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# Approximation for periodic functions via statistical $\sigma-$convergence

Kamil Demirci[1,*] and Fadime Dirik[1]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Sinop University, TR–57 000 Sinop, Turkey*

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**Abstract.** In this study, using the concept of statistical $\sigma-$convergence which is stronger than convergence and statistical convergence we prove a Korovkin-type approximation theorem for sequences of positive linear operators defined on $C^*$ which is the space of all $2\pi$-periodic and continuous functions on $\mathbb{R}$, the set of all real numbers. We also study the rates of statistical $\sigma-$convergence of approximating positive linear operators.

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**Key words**: statistical $\sigma-$convergence, statistical convergence, positive linear operator, Korovkin-type approximation theorem, periodic functions, Fejer polynomials

## 1. Introduction

For a sequence $\{L_n\}$ of positive linear operators on $C(X)$, which is the space of real valued continuous functions on a compact subset $X$ of real numbers, Korovkin [14] first introduced the necessary and sufficient conditions for the uniform convergence of $L_n(f)$ to a function $f$ by using the test function $f_i$ defined by $f_i(x) = x^i$, $(i = 0, 1, 2)$ (see, for instance, [3]). Later many researchers investigated these conditions for various operators defined on different spaces. Using the concept of statistical convergence in the approximation theory provides us with many advantages. In particular, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence of various sequences of linear operators such as the interpolation operator of Hermite-Fejér [4], because these types of operators do not converge at points of simple discontinuity. Furthermore, in recent years, with the help of the concept of uniform statistical convergence, which is a regular (non-matrix) summability transformation, various statistical approximation results have been proved [1, 2, 6, 7, 8, 9, 12, 13]. Also, a Korovkin-type approximation theorem has been studied via statistical convergence in the space $C^*$ which is the space of all $2\pi$-periodic and continuous functions on $\mathbb{R}$ in [5]. Then, it was demonstrated that those results are more powerful than the classical Korovkin theorem. Recently various kinds of statistical convergence stronger than the statistical convergence have been introduced by Mursaleen and Edely [15].

We now recall some basic definitions and notations used in the paper.

*Corresponding author. *Email addresses:* kamild@sinop.edu.tr (K. Demirci), dirikfadime@gmail.com (F. Dirik)

 

Let $K$ be a subset of $\mathbb{N}$, the set of natural numbers. Then the natural density of $K$, denoted by $\delta(K)$, is given by:

$$\delta(K) := \lim_n \frac{1}{n} \left|\{k \leq n \ : k \in K \ \}\right|$$

whenever the limit exists, where $|B|$ denotes the cardinality of the set $B$. Then a sequence $x = \{x_k\}$ of numbers is statistically convergent to $L$ provided that, for every $\varepsilon > 0$, $\delta \{k : |x_k - L| \geq \varepsilon\} = 0$ holds. In this case we write $st - \lim x_k = L$.

Notice that every convergent sequence is statistically convergent to the same value, but its converse is not true. Such an example may be found in [10, 11, 17].

Let $\sigma$ be a mapping of the set of $\mathbb{N}$ into itself. A continuous linear functional $\varphi$ defined on the space $l_\infty$ of all bounded sequences is called an invariant mean (or $\sigma-$mean) [16] if it is nonnegative, normal and $\varphi(x) = \varphi((x_{\sigma(n)}))$.

A sequence $x = \{x_k\}$ is said to be statistically $\sigma-$convergent to $L$ if for every $\varepsilon > 0$ the set $K_\varepsilon(\varphi) := \{k \in \mathbb{N} : \varphi(|x_k - L|) \geq \varepsilon\}$ has natural density zero, i.e. $\delta(K_\varepsilon(\varphi)) = 0$. In this case we write $\delta(\sigma) - \lim x_k = L$. That is,

$$\lim_n \frac{1}{n} \left|\{p \leq n \ : \ |t_{pm}(x_m) - L| \geq \varepsilon\}\right| = 0, \text{ uniformly in } m,$$

where

$$t_{pm}(x_m) := \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + ... + x_{\sigma^p(m)}}{p+1}, \ t_{-1,m}(x_m) = 0$$

(for details, see [15]). Using the above definitions, the next result follows immediately.

**Lemma 1.** *Statistical convergence implies statistical $\sigma-$convergence.*

However, one can construct an example which guarantees that the converse of Lemma 1 is not always true. Such an example was given in [15] as follows:

**Example 1.** *Consider the case $\sigma(n) = n+1$ and the sequence $u = \{u_m\}$ defined as*

$$u_m = \begin{cases} 1, & \text{if } m \text{ is odd,} \\ -1, & \text{if } m \text{ is even,} \end{cases} \tag{1}$$

*is statistically $\sigma-$convergence ($\delta(\sigma) - \lim u_m = 0$) but it is neither convergent nor statistically convergent.*

With the above terminology, our primary interest in the present paper is to obtain a Korovkin-type approximation theorem by means of the concept of statistical $\sigma-$convergence. Also, by considering Lemma 1 and the above Example 1, we will construct a sequence of positive linear operators such that while our new results work, their classical and statistical cases do not work. We also compute the rates of statistical $\sigma-$convergence of the sequence of positive linear operators.

## 2. A Korovkin-type approximation theorem

We denote by $C^*$ the space of all $2\pi$-periodic and continuous functions on $\mathbb{R}$. This space is equipped with the supremum norm

$$\|f\|_{C^*} = \sup_{x\in\mathbb{R}} |f(x)|, \ (f \in C^*).$$

Let $L$ be a linear operator from $C^*$ into $C^*$. Then, as usual, we say that $L$ is a positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ at a point $x \in \mathbb{R}$ by $L(f(u);x)$ or, briefly, $L(f;x)$.

We now recall the classical and statistical cases (for $A = C_1$, the Cesáro matrix) of the Korovkin-type results introduced in [14, 5], respectively.

**Theorem 1.** *Let $\{L_m\}$ be a sequence of positive linear operators acting from $C^*$ into $C^*$. Then, for all $f \in C^*$,*

$$\lim \|L_m(f;x) - f(x)\|_{C^*} = 0$$

*if and only if*

$$\begin{aligned} \lim \|L_m(1;x) - 1\|_{C^*} &= 0, \\ \lim \|L_m(\cos u;x) - \cos x\|_{C^*} &= 0, \\ \lim \|L_m(\sin u;x) - \sin x\|_{C^*} &= 0. \end{aligned}$$

**Theorem 2.** *Let $\{L_m\}$ be a sequence of positive linear operators acting from $C^*$ into $C^*$. Then, for all $f \in C^*$, we have*

$$st - \lim \|L_m(f;x) - f(x)\|_{C^*} = 0$$

*if and only if*

$$\begin{aligned} st - \lim \|L_m(1;x) - 1\|_{C^*} &= 0, \\ st - \lim \|L_m(\cos u;x) - \cos x\|_{C^*} &= 0, \\ st - \lim \|L_m(\sin u;x) - \sin x\|_{C^*} &= 0. \end{aligned}$$

**Theorem 3.** *Let $\{L_m\}$ be a sequence of positive linear operators acting from $C^*$ into $C^*$. Then, for all $f \in C^*$*

$$\delta(\sigma) - \lim \|L_m(f;x) - f(x)\|_{C^*} = 0 \tag{2}$$

*if and only if the following statements hold:*

$$\delta(\sigma) - \lim \|L_m(1;x) - 1\|_{C^*} = 0, \tag{3}$$
$$\delta(\sigma) - \lim \|L_m(\cos u;x) - \cos x\|_{C^*} = 0, \tag{4}$$
$$\delta(\sigma) - \lim \|L_m(\sin u;x) - \sin x\|_{C^*} = 0. \tag{5}$$

**Proof**. Under the hypotheses, since 1, $\cos u$ and $\sin u$ belong to $C^*$, the necessity is clear. Assume now that (2) holds. Let $f \in C^*$ and $I$ be a closed subinterval of length $2\pi$ of $\mathbb{R}$. Fix $x \in I$. As in the proof of Theorem 1 in [5], it follows from the continuity of $f$ that

$$|f(u) - f(x)| < \varepsilon + \frac{2M_f}{\sin^2 \frac{\delta_1}{2}} \sin^2 \frac{u-x}{2}$$

which gives

$$\begin{aligned}
|t_{pm}(L_m(f;x)) - f(x)| &\leq \frac{L_m(|f(u) - f(x)|;x) + L_{\sigma(m)}(|f(u) - f(x)|;x)}{p+1} + \dots \\
&\quad + \frac{L_{\sigma^p(m)}(|f(u) - f(x)|;x)}{p+1} + |f(x)|\,|t_{pm}(L_m(1;x)) - 1| \\
&\leq (\varepsilon + |f(x)|)\,||t_{pm}(L_m(1;x)) - 1|| + \varepsilon \\
&\quad + \frac{M_f}{\sin^2 \frac{\delta_1}{2}} \{|t_{pm}(L_m(1;x)) - 1| \\
&\quad + |\cos x|\,|t_{pm}(L_m(\cos u;x)) - \cos x| \\
&\quad + |\sin x|\,|t_{pm}(L_m(\sin u;x)) - \sin x|\} \\
&< \varepsilon + \left(\varepsilon + |f(x)| + \frac{M_f}{\sin^2 \frac{\delta_1}{2}}\right) \{|t_{pm}(L_m(1;x)) - 1| \\
&\quad + |t_{pm}(L_m(\cos u;x)) - \cos x| \\
&\quad + |t_{pm}(L_m(\sin u;x)) - \sin x|\},
\end{aligned}$$

where $M_f = \|f\|_{C^*}$. Then, we obtain

$$\begin{aligned}
\|t_{pm}(L_m(f)) - f\|_{C^*} < \varepsilon + K \big\{ &\|t_{pm}(L_m(1;x)) - 1\|_{C^*} \\
&+ \|t_{pm}(L_m(\cos u;x)) - \cos x\|_{C^*} \\
&+ \|t_{pm}(L_m(\sin u;x)) - \sin x\|_{C^*} \big\},
\end{aligned} \tag{6}$$

where

$$K := \sup_{x \in I} \left\{ \varepsilon + |f(x)| + \frac{M_f}{\sin^2 \frac{\delta_1}{2}} \right\}.$$

Now given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. By (6), it is easy to see that

$$\begin{aligned}
&\left|\left\{p \leq n : \|t_{pm}(L_m(f)) - f\|_{C^*} \geq r\right\}\right| \\
&\quad \leq \left|\left\{p \leq n : \|t_{pm}(L_m(1;x)) - 1\|_{C^*} \geq \frac{r-\varepsilon}{3K}\right\}\right| \\
&\qquad + \left|\left\{p \leq n : \|t_{pm}(L_m(\cos u;x)) - \cos x\|_{C^*} \geq \frac{r-\varepsilon}{3K}\right\}\right| \\
&\qquad + \left|\left\{p \leq n : \|t_{pm}(L_m(\sin u;x)) - \sin x\|_{C^*} \geq \frac{r-\varepsilon}{3K}\right\}\right|.
\end{aligned}$$

Now using (3), (4) and (5), we get (2) and the proof is complete. $\square$

**Remark 1.** *We now show that our result in Theorem 3 is stronger than its classical and statistical versions. Now define the Fejer operators $F_m$ as follows:*

$$F_m(f;x) = \frac{1}{m\pi}\int_{-\pi}^{\pi} f(y)\,\frac{\sin^2\left(\frac{m}{2}(u-x)\right)}{2\sin^2\left[\left(\frac{u-x}{2}\right)\right]}du, \tag{7}$$

*where $m \in \mathbb{N}$, $f \in C^*[-\pi,\pi]$ . Then, we get (see [14])*

$$F_m(1;x) = 1,\ F_m(\cos u;x) = \frac{m-1}{m}\cos x,\ F_m(\sin u;x) = \frac{m-1}{m}\sin x.$$

*Now using (1) and (7), we introduce the following positive linear operators defined on the space $C^*[-\pi,\pi]$ :*

$$L_m(f;x) = (1+u_m)\,F_m(f;x). \tag{8}$$

*Since $\delta(\sigma) - \lim u_m = 0$, we conclude that*

$$\delta(\sigma) - \lim \|L_m(1;x) - 1\|_{C^*[-\pi,\pi]} = 0,$$
$$\delta(\sigma) - \lim \|L_m(\cos u;x) - \cos x\|_{C^*[-\pi,\pi]} = 0,$$
$$\delta(\sigma) - \lim \|L_m(\sin u;x) - \sin x\|_{C^*[-\pi,\pi]} = 0.$$

*Then, by Theorem 3, we obtain for all $f \in C^*[-\pi,\pi]$,*

$$\delta(\sigma) - \lim \|L_m(f;x) - f(x)\|_{C^*[-\pi,\pi]} = 0.$$

*However, since u is not convergent and statistically convergent, we conclude that classical (Theorem 1) and statistical (Theorem 2) versions of our result do not work for the operators $L_m$ in (8) while our Theorem 3 still does.*

## 3. Rate of statistical $\sigma-$convergence

In this section, we study the rates of statistical $\sigma-$convergence of a sequence of positive linear operators defined $C^*$ into $C^*$ with the help of modulus of continuity.

**Definition 1.** *A sequence $\{x_m\}$ is statistically $\sigma$-convergent to a number $L$ with the rate of $\beta \in (0,1)$ if for every $\varepsilon > 0$,*

$$\lim_n \frac{|\{p \le n : |t_{pm}(x_m) - L| \ge \varepsilon\}|}{n^{1-\beta}} = 0,\ \textit{uniformly in } m.$$

*In this case, it is denoted by*

$$x_m - L = o(n^{-\beta})\,(\delta(\sigma)).$$

Using this definition, we obtain the following auxiliary result.

**Lemma 2.** *Let $\{x_m\}$ and $\{y_m\}$ be sequences. Assume that $x_m - L_1 = o(n^{-\beta_1})\,(\delta(\sigma))$ and $y_m - L_2 = o(n^{-\beta_2})\,(\delta(\sigma))$ . Then we have*

$(i)$ $(x_m - L_1) \mp (y_m - L_2) = o(n^{-\beta})\,(\delta\,(\sigma))$, *where* $\beta := \min\{\beta_1, \beta_2\}$,

$(ii)$ $\lambda(x_m - L_1) = o(n^{-\beta_1})\,(\delta\,(\sigma))$, *for any real number* $\lambda$.

**Proof**. $(i)$ Assume that $x_m - L_1 = o(n^{-\beta_1})\,(\delta\,(\sigma))$ and $y_m - L_2 = o(n^{-\beta_2})\,(\delta\,(\sigma))$. Then, for $\varepsilon > 0$, observe that

$$\begin{aligned}
&\frac{|\{p \le n : \ |(t_{pm}(x_m) - L_1) \mp (t_{pm}(y_m) - L_2)| \ge \varepsilon\}|}{n^{1-\beta}} \\
&\quad \le \frac{\left|\left\{p \le n : \ |t_{pm}(x_m) - L_1| \ge \frac{\varepsilon}{2}\right\}\right| + \left|\left\{p \le n : \ |t_{pm}(y_m) - L_2| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta}} \\
&\quad \le \frac{\left|\left\{p \le n : \ |t_{pm}(x_m) - L_1| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_1}} + \frac{\left|\left\{p \le n : \ |t_{pm}(y_m) - L_2| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_2}}
\end{aligned} \tag{9}$$

Now by taking the limit as $n \to \infty$ in (9) and using the hypotheses, we conclude that

$$\lim_n \frac{|\{p \le n : \ |(t_{pm}(x_m) - L_1) \mp (t_{pm}(y_m) - L_2)| \ge \varepsilon\}|}{n^{1-\beta}} = 0, \text{ uniformly in } m,$$

which completes the proof of $(i)$. Since the proof of $(ii)$ is similar, we omit it. □

Now we remind of the concept of modulus of continuity. For $f \in C^*$, the modulus of continuity of $f$, denoted by $\omega\,(f; \delta_1)$, is defined to be

$$\omega\,(f; \delta_1) = \sup_{|u-x|<\delta_1} |f\,(u) - f\,(x)|\,.$$

It is also well known that for any $\delta_1 > 0$,

$$|f\,(u) - f\,(x)| \le \omega\,(f; \delta_1) \left( \frac{|u-x|}{\delta_1} + 1 \right). \tag{10}$$

Then we have the following result.

**Theorem 4.** *Let* $\{L_m\}$ *be a sequence of positive linear operators acting from* $C^*$ *into* $C^*$. *Assume that the following conditions holds:*

*(i)* $\|L_m(1; x) - 1\|_{C^*} = o(n^{-\beta_1})\,(\delta\,(\sigma))$,

*(ii)* $w(f, \alpha_{pm}) = o(n^{-\beta_2})\,(\delta\,(\sigma))$, *where* $\alpha_{pm} := \sqrt{\|t_{pm}\,(L_m(\varphi; x))\|_{C^*}}$ *with* $\varphi(u) = \sin^2 \frac{u-x}{2}$.

*Then we have, for all* $f \in C^*$,

$$\|L_m(f; x) - f(x)\|_{C^*} = o(n^{-\beta})\,(\delta\,(\sigma))),$$

*where* $\beta = \min\{\beta_1, \beta_2\}$.

**Proof**. Let $f \in C^*$ and fix $x \in [-\pi, \pi]$. Then, we may write, for all $m \in \mathbb{N}$, that

$$
\begin{aligned}
&|t_{pm}(L_m(f;x)) - f(x)| \\
&\quad \leq \frac{L_m(|f(u)-f(x)|;x) + L_{\sigma(m)}(|f(u)-f(x)|;x)}{p+1} + \dots \\
&\quad\quad + \frac{L_{\sigma^p(m)}(|f(u)-f(x)|;x)}{p+1} + |f(x)|\,|t_{pm}(L_m(1;x)) - 1| \\
&\quad \leq \frac{L_m\left(1+\frac{(u-x)^2}{\delta_1^2};x\right)}{p+1}\omega(f;\delta_1) \\
&\quad\quad + \frac{L_{\sigma(m)}\left(1+\frac{(u-x)^2}{\delta_1^2};x\right) + \dots + L_{\sigma^p(m)}\left(1+\frac{(u-x)^2}{\delta_1^2};x\right)}{p+1}\omega(f;\delta_1) \\
&\quad\quad + |f(x)|\,|t_{pm}(L_m(1;x)) - 1| \\
&\quad \leq \frac{L_m\left(1+\frac{\pi^2}{\delta_1^2}\sin^2\frac{u-x}{2};x\right)}{p+1}\omega(f;\delta_1) \\
&\quad\quad + \frac{L_{\sigma(m)}\left(1+\frac{\pi^2}{\delta_1^2}\sin^2\frac{u-x}{2};x\right) + \dots + L_{\sigma^p(m)}\left(1+\frac{\pi^2}{\delta_1^2}\sin^2\frac{u-x}{2};x\right)}{p+1}\omega(f;\delta_1) \\
&\quad\quad + |f(x)|\,|t_{pm}(L_m(1;x)) - 1| \\
&\quad = \left(t_{pm}(L_m(1;x)) + \frac{\pi^2}{\delta_1^2}t_{pm}(L_m(\varphi;x))\right)\omega(f;\delta_1) \\
&\quad\quad + |f(x)|\,|t_{pm}(L_m(1;x)) - 1|.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
\|t_{pm}(L_m(f;x)) - f(x)\|_{C^*} \leq{}& \|f\|_{C^*}\|t_{pm}(L_m(1;x)) - 1\|_{C^*} + \left(1+\pi^2\right)w(f,\alpha_{pm}) \\
&+ w(f,\alpha_{pm})\|t_{pm}(L_m(1;x)) - 1\|_{C^*},
\end{aligned}
$$

where $\delta_1 := \alpha_{pm} := \sqrt{\|t_{pm}(L_m(\varphi;x))\|_{C^*}}$. Then we obtain

$$
\begin{aligned}
\|t_{pm}(L_m(f;x)) - f(x)\|_{C^*} \leq K\big\{&\|t_{pm}(L_m(1;x)) - 1\|_{C^*} + w(f,\alpha_{pm}) \\
&+ w(f,\alpha_{pm})\|t_{pm}(L_m(1;x)) - 1\|_{C^*}\big\}, \qquad (11)
\end{aligned}
$$

where $K = \max\left\{\|f\|_{C^*}, 1+\pi^2\right\}$. Then, we have, from (11)

$$
\begin{aligned}
&\frac{\left|\left\{p \leq n : \|t_{pm}(L_m(f;x)) - f(x)\|_{C^*} \geq \varepsilon\right\}\right|}{n^{1-\beta}} \\
&\quad \leq \frac{\left|\left\{p \leq n : \|t_{pm}(L_m(1;x)) - 1\|_{C^*} \geq \frac{\varepsilon}{3K}\right\}\right|}{n^{1-\beta_1}} \\
&\quad\quad + \frac{\left|\left\{p \leq n : w(f,\alpha_{pm}) \geq \frac{\varepsilon}{3K}\right\}\right|}{n^{1-\beta_2}} + \frac{\left|\left\{p \leq n : w(f,\alpha_{pm}) \geq \sqrt{\frac{\varepsilon}{3K}}\right\}\right|}{n^{1-\beta_2}} \\
&\quad\quad + \frac{\left|\left\{p \leq n : \|t_{pm}(L_m(1;x)) - 1\|_{C^*} \geq \sqrt{\frac{\varepsilon}{3K}}\right\}\right|}{n^{1-\beta_1}} \qquad (12)
\end{aligned}
$$

where $\beta = \min\{\beta_1, \beta_2\}$. Letting $n \to \infty$ in (12), we conclude from $(i)$ and $(ii)$ that

$$\lim_n \frac{\left|\left\{p \le n : \|t_{pm}(L_m(f;x)) - f(x)\|_{C^*} \ge \varepsilon\right\}\right|}{n^{1-\beta}} = 0, \text{ uniformly in } m,$$

which means

$$\|L_m(f;x) - f(x)\|_{C^*} = o(n^{-\beta})(\delta(\sigma)).$$

The proof is completed. $\square$

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