, then for each $t \in I = \{1, 2, 3\}$*

$$d_m(A, B) = \left(a'p_j + b' \sum_{i \in I \setminus \{j\}} p_i \right) \geq \left(a'p_t + b' \sum_{i \in I \setminus \{t\}} p_i \right). \tag{8}$$

Proof. If $\Delta_{AB} = p_j$, then it is obvious that $\delta_{AB} = \sum_{i \in I\text{-}\{j\}} p_i$. Therefore, $d_m(A, B) = a'p_j + b' \sum_{i \in I \setminus \{j\}} p_i$. Let $L_1 = d_m(A, B)$ and $L_2 = a'p_t + b' \sum_{i \in I \setminus \{t\}} p_i$ for some $t \in I$. Then

$$\begin{aligned} L_1 - L_2 &= a' \left(p_j - p_t \right) + b' \left(\sum_{i \in I \setminus \{j\}} p_i - \sum_{i \in I \setminus \{t\}} p_i \right) \\ &= a' \left(p_j - p_t \right) + b' \left(p_t - p_j \right) \\ &= \left(a' - b' \right) \left(p_j - p_t \right). \end{aligned}$$

Since $(a' - b') \geq 0$ and $(p_j - p_t) \geq 0$, $L_1 \geq L_2$ for each $t \in I$, as claimed. □

The m-distance function determines a metric in $\mathbb{R}^3$ by the following theorem.

Theorem 3. *The m-distance function in $\mathbb{R}^3$ determines a metric.*

Proof. Clearly, $d_m(A, B) = 0$ if and only if $A = B$, and $d_m(A, B) = d_m(B, A)$ for all A, B in $\mathbb{R}^3$. So, we only will show that d_m satisfies the triangle inequality, that is $d_m(A, B) \leq d_m(A, C) + d_m(C, B)$ for points $A = (x_1, x_2, x_3)$, $B = (y_1, y_2, y_3)$ and $C = (z_1, z_2, z_3)$ in $\mathbb{R}^3$.
Let

$$\begin{aligned} p_{1i} &= \left| (-1)^j (x_i - y_i) + m(x_j - y_j) \right|, \\ p_{2i} &= \left| (-1)^j (x_i - z_i) + m(x_j - z_j) \right|, \\ p_{3i} &= \left| (-1)^j (y_i - z_i) + m(y_j - z_j) \right| \end{aligned}$$

for $i, j \in \{1,2\}, i \neq j$ and let

$$p_{13} = |x_3 - y_3|\,, \quad p_{23} = |x_3 - z_3| \quad \text{and} \quad p_{33} = |y_3 - z_3|\,.$$

Since

$$\begin{aligned} |(x_1 - y_1) + m(x_2 - y_2)| &= |(x_1 - z_1 + z_1 - y_1) + m(x_2 - z_2 + z_2 - y_2)| \\ &\leq |(x_1 - z_1) + m(x_2 - z_2)| + |(z_1 - y_1) + m(z_2 - y_2)| \\ |m(x_1 - y_1) - (x_2 - y_2)| &\leq |m(x_1 - z_1) - (x_2 - z_2)| + |m(z_1 - y_1) - (z_2 - y_2)|\,, \end{aligned}$$

and

$$|x_3 - y_3| \leq |x_3 - y_3| + |y_3 - z_3|\,,$$

we have $p_{1i} \leq (p_{2i} + p_{3i})$ for each $i \in I$. If $\Delta_{AB} = \max\{p_{11}, p_{12}, p_{13}\} = p_{1j}$, then

$$\begin{aligned} d_m(A,B) &= a' p_{1j} + b' \sum_{i \in I \setminus \{j\}} p_{1i} \\ &\leq a'(p_{2j} + p_{3j}) + b' \sum_{i \in I \setminus \{j\}} (p_{2i} + p_{3i}) \\ &= a' p_{2j} + b' \sum_{i \in I\text{-}\{j\}} p_{2i} + a' p_{3j} + b' \sum_{i \in I \setminus \{j\}} p_{3i}. \end{aligned}$$

Since

$$d_m(A,C) \geq a' p_{2j} + b' \sum_{i \in I \setminus \{j\}} p_{2i}$$

and

$$d_m(B,C) \geq a' p_{3j} + b' \sum_{i \in I \setminus \{j\}} p_{3i}$$

by Lemma 1, we have the inequality $d_m(A,B) \leq d_m(A,C) + d_m(C,B)$. □

Let $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ be two points in the three-dimensional Cartesian coordinate space, and let α_X, α'_X, α''_X be planes through the point X and perpendicular to vectors $(1, m, 0)$, $(-m, 1, 0)$, $(0, 0, 1)$, respectively. Since the Euclidean distances (d_E) from the point A to planes α_B, α'_B, α''_B (or from the point B to planes α_A, α'_A, α''_A) are

$$\begin{aligned} l_1 &= d_E(A, \alpha_B) = |(x_1 - y_1) + m(x_2 - y_2)| \diagup \sqrt{1+m^2} \\ l_2 &= d_E(A, \alpha'_B) = |m(x_1 - y_1) - (x_2 - y_2)| \diagup \sqrt{1+m^2} \\ l_3 &= d_E(A, \alpha''_B) = |x_3 - y_3|\,, \end{aligned}$$

respectively, the m-distance between points A and B can be given by

$$d_m(A,B) = a \max\{l_1, l_2, l_3\} + b \min\{l_1 + l_2, l_1 + l_3, l_2 + l_3\}. \tag{9}$$

According to this fact, the m-distance between points A and B is a constant a multiple of Euclidean length of one of the shortest paths from A to B composed of line segments (see Figure 6). In the following figures the rectangular prism having diagonal AB, has sides AA', $A'B'$, BB' with direction vectors $(1,m,0)$, $(-m,1,0)$, $(0,0,1)$, respectively.

Now, we give a relation among the m-distance and the Euclidean distance between two points in the Cartesian coordinate space:

Proposition 8. *Let $A=(x_1,x_2,x_3)$ and $B=(y_1,y_2,y_3)$ be two points in $\mathbb{R}^3$, and l the line through points A and B. If l has direction vector (p,q,r), then $d_E(A,B) = \mu_{AB} d_m(A,B)$, where $\mu_{AB} = \sqrt{p^2+q^2+r^2}/(a' \max\{p_1,p_2,p_3\} + b' \min\{p_1+p_2, p_1+p_3, p_2+p_3\})$ with $p_1 = |p+mq|$, $p_2 = |mp-q|$, $p_3 = |r|\sqrt{1+m^2}$.*

Proof. If l has direction vector (p,q,r), then $x_1-y_1=\lambda p$, $x_2-y_2=\lambda q$, and $x_3-y_3=\lambda r$ for some $\lambda \in \mathbb{R}$. Using this fact and coordinate definitions of $d_m(A,B)$ and $d_E(A,B)$, one can derive the equation in the proposition by a straightforward calculation. □

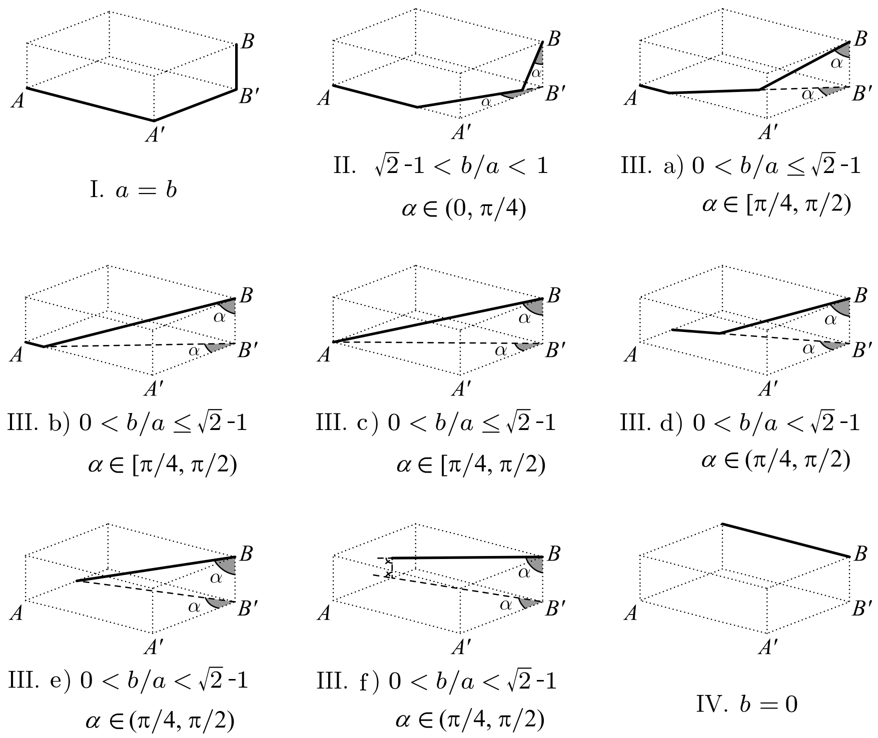

Figure 6: *Shortest paths between two points with respect to the m-distance*

Let us denote the three-dimensional Cartesian coordinate space endowed with the m-metric by $\mathbb{R}^3_m$. There is an open problem here: What is the group of isometries of $\mathbb{R}^3_m$? A special case of this question was answered in [8].

6. The m-distance in $\mathbb{R}^n$

The m-distance between points A and B in $\mathbb{R}^n$ can be defined as follows:

Definition 3. *Let $A = (x_1, x_2, ..., x_n)$ and $B = (y_1, y_2, ..., y_n)$ be two points in $\mathbb{R}^n$. For each real numbers a, b, and m such that $a \geq b \geq 0 \neq a$, the function $d_m : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by*

$$d_m(A, B) = (a\Delta_{AB} + b\delta_{AB}) \diagup \sqrt{1 + m^2}, \tag{10}$$

where Δ_{AB}=$\max\{p_1, p_2, ..., p_n\}$=$p_j$ and δ_{AB}=$\sum\limits_{i \in I\text{-}\{j\}} p_i$, I=$\{1, 2, ..., n\}$, with

$$p_1 = |(x_1 - y_1) + m(x_2 - y_2)|, \; p_2 = |m(x_1 - y_1) - (x_2 - y_2)|, \text{ and } p_k = |x_k - y_k|$$

for $k \in \{3, 4, ..., n\}$, is called the m-distance function in $\mathbb{R}^n$, and the number $d_m(A, B)$ is called the m-distance between points A and B.

The following lemma helps us to show that the m-distance function defined above determines a metric in $\mathbb{R}^n$.

Lemma 2. *Let $A = (x_1, x_2, ..., x_n)$ and $B = (y_1, y_2, ..., y_n)$ be two points in $\mathbb{R}^n$. For each $t \in I = \{1, 2, ..., n\}$*

$$d_m(A, B) \geq a' p_t + b' \sum_{i \in I \setminus \{t\}} p_i. \tag{11}$$

Proof. Let $L_1 = d_m(A, B) = a' p_j + b' \sum\limits_{i \in I\text{-}\{j\}} p_i$, $L_2 = a' p_t + b' \sum\limits_{i \in I\text{-}\{t\}} p_i$ for $t \in I$. Then

$$\begin{aligned} L_1 - L_2 &= a'(p_j - p_t) + b' \left(\sum_{i \in I \setminus \{j\}} p_i - \sum_{i \in I \setminus \{t\}} p_i \right) \\ &= a'(p_j - p_t) + b'(p_t - p_j) \\ &= (a' - b')(p_j - p_t). \end{aligned}$$

Since $(a' - b') \geq 0$ and $(p_j - p_t) \geq 0$, $L_1 \geq L_2$ for each $t \in I$, as claimed. □

The m-distance function defined above determines a metric in $\mathbb{R}^n$ by the following theorem.

Theorem 4. *The m-distance function in $\mathbb{R}^n$ determines a metric.*

Proof. Clearly, $d_m(A, B) = 0$ if and only if $A = B$, and $d_m(A, B) = d_m(B, A)$ for all A, B in $\mathbb{R}^n$. So, we will only show that d_m satisfies the triangle inequality, that is $d_m(A, B) \leq d_m(A, C) + d_m(C, B)$ for points $A = (x_1, x_2, ..., x_n)$, $B = (y_1, y_2, ..., y_n)$ and $C = (z_1, z_2, ..., z_n)$ in $\mathbb{R}^n$. Let

$$\begin{aligned} p_{1i} &= \left| (-1)^j (x_i - y_i) + m(x_j - y_j) \right|, \\ p_{2i} &= \left| (-1)^j (x_i - z_i) + m(x_j - z_j) \right|, \\ p_{3i} &= \left| (-1)^j (y_i - z_i) + m(y_j - z_j) \right| \end{aligned}$$

for $i, j \in \{1, 2\}, i \neq j$ and let

$$p_{1k} = |x_k - y_k|, \quad p_{2k} = |x_k - z_k| \quad \text{and} \quad p_{3k} = |y_k - z_k|$$

for $k \in \{3, 4, ..., n\}$. Since

$$\begin{aligned} |(x_1 - y_1) + m(x_2 - y_2)| &= |(x_1 - z_1 + z_1 - y_1) + m(x_2 - z_2 + z_2 - y_2)| \\ &\leq |(x_1 - z_1) + m(x_2 - z_2)| + |(z_1 - y_1) + m(z_2 - y_2)| \\ |m(x_1 - y_1) - (x_2 - y_2)| &\leq |m(x_1 - z_1) - (x_2 - z_2)| + |m(z_1 - y_1) - (z_2 - y_2)|, \end{aligned}$$

and

$$|x_k - y_k| \leq |x_k - y_k| + |y_k - z_k|,$$

we have $p_{1i} \leq (p_{2i} + p_{3i})$ for each $i \in I$. If $\Delta_{AB} = \max\{p_{11}, p_{12}, ..., p_{1n}\} = p_{1j}$, then

$$\begin{aligned} d_m(A, B) &= a' p_{1j} + b' \sum_{i \in I \setminus \{j\}} p_{1i} \\ &\leq a'(p_{2j} + p_{3j}) + b' \sum_{i \in I \setminus \{j\}} (p_{2i} + p_{3i}) \\ &= a' p_{2j} + b' \sum_{i \in I\text{-}\{j\}} p_{2i} + a' p_{3j} + b' \sum_{i \in I \setminus \{j\}} p_{3i}. \end{aligned}$$

Since $d_m(A, C) \geq a' p_{2j} + b' \sum_{i \in I \setminus \{j\}} p_{2i}$ and $d_m(B, C) \geq a' p_{3j} + b' \sum_{i \in I \setminus \{j\}} p_{3i}$ by Lemma 2, we have the inequality $d_m(A, B) \leq d_m(A, C) + d_m(C, B)$. □

Remark 2. *In $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$ the m-distance function involves the taxicab, Chinese checker, maximum and alpha distance functions and their generalizations as special cases: If $a = b = 1$ in d_m, then*

$$d_m(A, B) = (\Delta_{AB} + \delta_{AB}) \big/ \sqrt{1 + m^2},$$

which is a generalization of the taxicab distance d_T. Similarly, if $a = 1$ and $b = \sqrt{2} - 1$ in d_m, then we have

$$d_m(A, B) = \left(\Delta_{AB} + (\sqrt{2} - 1)\delta_{AB}\right) \big/ \sqrt{1 + m^2},$$

which is a generalization of the Chinese checker distance d_C. If $a = 1$ and $b = 0$ in d_m, then we have

$$d_m(A, B) = \Delta_{AB} \big/ \sqrt{1 + m^2},$$

which is a generalization of the maximum distance d_L. Finally, if $a = 1$ and $b = (\sec\alpha - \tan\alpha)$, $\alpha \in [0, \pi/2)$, in d_m, then we have

$$d_m(A, B) = (\Delta_{AB} + (\sec\alpha - \tan\alpha)\delta_{AB}) \big/ \sqrt{1 + m^2},$$

which is a generalization of the α-distance d_α. If we denote these distance families by $d_{T(m)}$, $d_{C(m)}$, $d_{L(m)}$ and $d_{\alpha(m)}$, respectively, then it is obvious that $d_{T(0)}(A, B) =$

$d_T(A,B)$, $d_{C(0)}(A,B) = d_C(A,B)$, $d_{L(0)}(A,B) = d_L(A,B)$ and $d_{\alpha(0)}(A,B) = d_\alpha(A,B)$. It is also easy to observe that if $\delta_{AB} > 0$ and $\alpha \in (0, \pi/4)$, then

$$d_{L(m)}(A,B) < d_{C(m)}(A,B) < d_{\alpha(m)}(A,B) < d_{T(m)}(A,B).$$

If $\delta_{AB} > 0$ and $\alpha \in (\pi/4, \pi/2)$, then

$$d_{L(m)}(A,B) < d_{\alpha(m)}(A,B) < d_{C(m)}(A,B) < d_{T(m)}(A,B).$$

In addition to inequalities above, if $\delta_{AB} = 0$, then A and B lie on a line parallel to one of axes of direction, and it follows that

$$d_{L(m)}(A,B) = d_{\alpha(m)}(A,B) = d_{C(m)}(A,B) = d_{T(m)}(A,B) = a d_E(A,B).$$

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