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Approximation by complex Lorentz polynomials

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Abstract. In this paper we obtain a quantitative estimate in the Voronovskaja's theorem and the exact orders in simultaneous approximation by the complex Lorentz polynomials attached to analytic functions in compact disks. Also, we study the approximation properties of the iterates of these polynomials.

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1. Introduction

In the recent book [1] (see also the papers cited therein), estimates for the convergence in Voronovskaja's theorem and the approximation orders in simultaneous approximation for several important classes of complex Bernstein-type operators attached to an analytic function f in closed disks were obtained.

The goal of the present paper is to extend these types of results to the complex Lorentz polynomials.

These polynomials were introduced in [2, p. 43, formula (2)] under the name of degenerate Bernstein polynomials, by the formula attached to any analytic function f in a domain containing the origin,

$$L_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{z}{n}\right)^k f^{(k)}(0), n \in \mathbb{N}.$$

In the same book [2], on pages 121–124 some qualitative approximation results are studied.

The plan of the present paper goes as follows. Section 2 deals with upper estimates in simultaneous approximation by these polynomials. In Section 3 we obtain a Voronovskaja result with a quantitative estimate and in Section 4 one obtains exact estimates in simultaneous approximation for these operators. Section 5 presents a quantitative approximation result for the iterates of the complex polynomials $L_n(f)(z)$. All the quantitative estimates are obtained in compact disks centered at the origin.

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2. Upper approximation estimates

The main result of this section is the following.

Theorem 1. *For $R > 1$ and denoting $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$, suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

(i) Let $1 \leq r < R$ be arbitrary fixed. For all $|z| \leq r$ and $n \in \mathbb{N}$, we have the upper estimate

$$|L_n(f)(z) - f(z)| \leq \frac{M_r(f)}{n},$$

where

$$M_r(f) = \frac{1}{2} \sum_{k=2}^{\infty} |c_k| k(k-1) r^k < \infty.$$

(ii) For the simultaneous approximation by complex Lorentz polynomials, we have: if $1 \leq r < r_1 < R$ are arbitrary fixed, then for all $|z| \leq r$, $p \in \mathbb{N}$ and $n \in \mathbb{N}$ we have

$$|L_n^{(p)}(f)(z) - f^{(p)}(z)| \leq \frac{p! r_1 M_{r_1}(f)}{n(r_1 - r)^{p+1}},$$

where $M_{r_1}(f)$ is given as at the above point (i).

Proof. (i) Denoting $e_j(z) = z^j$, we easily get that $L_n(e_0)(z) = 1$, $L_n(e_1)(z) = e_1(z)$ and that for all $j, n \in \mathbb{N}$, $j \geq 2$, we have

$$L_n(e_j)(z) = \binom{n}{j} j! \cdot \frac{z^j}{n^j} = z^j \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) ... \left(1 - \frac{j-1}{n}\right).$$

Also, since an easy computation shows that

$$L_n(f)(z) = \sum_{j=0}^{\infty} c_j L_n(e_j)(z), \text{ for all } |z| \leq r,$$

and taking $L_n(e_0)(z) = 1$, $L_n(e_1)(z) = z$ into account, we immediately obtain

$$\begin{aligned} |L_n(f)(z) - f(z)| &\leq \sum_{j=0}^{\infty} |c_j| \cdot |L_n(e_j)(z) - e_j(z)| \\ &\leq \sum_{j=2}^{\infty} |c_j| r^j \left| \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) ... \left(1 - \frac{j-1}{n}\right) - 1 \right|, \end{aligned}$$

for all $|z| \leq r$.

Taking into account that a simple inequality

$$1 - \Pi_{j=1}^{k-1} x_j \leq \sum_{j=1}^{k-1} (1 - x_j),$$

holds if $0 \le x_j \le 1$, for all $j = 1, ..., k-1$, by taking $x_j = 1 - \frac{j}{n}$ we obtain

$$1 - \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) ... \left(1 - \frac{k-1}{n}\right) \le \sum_{j=1}^{k-1}\left[1 - \frac{n-j}{n}\right] = \frac{k(k-1)}{2n},$$

which implies the desired estimate.

(ii) Denoting by γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \le r$ and $v \in \gamma$, we have $|v - z| \ge r_1 - r$, by the Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}|L_n^{(p)}(f)(z) - f^{(p)}(z)| &= \frac{p!}{2\pi}\left|\int_\gamma \frac{L_n(f)(v) - f(v)}{(v-z)^{p+1}}dv\right| \\ &\le \frac{M_{r_1}(f)}{n}\frac{p!}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{p+1}} = \frac{M_{r_1}(f)}{n} \cdot \frac{p! r_1}{(r_1 - r)^{p+1}},\end{aligned}$$

which proves (ii) and the theorem. □

3. Quantitative Voronovskaja type theorem

The following Voronovskaja-type result holds.

Theorem 2. *For $R > 1$, let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in \mathbb{D}_R$, and let $1 \le r < R$ be arbitrary fixed. We have*

$$\left|L_n(f)(z) - f(z) + \frac{z^2}{2n}f''(z)\right| \le \frac{1}{2n^2}\sum_{k=2}^{\infty}|c_k| r^k (k-1)^2(k-2)^2, \text{ for all } n \in \mathbb{N}, |z| \le r,$$

where

$$\sum_{k=2}^{\infty}|c_k| r^k (k-1)^2 (k-2)^2 < \infty.$$

Proof. We have

$$\begin{aligned}&\left|L_n(f)(z) - f(z) + \frac{z^2}{2n}f''(z)\right| \\ &= \left|\sum_{k=0}^{\infty} c_k\left[L_n(e_k)(z) - e_k(z) + \frac{k(k-1)}{2n}e_k(z)\right]\right| \\ &= \left|\sum_{k=2}^{\infty} c_k z^k \left[\frac{(n-1)(n-2)...(n-(k-1))}{n^{k-1}} - 1 + \frac{k(k-1)}{2n}\right]\right| \\ &\le \sum_{k=2}^{\infty}|c_k| r^k \left|\frac{(n-1)(n-2)...(n-(k-1))}{n^{k-1}} - 1 + \frac{k(k-1)}{2n}\right|,\end{aligned}$$

for all $|z| \le r$ and $n \in \mathbb{N}$.

In what follows, we will prove by mathematical induction with respect to k that

$$0 \leq E_{n,k} \leq \frac{(k-1)^2(k-2)^2}{2n^2}, \tag{1}$$

for all $k \geq 2$ (here $n \in \mathbb{N}$ is arbitrary fixed), where

$$E_{n,k} = \frac{(n-1)(n-2)...(n-(k-1))}{n^{k-1}} - 1 + \frac{k(k-1)}{2n}.$$

Indeed, for $k = 2$ it is trivial. Suppose that it is valid for arbitrary k. We are going to prove that it remains valid for $k+1$ too, that is

$$0 \leq \frac{(n-1)(n-2)...(n-k)}{n^k} - 1 + \frac{k(k+1)}{2n} \leq \frac{k^2(k-1)^2}{2n^2}. \tag{2}$$

For this purpose, we take into account that

$$\begin{aligned} E_{n,k+1} &= \frac{(n-1)(n-2)...(n-k)}{n^k} - 1 + \frac{k(k+1)}{2n} \\ &= \frac{(n-1)(n-2)...(n-(k-1))}{n^{k-1}}\left(1 - \frac{k}{n}\right) \\ &\quad -1 + \frac{k(k-1)}{2n} + \frac{k(k+1)}{2n} - \frac{k(k-1)}{2n} \\ &= E_{n,k} + \frac{k}{n}\left(1 - \frac{(n-1)(n-2)...(n-(k-1))}{n^{k-1}}\right). \end{aligned}$$

By (1) it is immediate that $E_{n,k+1} \geq 0$. Also, by the same relationship (1) and taking into account the simple inequality used at the end of the proof of Theorem 1, (i), we get

$$\begin{aligned} E_{n,k+1} &\leq \frac{(k-1)^2(k-2)^2}{2n^2} + \frac{k}{n}\left(1 - \frac{(n-1)(n-2)...(n-(k-1))}{n^{k-1}}\right) \\ &\leq \frac{(k-1)^2(k-2)^2}{2n^2} + \frac{k}{n} \cdot \frac{k(k-1)}{2n} = \frac{1}{2n^2}[(k-1)^2(k-2)^2 + k^2(k-1)]. \end{aligned}$$

Looking at (2), in fact it remains to prove that

$$(k-1)^2(k-2)^2 + k^2(k-1) \leq k^2(k-1)^2,$$

which is after simple calculation equivalent to the inequality $0 \leq 3k^2 - 8k + 4$, which is obviously valid for all $k \geq 2$.

In conclusion, (2) is valid, which implies that (1) is valid and this proves the theorem. □

4. Exact approximation estimates

The first main result of this section is the following.

Theorem 3. *Let $R > 1$, $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in \mathbb{D}_R$, and $1 \leq r < R$ be arbitrary fixed. If f is not a polynomial of degree ≤ 1, then for all $n \in \mathbb{N}$ and $|z| \leq r$ we have*

$$\|L_n(f) - f\|_r \geq \frac{C_r(f)}{n},$$

where the constant $C_r(f)$ depends only on f. Here $\|f\|_r$ denotes $\max_{|z|\leq r}\{|f(z)|\}$.

Proof. For all $|z| \leq r$ and $n \in \mathbb{N}$ we have

$$L_n(f)(z) - f(z) = \frac{1}{n}\left\{-\frac{z^2}{2}f''(z) + \frac{1}{n}\left[n^2\left(L_n(f)(z) - f(z) + \frac{z^2}{2n}f''(z)\right)\right]\right\}.$$

In what follows we will apply to this identity the following obvious property:

$$\|F + G\|_r \geq |\,\|F\|_r - \|G\|_r\,| \geq \|F\|_r - \|G\|_r.$$

It follows

$$\|L_n(f) - f\|_r \geq \frac{1}{n}\left\{\left\|\frac{e_1^2}{2}f''\right\|_r - \frac{1}{n}\left[n^2\left\|L_n(f) - f + \frac{e_1^2}{2n}f''\right\|_r\right]\right\}.$$

Since by hypothesis f is not a polynomial of degree ≤ 1 in $\mathbb{D}_R$, we get $\left\|\frac{e_1^2}{2}f''\right\|_r > 0$.

Indeed, supposing the contrary it follows that $\frac{z^2}{2}f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r = \{z \in \mathbb{C}; |z| \leq r\}$, which implies $f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r \setminus \{0\}$. Since f is supposed to be analytic, from the identity theorem of analytic (holomorphic) functions this necessarily implies that $f''(z) = 0$, for all $z \in \mathbb{D}_R$, i.e. that f is a polynomial of degree ≤ 1, which is a contradiction.

But by Theorem 2 we have

$$n^2\left\|L_n(f) - f + \frac{e_1^2}{2n}f''\right\|_r \leq \frac{1}{2}\cdot\sum_{k=2}^{\infty}|c_k|r^k(k-1)^2(k-2)^2.$$

Therefore, there exists an index n_0 depending only on f and r, such that for all $n > n_0$ we have

$$\left\|\frac{e_1^2}{2}f''\right\|_r - \frac{1}{n}\left[n^2\left\|L_n(f) - f + \frac{e_1^2}{2n}f''\right\|_r\right] \geq \frac{1}{2}\left\|\frac{e_1^2}{2}f''\right\|_r,$$

which immediately implies that

$$\|L_n(f) - f\|_r \geq \frac{1}{n}\cdot\frac{1}{2}\left\|\frac{e_1^2}{2}f''\right\|_r, \quad \forall n > n_0.$$

For $n \in \{1, ..., n_0\}$ we obviously have $\|L_n(f) - f\|_r \geq \frac{M_{r,n}(f)}{n}$ with $M_{r,n}(f) = n \cdot \|L_n(f) - f\|_r > 0$ (if $\|L_n(f) - f\|_r$ would be equal to 0, this would imply that f is a linear function, a contradiction).

Therefore, finally we get $\|L_n(f)-f\|_r \geq \frac{C_r(f)}{n}$ for all $n \in \mathbb{N}$, where

$$C_r(f)=\min\left\{M_{r,1}(f),...,M_{r,n_0}(f),\frac{1}{2}\left\|\frac{e_1^2}{2}f''\right\|_r\right\},$$

which completes the proof. □

Combining now Theorem 3 with Theorem 1, (i) we immediately get the following.

Corollary 1. *Let $R>1$, $f:\mathbb{D}_R\to\mathbb{C}$ be analytic in $\mathbb{D}_R$, that is $f(z)=\sum_{k=0}^{\infty}c_kz^k$ for all $z\in\mathbb{D}_R$, and let $1\leq r<R$ be arbitrary fixed. If f is not a polynomial of degree ≤ 1, then for all $n\in\mathbb{N}$ we have*

$$\|L_n(f)-f\|_r \sim \frac{1}{n},$$

where the constants in the equivalence depend on f and r but are independent of n.

Concerning the simultaneous approximation we present the following.

Theorem 4. *Let $R>1$, $f:\mathbb{D}_R\to\mathbb{C}$ be analytic in $\mathbb{D}_R$, that is $f(z)=\sum_{k=0}^{\infty}c_kz^k$ for all $z\in\mathbb{D}_R$, and let $1\leq r<r_1<R$ be arbitrary fixed. Also, let $p\in\mathbb{N}$. If f is not a polynomial of degree $\leq\max\{1,p-1\}$, then for all $n\in\mathbb{N}$ we have*

$$\|L_n^{(p)}(f)-f^{(p)}\|_r \sim \frac{1}{n},$$

where the constants in the equivalence depend on f, r, r_1 and p but are independent of n.

Proof. Since by Theorem 1, (ii) we have an upper estimate for $\|L_n^{(p)}(f)-f^{(p)}\|_r$, it remains to prove the lower estimate for $\|L_n^{(p)}(f)-f^{(p)}\|_r$. For this purpose, denoting by Γ the circle of radius r_1 and center 0, we have the inequality $|v-z|\geq r_1-r$ valid for all $|z|\leq r$ and $v\in\Gamma$. The Cauchy's formula is expressed by

$$L_n^{(p)}(f)(z)-f^{(p)}(z)=\frac{p!}{2\pi i}\int_\Gamma \frac{L_n(f)(v)-f(v)}{(v-z)^{p+1}}dv.$$

Now, as in the proof of Theorem 1, (ii), for all $v\in\Gamma$ and $n\in\mathbb{N}$ we have

$$L_n(f)(v)-f(v)=\frac{1}{n}\left\{-\frac{v^2}{2}f''(v)+\frac{1}{n}\left[n^2\left(L_n(f)(v)-f(v)+\frac{v^2}{2n}f''(v)\right)\right]\right\},$$

which replaced in the above Cauchy's formula implies

$$\begin{aligned} L_n^{(p)}(f)(z) - f^{(p)}(z) &= \frac{1}{n}\left\{ \frac{p!}{2\pi i}\int_{\Gamma} -\frac{v^2 f''(v)}{2(v-z)^{p+1}}dv \right. \\ &\left. + \frac{1}{n}\cdot\frac{p!}{2\pi i}\int_{\Gamma}\frac{n^2\left(L_n(f)(v) - f(v) + \frac{v^2}{2n}f''(v)\right)}{(v-z)^{p+1}}dv \right\} \\ &= \frac{1}{n}\left\{ \left[-\frac{z^2}{2}f''(z)\right]^{(p)} \right. \\ &\left. + \frac{1}{n}\cdot\frac{p!}{2\pi i}\int_{\Gamma}\frac{n^2\left(L_n(f)(v) - f(v) + \frac{v^2}{2n}f''(v)\right)}{(v-z)^{p+1}}dv \right\}. \end{aligned}$$

Passing now to $\|\cdot\|_r$, for all $n \in \mathbb{N}$ it follows

$$\begin{aligned} \|L_n^{(p)}(f) - f^{(p)}\|_r \geq \frac{1}{n}&\left\{ \left\| \left[-\frac{e_1^2}{2}f''\right]^{(p)} \right\|_r \right. \\ &\left. - \frac{1}{n}\left\| \frac{p!}{2\pi}\int_{\Gamma}\frac{n^2\left(L_n(f)(v) - f(v) + \frac{v^2}{2n}f''(v)\right)}{(v-z)^{p+1}}dv \right\|_r \right\}, \end{aligned}$$

where by using Theorem 2, for all $n \in \mathbb{N}$ we get

$$\begin{aligned} &\left\| \frac{p!}{2\pi}\int_{\Gamma}\frac{n^2\left(L_n(f)(v) - f(v) + \frac{v^2}{2n}f''(v)\right)}{(v-z)^{p+1}}dv \right\|_r \\ &\leq \frac{p!}{2\pi}\cdot\frac{2\pi r_1 n^2}{(r_1 - r)^{p+1}}\left\| L_n(f) - f + \frac{e_1^2}{2n}f'' \right\|_{r_1} \\ &\leq \frac{1}{2}\sum_{k=2}^{\infty} r_1^k (k-1)^2(k-2)^2 \cdot \frac{p! r_1}{(r_1 - r)^{p+1}}. \end{aligned}$$

But by hypothesis on f, we have $\left\| -\left[\frac{e_1^2}{2}f''\right]^{(p)} \right\|_r > 0$. Indeed, supposing the contrary it follows that $-\frac{z^2}{2}f''(z)$ is a polynomial of degree $\leq p-1$.

Now, if $p = 1$ and $p = 2$, then the analyticity of f obviously implies that f is necessarily a polynomial of degree $\leq 1 = \max\{1, p-1\}$, which contradicts the hypothesis. If $p > 2$, then the analyticity of f obviously implies that f is necessarily a polynomial of degree $\leq p-1 = \max\{1, p-1\}$, which again contradicts the hypothesis.

In continuation, reasoning exactly as in the proof of Theorem 3, we immediately get the desired conclusion. □

5. Approximation by iterates

For f analytic in $\mathbb{D}_R$ that is of the form $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$, let us define the iterates of a complex Lorentz polynomial $L_n(f)(z)$, by $L_n^{(1)}(f)(z) =$

$L_n(f)(z)$ and $L_n^{(m)}(f)(z) = L_n[L_n^{(m-1)}(f)](z)$, for any $m \in \mathbb{N}$, $m \geq 2$.

Since we have

$$L_n(f)(z) = \sum_{k=0}^{\infty} c_k L_n(e_k)(z),$$

by recurrence for all $m \geq 1$, we easily get that

$$L_n^{(m)}(f)(z) = \sum_{k=0}^{\infty} c_k L_n^{(m)}(e_k)(z),$$

where $L_n^{(m)}(e_k)(z) = 1$ if $k = 0$, $L_n^{(m)}(e_k)(z) = z$ if $k = 1$ and

$$L_n^{(m)}(e_k)(z) = \left(1 - \frac{1}{n}\right)^m \left(1 - \frac{2}{n}\right)^m ... \left(1 - \frac{k-1}{n}\right)^m z^k, \text{ for } k \geq 2.$$

The main result of this section is the following

Theorem 5. *Let f be analytic in $\mathbb{D}_R$ with $R > 1$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Let $1 \leq r < R$. We have*

$$\|L_n^{(m)}(f) - f\|_r \leq \frac{m}{n} \sum_{k=2}^{\infty} |c_k| \frac{k(k-1)}{2} r^k,$$

and therefore if $\lim_{n\to\infty} \frac{m}{n} = 0$, *then*

$$\lim_{n\to\infty} \|L_n^{(m)}(f) - f\|_r = 0.$$

Proof. For all $|z| \leq r$ we easily obtain

$$|f(z) - L_n^{(m)}(f)(z)| \leq \sum_{k=2}^{\infty} |c_k| r^k \left[1 - \left(1 - \frac{1}{n}\right)^m \left(1 - \frac{2}{n}\right)^m ... \left(1 - \frac{k-1}{n}\right)^m\right].$$

Denoting $A_k = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) ... \left(1 - \frac{k-1}{n}\right)$, we get

$$1 - A_k^m = (1 - A_k)(1 + A + A^2 + ... + A^{m-1}) \leq m(1 - A_k)$$

and therefore since $1 - A_k \leq \frac{k(k-1)}{2n}$, for all $|z| \leq r$ we obtain

$$|f(z) - L_n^{(m)}(f)(z)| \leq m \sum_{k=2}^{\infty} |c_k| r^k [1 - A_k] \leq \frac{m}{n} \sum_{k=2}^{\infty} |c_k| r^k \frac{k(k-1)}{2},$$

which immediately proves the theorem. □

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