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# Intersections of two ruin probability functions*

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**Abstract.** In this paper we study intersections of ruin probability functions for two risk models. The number of intersection points is determined for some of the most widely used models. It is also shown that there exist risk processes with ruin probability functions intersecting in an arbitrary large number of points.

**AMS subject classifications**: 60G50, 60G51, 60G52

**Key words**: risk process, spectrally negative Lévy process, ruin probability, scale function, Mittag-Leffler function

## 1. Introduction

Ruin theory has always been a vital part of actuarial mathematics. At first glance, some of the theoretically derived results seem to have a limited scope in practical situations. Nevertheless, calculation of and approximation to ruin probabilities have been a constant source of inspiration and technique development in actuarial mathematics.

Assume an insurance company is willing to risk a certain amount $x$ in a certain branch of insurance. Because in some sense this part of the business starts with the capital $x$ we can safely call $x$ the initial capital. The actuary now has to make some decisions, for instance which premium should be charged and which type of reinsurance to take. The premium is often determined by company policies and by tariffs of rivals. A possible criterion for optimizing the reinsurance treaty would be to minimize the probability that the risk process ever becomes negative.

We begin our paper with one simple result which was the motivation to start solving a problem stated by H.J.Furrer in the paper about risk processes perturbed by $\alpha$-stable Lévy motion ([1],1998). In Remark 3 at the end of Section 2, Furrer mentioned an unsolved problem of comparing ruin probabilities as functions of the initial surplus for the given risk model and for two different stability parameters. We will consider this problem for a model with two sets of different parameters in order to decide which model has a smaller probability of ruin for the given initial surplus. The insurance company could use one model with smaller probability of ruin up to the intersection of ruin probability functions and change to the other

*For interpretation of color in all figures, the reader is referred to the web version of this article available at www.mathos.hr/mc.

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model, i.e. make arrangements for reinsurance in order to minimize the probability of ruin after it.

Consider a classical risk process $X = \{X_t : t \geq 0\}$ taking the form of a compound Poisson process with arrival rate $\lambda > 0$ and negative jumps, corresponding to claims, having a distribution with finite mean $\mu$ as well as a drift $c > 0$, corresponding to a steady income due to premiums. More precisely,

$$X_t = ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where $c > 0$, $N(t)$ is a Poisson process with rate $\lambda > 0$ modeling claim arrivals and $\{Y_i\}$ is a sequence of i.i.d. nonnegative random variables with finite mean $\mu$ modeling individual claims. It is usual to assume the *net profit condition* $c - \lambda\mu > 0$ which says that the income due to premiums is greater than the expected loss.

In the case of exponentially distributed claims, $Y_i \sim Exp(1/\mu)$, one has an explicit formula for the ruin probability with the initial capital $x$ (see [11, Section 5.3])

$$\phi(x) := \mathbf{P}(X_t < 0 \text{ for some } t \geq 0 | X_0 = x) = \frac{\lambda\mu}{c} e^{-\left(\frac{1}{\mu} - \frac{\lambda}{c}\right)x}. \tag{1}$$

Assume now that we want to compare ruin probabilities for classical risk processes with different parameters:

$$X_t^{(j)} = c_j t - \sum_{i=1}^{N_j(t)} Y_i^{(j)}, \quad t \geq 0,$$

where $c_j > 0$ is a positive drift, $N_j(t)$ is a Poisson process with rate $\lambda_j > 0$ and $Y_1^{(j)}, Y_2^{(j)}, \ldots$ are i.i.d. random variables with common distribution function $Exp(1/\mu_j)$, $\mathbf{E}(Y_1^{(j)}) = \mu_j$ and $\gamma_j = c_j - \lambda_j \mu_j > 0$, $j = 1, 2$.

It is elementary to derive from (1) the following simple result.

**Proposition 1.** *Let $\phi_j$ be a ruin probability function for the process $X_t^{(j)}$, $j = 1, 2$.*

*(i) If $\frac{\lambda_1\mu_1}{c_1} \leq \frac{\lambda_2\mu_2}{c_2}$ and $\frac{1}{\mu_1} - \frac{\lambda_1}{c_1} \geq \frac{1}{\mu_2} - \frac{\lambda_2}{c_2}$, then $\phi_1(x) \leq \phi_2(x)$ for $x > 0$.*

*(ii) If $\frac{\lambda_1\mu_1}{c_1} < \frac{\lambda_2\mu_2}{c_2}$ and $\frac{1}{\mu_1} - \frac{\lambda_1}{c_1} < \frac{1}{\mu_2} - \frac{\lambda_2}{c_2}$, then $\phi_1(x) < \phi_2(x)$ for $x < x_0$ and $\phi_1(x) > \phi_2(x)$ for $x > x_0$.*

*Here $x_0$ represents the unique intersection*

$$x_0 = \frac{1}{\frac{1}{\mu_2} - \frac{1}{\mu_1} + \frac{\lambda_1}{c_1} - \frac{\lambda_2}{c_2}} \ln \frac{\lambda_2\mu_2 c_1}{\lambda_1\mu_1 c_2}$$

*of ruin probability functions $\phi_1$ and $\phi_2$.*

Note that the ruin probability functions $\phi_1$ and $\phi_2$ either do not intersect or have precisely one strictly positive point of intersection. Now we give examples for the cases (i) and (ii).

**Example 1.** *Let* $X_t^{(1)} = 4t - \sum_{i=1}^{N_1(t)} Y_i^{(1)}$ *and* $X_t^{(2)} = 5t - \sum_{i=1}^{N_2(t)} Y_i^{(2)}$, $t \geq 0$, *where* $\lambda_1 = 2$, $\lambda_2 = 4$, $\mu_1 = \mu_2 = \frac{1}{2}$ *and let* $X_t^{(3)} = 2t - \sum_{i=1}^{N_3(t)} Y_i^{(3)}$ *and* $X_t^{(4)} = 3t - \sum_{i=1}^{N_4(t)} Y_i^{(4)}$, $t \geq 0$, *where* $\lambda_3 = 2$, $\lambda_4 = 1$, $\mu_3 = \frac{1}{2}$ *i* $\mu_4 = 1$. *The left part of Figure 1 shows the ruin probability function for the process* $X_t^{(1)}$

$$\phi_1(x) = \frac{1}{4}e^{-\frac{3}{2}x}$$

*and the process* $X_t^{(2)}$

$$\phi_2(x) = \frac{2}{5}e^{-\frac{6}{5}x}$$

*and the right part represents the ruin probability functions for the process* $X_t^{(3)}$

$$\phi_3(x) = \frac{1}{2}e^{-x}$$

*and the process* $X_t^{(4)}$

$$\phi_4(x) = \frac{1}{3}e^{-\frac{2}{3}x}.$$

*Here the* $x$*-axis represents the initial capital and the* $y$*-axis stands for the value of the probability of ruin.*

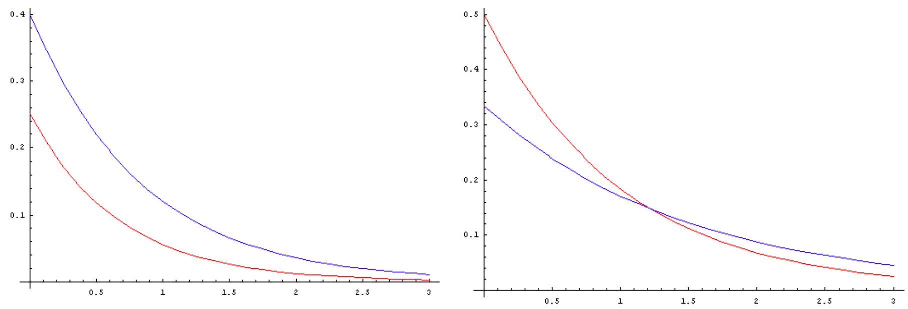


Figure 1: *The ruin probability functions for two classical risk processes with exponential claims*

In the last few years there were many papers concerning spectrally negative Lévy processes (see [2] and [9]) and it has become standard to model the generalized risk process by a spectrally negative Lévy process (see for example [3, 4, 7] and [10]). In this paper we will also consider spectrally negative Lévy processes. These are the Lévy processes with no positive jumps which are not the negative of a subordinator. Namely, for Lévy processes with jumps in only one direction many calculations can be carried out explicitly. To be more specific, we will consider $\alpha$-stable spectrally negative Lévy processes. A random variable $Y$ is said to have a stable distribution if for all $n \geq 1$ it observes the distributional equality

$$Y_1 + ... + Y_n = a_nY + b_n$$

where $Y_1, ..., Y_n$ are independent copies of $Y$ , $a_n > 0$ and $b_n \in \mathbf{R}$. It turns out that necessarily $a_n = n^{1/\alpha}$ for $\alpha \in (0, 2]$.

In this paper we compare ruin probabilities with respect to the initial surplus for the $\alpha$-stable spectrally negative Lévy processes with two different positive drifts, for two $\alpha$-stable spectrally negative Lévy processes with two different stability parameters $\alpha \in (1, 2]$, two scale parameters and positive drifts, and for two classical risk processes perturbed by $\alpha$-stable spectrally negative Lévy motion. For those models the point zero is regular for $(-\infty, 0)$, hence ruin with zero initial surplus is certain (see [8, p.142]). As a consequence, if $\phi(x)$ denotes the ruin probability for a process in this model, it holds that $\phi(0) = 1$. Hence, ruin probability functions for any two processes within this model will intersect at zero. We give conditions on parameters of compared models under which the point zero is the only intersection point and conditions under which there is at least one additional intersection point. Numerical computations for considered models strongly suggest that when there exists a strictly positive intersection point, it is unique, i.e. there are no other positive intersection points. We have not been able to analytically determine this fact. The approximate value for the positive intersection can be determined by using *MATHEMATICA*©  or any similar programming package.

The fact that numerical computations suggest that in the considered models there are at most two intersection points led us to consider the following problem: do two ruin probabilities as functions of the initial surplus for two arbitrary risk models always have only two points in common. The answer to this question turned out to be negative. In Theorem 5 we prove that given any positive integer $n$, there are two risk models with ruin probability functions intersecting in at least $n$ points.

In the remaining part of the introduction we set up notations and recall some facts that will be needed in the rest of the paper.

Let $\{X_t : t \geq 0\}$ be a spectrally negative Lévy process. The Laplace exponent of $X$ is defined by

$$\psi(\lambda) := \frac{1}{t} \ln \mathbf{E}(e^{\lambda X_t}),$$

and is finite at least for all $\lambda \geq 0$. The function $\psi : [0, \infty) \to \mathbf{R}$ is zero at zero and tends to infinity at infinity. It is infinitely differentiable and strictly convex. The first passage time below a level $x = 0$ is defined by

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}$$

and the ruin probability function by

$$\phi(x) = \mathbf{P}_x(\tau_0^- < \infty),$$

where $\mathbf{P}_x = \mathbf{P}(.|X_0 = x)$. It represents the probability for a process to fall below the level $x = 0$ when it started at $x > 0$.

A key object in the fluctuation theory of spectrally negative Lévy processes and their applications is the scale function. From [8, Theorem 8.1], it follows that for every spectrally negative Lévy process $\{X_t : t \geq 0\}$ there exists the function $W : \mathbf{R} \to [0, \infty)$, called a *scale function* characterized as follows. It is a unique function such that $W(x) = 0$ for $x < 0$, it is strictly increasing and continuous on $[0, \infty)$ and with Laplace transform given by

$$\int_0^\infty e^{-\lambda x} W(x) \mathrm{d}x = \frac{1}{\psi(\lambda)}, \quad \lambda > 0. \tag{2}$$

Theorem 8.1 in [8] also gives the relation between the ruin probability function $\phi$ and the scale function:

$$\phi(x) = \begin{cases} 1 - \psi'(0+) W(x), & \psi'(0+) > 0 \\ 1, & \psi'(0+) \leq 0. \end{cases} \tag{3}$$

We will need the scale function $W(x)$ for the $\alpha$-stable spectrally negative Lévy process $Z_t^\alpha$ with positive drift $c > 0 :\ X_t = Z_t^\alpha + ct$, where $\alpha \in (1, 2]$ is the stability parameter (see e.g. [2]):

$$W(x) = \frac{1}{c}\Big(1 - E_{\alpha-1}(-cx^{\alpha-1})\Big), \tag{4}$$

where

$$E_{\alpha-1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + (\alpha - 1)k)} \tag{5}$$

is the Mittag-Leffler function with index $\alpha - 1$ and

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t, \quad x > 0$$

is the Gamma function for which $\Gamma(n) = (n-1)!$, $n \in \mathbf{N}$.

Our proofs use the Tauberian theorem and the monotone density theorem (see for example Theorems 5.13 and 5.14 from [8]). Suppose that $U$ is a measure supported on $[0, \infty)$ with Laplace transform

$$\Lambda(\theta) = \int_0^\infty e^{-\theta x} U(\mathrm{d}x), \quad \theta \geq 0.$$

**Remark 1.** *The notation $f \sim g$ for functions $f$ and $g$ means that* $\lim \frac{f(x)}{g(x)} = 1$.

**Theorem 1.** *Suppose $L > 0$, $\rho \geq 0$ are positive constants and $U$ is a measure on $[0, \infty)$ with Laplace transform $\Lambda$.*

*(a) (Tauberian theorem) The following two statements are equivalent:*

*(i) $\Lambda(\theta) \sim L\theta^{-\rho}$, as $\theta \to 0$,*

*(ii) $U(x) \sim \frac{L}{\Gamma(1+\rho)} x^\rho$, as $x \to \infty$, where $U(x) = U([0, x])$*

*(b) (monotone density theorem) Suppose $\rho > 0$ and that the measure $U$ has a monotone density $u$. Then the following two statements are equivalent:*

*(i) $\Lambda(\theta) \sim L\theta^{-\rho}$, as $\theta \to 0$,*

*(ii)* $u(x) \sim \frac{L}{\Gamma(\rho)} x^{\rho-1}$, *as* $x \to \infty$.

**Remark 2.** *The statements of Theorem 1 are still valid when the limits in parts (i) and (ii) are simultaneously changed to* $\theta \to \infty$ *and* $x \to 0$.

Our paper is organized as follows. In Section 2 we give proofs that intersections for two ruin probabilities as functions of the initial surplus in already mentioned models exist. For some of the risk models we include pictures of the intersections. Those pictures were made by using *MATHEMATICA*©. In Section 3 for any positive integer $n$ we construct two ruin probability functions which intersect in at least $n$ positive points.

## 2. The models

First we compare the ruin probability functions for two processes which differ only in constant positive drift. Consider

$$X_t^{(1)} = X_t + c_1 t$$

and

$$X_t^{(2)} = X_t + c_2 t,$$

where $X = \{X_t : t \geq 0\}$ is an arbitrary spectrally negative Lévy process with positive drift $c_1 > 0$ and positive drift $c_2 > 0$. Let $\phi_1(x)$ and $\phi_2(x)$ be the ruin probability functions for $X^{(1)}$ and for $X^{(2)}$, respectively. Suppose that $0 < c_1 < c_2$. Then it holds that

$$\phi_1(x) > \phi_2(x)$$

for every $x > 0$. This is because $X_t^{(2)} > X_t^{(1)}$ $\mathbf{P}_x - a.s.$ and it is evident that the ruin probability functions for these processes do not have positive intersections. The case $0 < c_2 < c_1$ can be treated in the same way.

Let $X_t = \sigma Y_t + ct$ be the scaled spectrally negative Lévy process with drift, where $\sigma > 0$ is a scale parameter and $c > 0$ a positive drift. Also let $\phi^{Y_t + c_1 t}$ be the ruin probability function for the spectrally negative Lévy process with drift $Y_t + c_1 t$, $c_1 > 0$, and $\phi_{X_t}$ the ruin probability function for the process $X_t$.

**Lemma 1.** *The ruin probability function for the process* $X$ *satisfies*

$$\phi^{X_t}(x) = \phi^{Y_t + \frac{c}{\sigma} t}\left(\frac{x}{\sigma}\right). \tag{6}$$

**Proof**. We calculate

$$\begin{aligned}\phi^{X_t}(x) &= P_x(X_t < 0) = P_x(\sigma Y_t + ct < 0 \text{ for some } t > 0)\\ &= P_{\frac{x}{\sigma}}\left(Y_t + \frac{c}{\sigma} t < 0 \text{ for some } t > 0\right) = \phi^{Y_t + \frac{c}{\sigma} t}\left(\frac{x}{\sigma}\right).\end{aligned}$$

□

### 2.1. Two $\alpha$-stable spectrally negative Lévy processes

We consider $\{X_t : t \geq 0\}$, $X_t = Z_t^\alpha + ct$, where $c > 0$ is a positive drift and $Z_t^\alpha$ an $\alpha$-stable spectrally negative Lévy process with stability parameter $\alpha \in (1, 2]$, where the case $\alpha = 2$ corresponds to Brownian motion with double speed. The Laplace exponent for the process $X_t$ equals:

$$\psi(x) = x^\alpha + cx. \tag{7}$$

Using relations (3),(4),(5) and (7) we get the ruin probability function $\phi$ for the process $X_t$:

$$\phi(x) = \sum_{k=0}^{\infty} (-1)^k \frac{c^k x^{(\alpha-1)k}}{\Gamma(1 + (\alpha - 1)k)} = E_{\alpha-1}(-cx^{\alpha-1}) \tag{8}$$

where $E_{\alpha-1}$ is the Mittag-Leffler function with index $\alpha - 1$.

In the Brownian motion case, $\alpha = 2$, the ruin probability function is equal to $\phi(x) = E_1(-cx) = e^{-cx}$.

Let $\{X_t : t \geq 0\}$, $X_t = \sigma Z_t^\alpha + ct$ be a scaled $\alpha$-stable spectrally negative Lévy process with drift, where $\sigma > 0$ is a scale parameter, $c > 0$ a positive drift and $\alpha \in (1, 2]$ a stability parameter. Using (6) and (8) we calculate the ruin probability function for the process $X_t$:

$$\phi^{X_t}(x) = \phi^{Z_t^\alpha + \frac{c}{\sigma}t}\left(\frac{x}{\sigma}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{c}{\sigma^\alpha})^k x^{(\alpha-1)k}}{\Gamma(1 + (\alpha - 1)k)} = E_{\alpha-1}\left(-\frac{c}{\sigma^\alpha} x^{\alpha-1}\right). \tag{9}$$

We determine the Laplace exponent for the process $X_t$:

$$\psi^{X_t}(x) = \frac{1}{t} \ln E(e^{xX_t}) = \psi^{Z_t^\alpha + \frac{c}{\sigma}t}(x\sigma) = (\sigma x)^\alpha + cx. \tag{10}$$

**Lemma 2.** *The ruin probability function $\phi$ for the process $X_t$ satisfies*

$$\phi(x) \sim \frac{\sigma^\alpha x^{-(\alpha-1)}}{c\Gamma(1 - (\alpha - 1))}, \quad x \to \infty. \tag{11}$$

**Proof**. Using (2),(3) and (10) we calculate Laplace transform for the ruin probability function

$$\begin{aligned}\mathbf{L}\phi(\lambda) &= \int_0^\infty e^{-\lambda x} \phi(x) \mathrm{d}x = \int_0^\infty e^{-\lambda x}(1 - cW(x)) \mathrm{d}x \\ &= \frac{1}{\lambda} - \frac{c}{\psi(x)} = \frac{1}{\lambda} - \frac{c}{(\sigma\lambda)^\alpha + c\lambda} = \frac{\sigma^\alpha \lambda^\alpha}{\sigma^\alpha \lambda^{\alpha+1} + c\lambda^2} = \frac{\lambda^{\alpha-2}}{\lambda^{\alpha-1} + \frac{c}{\sigma^\alpha}}.\end{aligned}$$

We see that

$$\mathbf{L}\phi(\lambda) \sim \frac{\sigma^\alpha}{c} \lambda^{\alpha-2}, \quad \lambda \to 0.$$

Now (11) follows from the monotone density theorem. □

We define the survival probability function for the process $X_t$ by

$$\chi(x) := 1 - \phi(x),$$

where $\phi$ is the probability of ruin.

**Lemma 3.** *The probability of survival $\chi$ for the process $X_t$ satisfies*

$$\chi(x) \sim \frac{cx^{\alpha-1}}{\sigma^\alpha \Gamma(\alpha)}, \quad x \to 0. \tag{12}$$

**Proof**. From (8) we calculate the survival probability function for the process $X_t$:

$$\chi(x) = 1 - \phi(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(\frac{c}{\sigma^\alpha})^k x^{(\alpha-1)k}}{\Gamma(1+(\alpha-1)k)}.$$

Now we find the Laplace transform for the measure $\mathrm{d}\chi(x)$:

$$\begin{aligned}
\mathbf{L}\chi(\lambda) &= \int_0^\infty e^{-\lambda x} \mathrm{d}\chi(x) \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(\frac{c}{\sigma^\alpha})^k (\alpha-1)k}{\Gamma(1+(\alpha-1)k)} \int_0^\infty e^{-\lambda x} x^{(\alpha-1)k-1} \mathrm{d}x \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(\frac{c}{\sigma^\alpha})^k (\alpha-1)k}{\Gamma(1+(\alpha-1)k)} \frac{\Gamma((\alpha-1)k)}{\lambda^{(\alpha-1)k}} = \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{c}{\sigma^\alpha})^{k+1}}{\lambda^{(\alpha-1)(k+1)}} \\
&= \frac{c}{\sigma^\alpha \lambda^{\alpha-1}} \sum_{k=0}^{\infty} \left( -\frac{c}{\sigma^\alpha \lambda^{\alpha-1}} \right)^k = \frac{c}{\sigma^\alpha \lambda^{\alpha-1}} \frac{\sigma^\alpha \lambda^{\alpha-1}}{\sigma^\alpha \lambda^{\alpha-1} + c} = \frac{c}{\sigma^\alpha \lambda^{\alpha-1} + c},
\end{aligned}$$

where $\lambda > \frac{1}{\sigma} (\frac{c}{\sigma})^{\frac{1}{\alpha-1}}$, and we used

$$\int_0^\infty e^{-\lambda x} x^\rho \mathrm{d}x = \frac{\Gamma(1+\rho)}{\lambda^{1+\rho}}.$$

We conclude that

$$\mathbf{L}\chi(\lambda) \sim \frac{c}{\sigma^\alpha} \lambda^{-(\alpha-1)}, \quad \lambda \to \infty.$$

Now (12) follows from the Tauberian theorem and Remark 2. □

We consider the processes

$$X_t^{(1)} = \sigma_1 Z_t^{\alpha_1} + c_1 t \tag{13}$$

and

$$X_t^{(2)} = \sigma_2 Z_t^{\alpha_2} + c_2 t, \tag{14}$$

where $Z_t^{\alpha_1}$ and $Z_t^{\alpha_2}$ are spectrally negative Lévy processes with stability parameters $\alpha_1, \alpha_2 \in (1, 2]$, $\sigma_1, \sigma_2 > 0$ are scale parameters and $c_1, c_2 > 0$ are positive drifts, respectively.

**Proposition 2.** *Suppose $\alpha_1 = \alpha_2 = \alpha$. Let $\phi_1$ and $\phi_2$ be ruin probability functions for processes $X^{(1)}$ and $X^{(2)}$, respectively.*

(a) *If $\frac{c_1}{\sigma_1^\alpha} < \frac{c_2}{\sigma_2^\alpha}$, then $\phi_1(x) > \phi_2(x)$ for all $x > 0$.*

(b) *If $\frac{c_1}{\sigma_1^\alpha} = \frac{c_2}{\sigma_2^\alpha}$, then $\phi_1(x) = \phi_2(x)$ for all $x > 0$.*

**Proof**. Let $X_t = Z_t^\alpha + t$. The ruin probability function for the process $X_t$ is given by $\phi(x) = E_{\alpha-1}(-x^{\alpha-1})$ and according to (9), the ruin probability functions for the processes $X_t^{(1)}$ and $X_t^{(2)}$ satisfy

$$\phi_1(x) = E_{\alpha-1}\left(-\frac{c}{\sigma_1^\alpha}x^{\alpha-1}\right)$$

and

$$\phi_2(x) = E_{\alpha-1}\left(-\frac{c}{\sigma_2^\alpha}x^{\alpha-1}\right)$$

for $x > 0$, respectively. Then we have

$$\phi_1(x) = \phi\left(\frac{c_1}{\sigma_1^\alpha}x\right)$$

and

$$\phi_2(x) = \phi\left(\frac{c_2}{\sigma_2^\alpha}x\right).$$

(a) Let $\frac{c_1}{\sigma_1^\alpha} < \frac{c_2}{\sigma_2^\alpha}$. The ruin probability function $\phi$ is decreasing, hence

$$\phi_1(x) = \phi\left(\frac{c_1}{\sigma_1^\alpha}x\right) > \phi\left(\frac{c_2}{\sigma_2^\alpha}x\right) = \phi_2(x)$$

for $x > 0$.

(b) In the case $\frac{c_1}{\sigma_1^\alpha} = \frac{c_2}{\sigma_2^\alpha}$ the processes $X_t^{(1)}$ and $X_t^{(2)}$ have the common ruin probability function.

□

The next theorem shows that the ruin probability functions for processes $X_t^{(1)}$ and $X_t^{(2)}$ in case $\alpha_1 \neq \alpha_2$ do have a positive intersection.

**Theorem 2.** *Suppose $\alpha_1 \neq \alpha_2$. Then the ruin probability functions for the processes $X_t^{(1)}$ and $X_t^{(2)}$ given by (13) and (14) have a positive intersection point.*

**Proof**. Let $\alpha_1 < \alpha_2$. According to Lemma 3, the survival probability functions $\chi_1$ and $\chi_2$ for the processes $X_t^{(1)}$ and $X_t^{(2)}$, respectively, satisfy

$$\chi_1(x) \sim \frac{c_1 x^{\alpha_1 - 1}}{\sigma_1^{\alpha_1}\Gamma(\alpha_1)}, \quad x \to 0$$

and

$$\chi_2(x) \sim \frac{c_2 x^{\alpha_2-1}}{\sigma_2^{\alpha_2}\Gamma(\alpha_2)}, \quad x \to 0.$$

We conclude that $\chi_1(x) > \chi_2(x)$ when $x > 0$ is small enough. Then the ruin probability functions for the processes (13) and (14), respectively, satisfy

$$\phi_1(x) < \phi_2(x)$$

if $x > 0$ is small enough. From Lemma 2 it follows that

$$\phi_1(x) \sim \frac{\sigma_1^{\alpha_1} x^{-\alpha_1+1}}{c_1\Gamma(2-\alpha_1)}, \quad x \to \infty$$

and

$$\phi_2(x) \sim \frac{\sigma_2^{\alpha_2} x^{-\alpha_2+1}}{c_2\Gamma(2-\alpha_2)}, \quad x \to \infty$$

where $\phi_1$ and $\phi_2$ are ruin probability functions for the processes $X_t^{(1)}$ and $X_t^{(2)}$, respectively. Then the ruin probability functions satisfy

$$\phi_1(x) > \phi_2(x)$$

when $x$ is large enough. Ruin probability functions are continuous; hence there exists a positive intersection $x > 0$. The case $\alpha_1 > \alpha_2$ can be proved in the same way. $\square$

We can see that the existence of more than one intersection depends only on the parameter $\alpha$. If the processes (13) and (14) have a common stability parameter $\alpha$, the ruin probability functions have one intersection in zero and when the stability parameters are not the same there also exists a positive intersection for ruin probability functions. Scale parameters and drift parameters are important only in the case when $\alpha_1 = \alpha_2 = \alpha$ and $\frac{c_1}{\sigma_1^\alpha} = \frac{c_2}{\sigma_2^\alpha}$. Then the processes $X_t^{(1)}$ and $X_t^{(2)}$ have the common ruin probability function.

**Example 2.** *Let $X_t^{(1)} = Z_t^{1.6} + 0.7t$ and $X_t^{(2)} = 0.8Z_t^{1.8} + 0.5t$, $t \geq 0$, be two processes where $Z_t^{1.6}$ is a stable spectrally negative Lévy process with stability parameter $\alpha = 1.6$ and $Z_t^{1.8}$ is a stable spectrally negative Lévy process with stability parameter $\alpha = 1.8$. Figure 2 shows ruin probability functions for the process $X^{(1)}$*

$$\phi_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{0.7^k x^{0.6k}}{\Gamma(1+0.6k)}$$

*and the process $X^{(2)}$*

$$\phi_2(x) = \sum_{k=0}^{\infty} (-1)^k \frac{0.75^k x^{0.8k}}{\Gamma(1+0.8k)}.$$

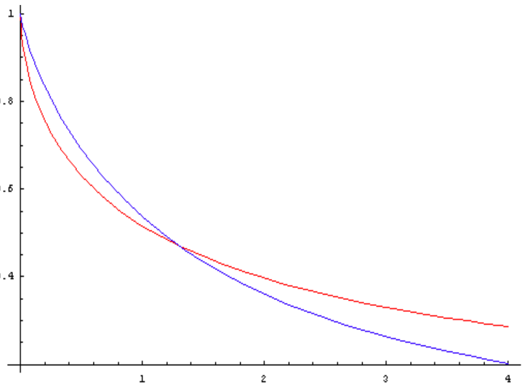


Figure 2: *The ruin probability functions for two $\alpha$-stable spectrally negative Lévy processes with drift*

## 2.2. Two classical risk processes perturbed by an $\alpha$-stable spectrally negative Lévy process

Recall that the classical Cramér-Lundberg risk process corresponds to a spectrally negative Lévy process $X$ taking the form of a compound Poisson process with arrival rate $\lambda > 0$ and negative jumps, corresponding to claims, having common distribution function $F$ with finite mean $\mu$ as well as a drift $c > 0$, corresponding to a steady income due to premiums. It is usual to assume the *net profit condition* $c - \lambda\mu > 0$ which says nothing other than $\psi'(0+) > 0$. Furrer (1998) added an $\alpha$-stable Lévy motion to the classical risk process which expresses either an additional uncertainty of the aggregate claims or of the premium income.

We consider the classical risk processes perturbed by an $\alpha$-stable spectrally negative Lévy motion

$$X_t = Z_t^\alpha + ct - \sum_{i=1}^{N(t)} Y_i,$$

where $Z_t^\alpha$ is a spectrally negative Lévy process with stability parameter $\alpha \in (1,2)$, $c > 0$ is a positive drift, $N(t)$ is a Poisson process with intensity $\lambda > 0$ and $Y_1, Y_2, \ldots$ are i.i.d. random variables with common distribution function $F$, $\mathbf{E}(Y_1) = \mu$. We assume the net profit condition $\gamma := c - \lambda\mu > 0$.

**Lemma 4.** *The survival probability $\chi$ for the process $X_t$ satisfies*

$$\chi(x) \sim \frac{\gamma x^{\alpha-1}}{\Gamma(\alpha)}, \quad x \to 0. \tag{15}$$

**Proof**. The Laplace exponent for the process $X_t$ is given by

$$\psi(x) = x^\alpha + cx - \lambda \int_0^\infty (1 - e^{-xt}) F(\mathrm{d}t).$$

Then $\psi'(0+) = \gamma > 0$. Using (2) and (3) we calculate the Laplace transform for survival probability function $\chi = \gamma W$

$$\mathbf{L}\chi(x) = \gamma \int_0^\infty e^{-xt} W(t)\mathrm{d}t = \frac{\gamma}{\psi(x)} = \frac{\gamma}{x^\alpha + cx - \lambda \int_0^\infty (1 - e^{-xt})F(\mathrm{d}t)}.$$

Therefore

$$\mathbf{L}\chi(x) \sim \gamma x^{-\alpha}, \quad x \to \infty.$$

Now (15) follows from the monotone density theorem and Remark 2. □

The next theorem shows that two classical risk processes perturbed by an $\alpha$-stable Lévy motion for different stability parameters $\alpha$ have a positive intersection.

We consider the processes

$$X_t^{(1)} = Z_t^{\alpha_1} + c_1 t - \sum_{i=1}^{N_1(t)} Y_i^{(1)} \tag{16}$$

and

$$X_t^{(2)} = Z_t^{\alpha_2} + c_2 t - \sum_{i=1}^{N_2(t)} Y_i^{(2)}, \tag{17}$$

where $t \geq 0$, $Z_t^{\alpha_1}$ and $Z_t^{\alpha_2}$ are spectrally negative Lévy processes with stability parameters $\alpha_1, \alpha_2 \in (1, 2]$, $\alpha_1 \neq \alpha_2$, $c_i > 0$ are positive drifts, $N_i(t)$ are Poisson processes with intensities $\lambda_i > 0$, $Y_1^{(i)}, Y_2^{(i)}, \ldots$ are i.i.d. random variables with common distribution function $F_i$, $\mathbf{E}(Y_1^{(i)}) = \mu_i$ and we assume the net profit condition $\gamma_i = c_i - \lambda_i \mu_i > 0$, $i = 1, 2$.

**Theorem 3.** *The ruin probability functions for the processes $X_t^{(1)}$ and $X_t^{(2)}$ given by (16) and (17) have a positive intersection point.*

**Proof**. Let $\alpha_1 < \alpha_2$. According to Lemma 4, the survival probability functions $\chi_1$ and $\chi_2$ for processes $X_t^{(1)}$ and $X_t^{(2)}$, respectively, satisfy

$$\chi_1(x) \sim \frac{\gamma_1 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)}, \quad x \to 0$$

and

$$\chi_2(x) \sim \frac{\gamma_2 x^{\alpha_2 - 1}}{\Gamma(\alpha_2)}, \quad x \to 0.$$

We conclude that ruin probability functions satisfy $\phi_1(x) < \phi_2(x)$ for $x > 0$ small enough.

Furrer showed ([1, Theorem 4]) that ruin probability functions $\phi_1$ and $\phi_2$ for processes $X_t^{(1)}$ and $X_t^{(2)}$ satisfy

$$\phi_1(x) \sim \frac{x^{-\alpha_1 + 1}}{\gamma_1 \Gamma(2 - \alpha_1)}, \quad x \to \infty$$

and

$$\phi_2(x) \sim \frac{x^{-\alpha_2+1}}{\gamma_2\Gamma(2-\alpha_2)}, \quad x \to \infty.$$

We see that ruin probability functions satisfy $\phi_1(x) > \phi_2(x)$ when $x$ is big enough. Since ruin probability functions are continuous, we conclude that there must be a positive intersection. The case $\alpha_1 > \alpha_2$ can be treated in the same way. □

## 3. Ruin probability functions with $n$ intersections

In this section we will show that there exist two spectrally negative Lévy processes such that their ruin probability functions have at least $n$ positive intersections for an arbitrary positive integer $n$.

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process with the Laplace exponent $\psi$ and scale function $W$. We assume that $X$ drifts to $+\infty$, which means that $\psi'(0+) > 0$. A function $\eta : [0,\infty) \to \mathcal{R}$ is a Bernstein function if $\eta(\lambda) \geq 0$ and $(-1)^{n-1}\eta^{(n)}(\lambda) \geq 0$ for every $n \in \mathbf{N}$ and every $\lambda$. A function $\nu : [0,\infty) \to \mathcal{R}$ is completely monotone if $\nu$ is continuous on $[0,\infty)$, infinitely differentiable on $(0,\infty)$ and satisfies $(-1)^n\nu^{(n)}(\lambda) \geq 0$ for all nonnegative integers $n$ and for all $t > 0$.

**Lemma 5.** *Let $\mu$ be a probability measure on $[0,\infty)$ such that $\mu(\{0\}) = 0$. Then the function*

$$\phi(x) := \int_{[0,\infty)} e^{-xt}\mu(dt) = \mathcal{L}\mu(x)$$

*is the ruin probability function of a spectrally negative Lévy process which drifts to $+\infty$.*

**Proof**. Let $\eta(x) := 1 - \phi(x)$. Then $\eta$ is a Bernstein function. Following [9] we are going to show that there exists a spectrally negative Lévy process $X$ with the scale function equal to $\eta$. Let $f(x) := x^2\mathcal{L}\eta(x)$. Then $f$ is a complete Bernstein function (see for example [5], p. 192). Now $f^*(x) := x/f(x)$ is a complete Bernstein and there exists a Bernstein function $\eta^*$ such that $f^*(x) := x^2\mathcal{L}\eta^*(x)$. According to [9, Corollary 2 in Section 5], $W(x) := \eta^*(x)$ and $W^*(x) := \eta(x)$ are scale functions for spectrally negative Lévy processes $X$ and $X^*$ with Laplace exponents

$$\psi(x) = \frac{x^2}{f^*(x)} = xf(x) \text{ and } \psi^*(x) = \frac{x^2}{f(x)} = xf^*(x).$$

Then we have

$$(\psi^*)'(0+) = \lim_{x\to 0}\frac{\psi^*(x)}{x} = f^*(0+) = \lim_{x\to 0}\frac{x}{f(x)} = \lim_{x\to 0}\frac{1}{x\mathcal{L}\eta(x)}.$$

When $(\psi^*)'(0+) > 0$, the spectrally negative Lévy process $X^*$ drifts to infinity and $\eta$ is its scale function. Now we calculate

$$\begin{aligned}\mathcal{L}\eta(x) &= \int_0^\infty e^{-xt}\mathrm{d}t - \mathcal{L}(\mathcal{L}\mu)(x)\\ &= \frac{1}{x} - \int_{[0,\infty)}\frac{1}{x+t}\mu(\mathrm{d}t).\end{aligned}$$

Then $\lim_{x\to 0} x\mathcal{L}\eta(x) = 1-\mu(\{0\})$. Since $\mu(\{0\})=0$, we know that $\eta := 1-\phi$ is the scale function for a spectrally negative Lévy process which drifts to $+\infty$. Let $X^*$ be this spectrally negative Lévy process with Laplace exponent $\psi^*$ and scale function $W^*$. Then $W^*(x)=\eta(x)=1-\phi(x)$, $(\psi^*)'(0+)=1$ and we conclude that

$$\mathbf{P}_x(\tau_0^- < \infty) = 1-(\psi^*)'(0+)W^*(x) = 1-\eta(x) = \phi(x)$$

equals the ruin probability for the process $X^*$. □

Now we explain a result by Karlin and Shapley from [6]. Consider probability measures $\mu$ on $[0,1]$. The measure $\delta_{t_0}$ is a point mass distribution on $[0,1]$ for every $0\le t_0\le 1$. Consider a convex combination of point masses on $[0,1]$

$$\mu_n = \sum_{j=1}^n a_j\delta_{t_j}, \quad 0\le t_1 < t_2 < ... < t_n \le 1, \quad a_j > 0, \quad \sum_{j=1}^n a_j = 1. \tag{18}$$

Let $\mathcal{D}$ be the set of all probability measures on $[0,1]$ and $\mathcal{D}_A$ the set of all convex combinations of point masses on $[0,1]$. The moment of $n$-th order $m_n(\mu)$ for measure $\mu\in\mathcal{D}$ is defined as

$$m_n(\mu) = \int_0^1 t^n\mu(\mathrm{d}t), \quad n = 0,1,2,....$$

Let $D^n = \{(m_1(\mu),...,m_n(\mu)) : \mu\in\mathcal{D}\}$ be the space of the moments of $n$-th order and $D_A^n = \{(m_1(\mu),...,m_n(\mu)) : \mu\in\mathcal{D}_A\}$ its subset for convex combinations of point masses.

Karlin and Shapley showed in [6] that

**Lemma 6.** *$D_A^n = D^n$.*

For our readers' convenience we sketch the proof following Karlin and Shapley.

**Proof**. Let $C^n$ be the curve traced out by $x(t) = (t,t^2,...,t^n)$ as $t$ runs between 0 and 1. It is clear from (18) that $D_A^n$ is exactly the set of points spanned by $C^n$. Moreover, $D_A^n$ is a closed set, since $C^n$ is closed and bounded. Further, for any $\mu$ in $\mathcal{D}$ there is a sequence of point masses $\mu_n\in\mathcal{D}_A$ such that

$$\lim_{n\to\infty}\int_0^1 f(t)\mu_n(\mathrm{d}t) = \int_0^1 f(t)\mu(\mathrm{d}t)$$

for every continuous function $f$. Taking $f(t)=t^k, k=1,...,n$ we see that $D^n$ is the closure of its subset $D_A^n$. But $D_A^n$ is already closed; hence $D^n = D_A^n$. □

Our next theorem solves the problem of $n$ positive intersections.

**Theorem 4.** *For every $n\in\mathbf{N}$ there exist two different spectrally negative Lévy processes which drift to $+\infty$ with ruin probability functions $\phi_1$ and $\phi_2$, respectively, such that $\phi_1$ and $\phi_2$ have at least n positive intersections.*

**Proof**. Let $0 < \epsilon < \frac{1}{2}$ and $\mu = \frac{1}{1-2\epsilon} l$, where $l$ is the Lebesgue measure on $[0,1]$. Then the probability measure $\mu$ on $[\epsilon, 1-\epsilon]$ is in $\mathcal{D}$ and according to Lemma 6, for every $n \in \mathbf{N}$ there exists a convex combination of point masses $\nu^{(n)} \in \mathcal{D}_A$ such that

$$\begin{aligned}\int_\epsilon^{1-\epsilon} x^k \nu^{(n)}(\mathrm{d}x) &= \int_\epsilon^{1-\epsilon} x^k \mu(\mathrm{d}x) \\ &= \frac{1}{1-2\epsilon} \int_\epsilon^{1-\epsilon} x^k \mathrm{d}x \\ &= \frac{1}{1-2\epsilon} \frac{(1-\epsilon)^{k+1} - \epsilon^{k+1}}{k+1}, \quad k = 0, 1, ..., n-1.\end{aligned}$$

Now we extend the measures $\nu^{(n)}$ and $\mu$ on $(0,1]$. We put $\nu^{(n)} = \mu = 0$ on $(0,\epsilon)$ and on $(1-\epsilon, 1]$. Let $\hat{\nu}^{(n)}(\mathrm{d}t)$ and $\hat{\mu}(\mathrm{d}t)$ be the image measures on $[0,\infty)$ of the measure $\nu^{(n)}(\mathrm{d}x)$ and the measure $\mu$ on $(0,1]$, respectively, under the map $t = -\ln x$. We define

$$\phi_1(\lambda) = \int_0^\infty e^{-\lambda t} \hat{\mu}(\mathrm{d}t) = \int_0^1 x^\lambda \mu(\mathrm{d}x)$$

and

$$\phi_2(\lambda) = \int_0^\infty e^{-\lambda t} \hat{\nu}^{(n)}(\mathrm{d}t) = \int_0^1 x^\lambda \nu^{(n)}(\mathrm{d}x).$$

According to Lemma 5, $\phi_1$ and $\phi_2$ are ruin probability functions for two spectrally negative Lévy processes drifting to $+\infty$. The measure $\nu^{(n)}$ is constructed to satisfy $\phi_1(k) = \phi_2(k), \quad k = 0, 1, ..., n-1.$ □

The next example is based on the construction given in Theorem 5.

**Example 3.** *Let $\epsilon = \frac{1}{10}$ and $\mu = \frac{5}{4} l$, where $l$ is the Lebesgue measure on $[0,1]$. Than $\mu$ is the probability measure on $[\frac{1}{10}, \frac{9}{10}]$. Let $\mu = 0$ on $(0, \frac{1}{10})$ and on $(\frac{9}{10}, 1]$. We define the function*

$$\phi_1(x) = \int_0^1 s^x \mu(ds) = \frac{5}{4} \int_{\frac{1}{10}}^{\frac{9}{10}} s^x ds = \frac{5}{4} \frac{(\frac{9}{10})^{x+1} - (\frac{1}{10})^{x+1}}{x+1},$$

*where $x \geq 0$. Under the map $s = -\ln t$ the measure $\mu$ on $(0,1]$ becomes an image measure $\hat{\mu}(dt)$ on $[0,\infty)$ and $\phi_1(x)$ is given as*

$$\phi_1(x) = \int_0^\infty e^{-xt} \hat{\mu}(dt).$$

*According to Lemma 5, $\phi_1$ is the ruin probability function for a spectrally negative Lévy process which drifts to $+\infty$. Let $\nu$ be a probability measure on $(0,1]$, $\nu = 0$ on $(0, \frac{1}{10})$ and on $(\frac{9}{10}, 1]$. We define*

$$\phi_2(x) = \int_0^1 s^x \nu(ds).$$

*Let us find the measure* $\nu$ *on* $(0,1]$ *such that the* $0^{th}$*,* $1^{st}$ *and* $2^{nd}$ *moments for* $\nu$ *and for* $\mu$ *are equal. We know that* $\int_0^1 s^1\mu(ds) = \frac{1}{2}$*,* $\int_0^1 s^2\mu(ds) = \frac{91}{300}$ *and* $\int_0^1 s^3\mu(ds) = \frac{41}{200}$*. We assume*

$$\nu = \alpha_1\delta_{\frac{3}{4}} + \alpha_2\delta_{\frac{1}{2}} + \alpha_3\delta_{\frac{1}{4}},$$

*where* $\alpha_1$*,* $\alpha_2$ *and* $\alpha_3$ *are positive real numbers. From equality of moments it follows that positive constants* $\alpha_1$*,* $\alpha_2$ *and* $\alpha_3$ *satisfy the linear system*

$$\begin{aligned}
\frac{3}{4}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{4}\alpha_3 &= \int_0^1 s^0\nu(ds) = \frac{1}{2},\\
\frac{9}{16}\alpha_1 + \frac{1}{4}\alpha_2 + \frac{1}{16}\alpha_3 &= \int_0^1 s^1\nu(ds) = \frac{91}{300},\\
\frac{81}{64}\alpha_1 + \frac{1}{16}\alpha_2 + \frac{1}{64}\alpha_3 &= \int_0^1 s^2\nu(ds) = \frac{41}{200}.
\end{aligned}$$

*This system has a unique solution* $\alpha_1 = \frac{32}{75}$*,* $\alpha_2 = \frac{11}{75}$ *and* $\alpha_3 = \frac{32}{75}$ *so that* $\nu$ *is given by*

$$\nu = \frac{32}{75}\delta_{\frac{3}{4}} + \frac{11}{75}\delta_{\frac{1}{2}} + \frac{32}{75}\delta_{\frac{1}{4}}.$$

*Under the map* $t = -\ln s$ *the measure* $\nu$ *becomes an image measure* $\hat{\nu}$ *on* $[0,\infty)$

$$\hat{\nu} = \frac{32}{75}\delta_{\ln\frac{4}{3}} + \frac{11}{75}\delta_{\ln 2} + \frac{32}{75}\delta_{\ln 4}.$$

*Let us find the function* $\phi_2(x)$

$$\phi_2(x) = \int_0^1 s^x\nu(ds) = \int_0^\infty e^{-xt}\hat{\nu}(dt) = \frac{32}{75}\left(\frac{4}{3}\right)^{-x} + \frac{11}{75}2^{-x} + \frac{32}{75}4^{-x}.$$

*According to Lemma 5, we conclude that* $\phi_2$ *is a ruin probability function for some spectrally negative Lévy process which drifts to* $+\infty$*. It is easy to see that* $\phi_1(0) = \phi_2(0) = \frac{1}{2}$*,* $\phi_1(1) = \phi_2(1) = \frac{91}{300}$ *and* $\phi_1(2) = \phi_2(2) = \frac{41}{200}$*. Now we have two ruin probability functions with at least three nonnegative intersections.*

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