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On the modular equations of degree m - a generalization

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Abstract. We give a general Lambert series expansion leading to a modular equation of degree m, with this we have a general Fourier series expansion for *dn* z. The Lambert series are essential for giving modular equations. We also give a simple proof of a consequence of an identity of Ramanujan leading to a Theorem of Jacobi.

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1. Introduction

There are twelve Jacobian elliptic functions, the complete listing is given in [9, pp. 511-512]. From Ramanujan's ${}_1\psi_1$-summation formula we have the following well-known identity:

$$\sum_{n=-\infty}^{\infty} \frac{\alpha^n}{1-tq^{2n}} = \frac{i}{2}\theta_1' \frac{\theta_1(x+y,q)}{\theta_1(x,q)\theta_1(y,q)}, \tag{1}$$

where $\alpha = e^{2ix}$,$t = e^{2iy}$,$|q| < 1$,$|q^2| < |\alpha| < 1$ and $|t| \neq 1$.

From this identity (1), by suitably choosing x and y, we can easily have the Fourier series expansion of these twelve Jacobian elliptic functions. For getting modular equations, Lambert series identities are essential. In his paper Shen [4] gave a collection of well-known Lambert series identities from which modular identities can be obtained.

In this paper we consider the Fourier series expansion of the Jacobian elliptic function *dn* z and prove a general theorem from which Shen's identities listed in [4] come easily as special cases and moreover we get new identities. This method can be applied to get identities from the Fourier series expansion of other Jacobian elliptic functions.

For writing the Fourier series expansion for *dn*z which is given in (5), we recall by definition

$$dnz = \frac{\theta_4(q)\theta_3(z/\theta_3^2)}{\theta_3(q)\theta_4(z/\theta_3^2)}. \tag{2}$$

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Taking $z = \frac{2Kz}{\pi}$, where $K = \frac{\pi}{2}\theta_3^2$, (2) can be written as

$$dn\left(\frac{2Kz}{\pi}\right) = \frac{\theta_4(q)\theta_3(z)}{\theta_3(q)\theta_4(z)}. \tag{3}$$

By specializing the parameter of (1), we have

$$\theta_3(q)\theta_4(q)\frac{\theta_3(z)}{\theta_4(z)} = 2\sum_{n=-\infty}^{\infty}\frac{q^n e^{2inz}}{1+q^{2n}} = 1 + 4\sum_{n=1}^{\infty}\frac{q^n cos2nz}{1+q^{2n}}. \tag{4}$$

Thus the Fourier series expansion of dnz is

$$dn\left(\frac{2Kz}{\pi}\right) = \frac{\pi}{2K} + \frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{q^n cos2nz}{1+q^{2n}}. \tag{5}$$

My general approach will, in a way, unify the identities given by Shen in [4].

2. Basic facts

We shall use the following standard $q-$notations, $|q^k| < 1$:

$$(a;q^k)_n = (1-a)(1-aq^k).......(1-aq^{k(n-1)}), n \geq 1$$

$$(a;q^k)_\infty = \prod_{m=0}^{\infty}(1-aq^{mk}),$$

$$(a;q^k)_0 = 1.$$

The following are Jacobi's expressions for the theta functions as infinite products [9, p.469]

$$\theta_1(z,q) = iq^{\frac{1}{4}}e^{-iz}\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-2}e^{2iz})(1-q^{2n}e^{-2iz}),$$

$$\theta_2(z,q) = 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n}e^{2iz})(1+q^{2n}e^{-2iz}),$$

$$\theta_3(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2iz})(1+q^{2n-1}e^{-2iz}),$$

and

$$\theta_4(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}e^{2iz})(1-q^{2n-1}e^{-2iz}),$$

where $q = e^{\pi i\tau}$ and $Im(\tau) > 0$. φ(q) is defined [2, p. 36] as:

$$\varphi(q) = 1 + 2\sum_{k=1}^{\infty}q^{k^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, |q| < 1.$$

Ramanujan's ${}_1\psi_1$- summation formula

$${}_1\psi_1(a;b;q,z) = \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, |b/a| < |z| < 1.$$

3. A general theorem for Lambert series

In this section we will prove a theorem which will unify different Lambert series identities of different modulus which are essential for getting modular equations. We shall prove the following general theorem and deduce special identities:

Theorem 1.

$$\begin{aligned}&\frac{(q;q)_\infty(-q^{2m};q^{2m})_\infty(-q^{m+2a}e^{2ib\pi};q^{2m})_\infty(-q^{m-2a}e^{-2ib\pi};q^{2m})_\infty}{(-q;q)_\infty(q^{2m};q^{2m})_\infty(q^{m+2a}e^{2ib\pi};q^{2m})_\infty(q^{m-2a}e^{-2ib\pi};q^{2m})_\infty}\frac{\theta_4^3(q^{2m})}{\theta_4(q)}\\&=1+2\sum_{n=1}^{\infty}q^{mn}\left(\frac{e^{2in(a\pi\tau+b\pi)}+e^{-2in(a\pi\tau+b\pi)}}{1+q^{2mn}}\right).\end{aligned}\tag{6}$$

Proof. We make $q\to q^m$ in (4) and then take $z=a\pi\tau+b\pi$, where m,a and b are rational numbers. First we simplify the left-hand side.

The left-hand side of (4) is equal to

$$\theta_3(q^m)\theta_4(q^m)\frac{\theta_3(a\pi\tau+b\pi,q^m)}{\theta_4(a\pi\tau+b\pi,q^m)}\tag{7}$$

$$\begin{aligned}&=(q^{2m};q^{2m})_\infty^2(q^{2m};q^{4m})_\infty^2\frac{(-q^me^{2i(a\pi\tau+b\pi)};q^{2m})_\infty(-q^me^{-2i(a\pi\tau+b\pi)};q^{2m})_\infty}{(q^me^{2i(a\pi\tau+b\pi)};q^{2m})_\infty(q^me^{-2i(a\pi\tau+b\pi)};q^{2m})_\infty}\\&=(q^{2m};q^{2m})_\infty^2(q^{2m};q^{4m})_\infty^2\frac{(-q^{m+2a}e^{2ib\pi};q^{2m})_\infty(-q^{m-2a}e^{-2ib\pi};q^{2m})_\infty}{(q^{m+2a}e^{2ib\pi};q^{2m})_\infty(q^{m-2a}e^{-2ib\pi};q^{2m})_\infty}\\&=\frac{(q^{2m};q^{2m})_\infty^3(q^{2m};q^{4m})_\infty^2(-q^{2m};q^{2m})_\infty}{(-q^{2m};q^{2m})_\infty(q^{2m};q^{2m})_\infty}\\&\quad\times\frac{(-q^{m+2a}e^{2ib\pi};q^{2m})_\infty(-q^{m-2a}e^{-2ib\pi};q^{2m})_\infty}{(q^{m+2a}e^{2ib\pi};q^{2m})_\infty(q^{m-2a}e^{-2ib\pi};q^{2m})_\infty}\end{aligned}$$

$$\begin{aligned}&=\frac{(q;q)_\infty(-q^{2m};q^{2m})_\infty(-q^{m+2a}e^{2ib\pi};q^{2m})_\infty(-q^{m-2a}e^{-2ib\pi};q^{2m})_\infty}{(-q;q)_\infty(q^{2m};q^{2m})_\infty(q^{m+2a}e^{2ib\pi};q^{2m})_\infty(q^{m-2a}e^{-2ib\pi};q^{2m})_\infty}\\&\quad\times\frac{\theta_4^3(q^{2m})}{\theta_4(q)}.\end{aligned}\tag{8}$$

The right-hand side of (4) is equal to

$$\begin{aligned}&1+2\sum_{n=1}^{\infty}q^{mn}\left(\frac{e^{2in(a\pi\tau+b\pi)}+e^{-2in(a\pi\tau+b\pi)}}{1+q^{2mn}}\right)\\&=1+2\sum_{n=1}^{\infty}q^{mn}\left(\frac{q^{2na}e^{2inb\pi}+q^{-2na}e^{-2inb\pi}}{1+q^{2mn}}\right).\end{aligned}\tag{9}$$

From (8) and (9), we have our Theorem 1. □

4. Identities leading to modular identity

In this section we deduce two identities and then give a modular identity.

From (7) and (9), we have

$$\theta_3(q^m)\theta_4(q^m)\frac{\theta_3(a\pi\tau+b\pi,q^m)}{\theta_4(a\pi\tau+b\pi,q^m)}=1+2\sum_{n=1}^{\infty}q^{mn}\left(\frac{q^{2na}e^{2inb\pi}+q^{-2na}e^{-2inb\pi}}{1+q^{2mn}}\right). \tag{10}$$

Letting $q\to q^3$ in (10) and taking $a=\frac{m}{3}$,$b=0$, we get

$$\theta_3(q^{3m})\theta_4(q^{3m})\frac{\theta_3(m\pi\tau,q^{3m})}{\theta_4(m\pi\tau,q^{3m})}=1+2\sum_{n=1}^{\infty}\frac{q^{5mn}+q^{mn}}{1+q^{6mn}}. \tag{11}$$

We first simplify the left-hand side of (11).

The left hand-side of (11) is equal to

$$\begin{aligned}
&(q^{6m};q^{6m})^2_\infty(-q^{3m};q^{6m})^2_\infty(q^{3m};q^{6m})^2_\infty\frac{(-q^{5m};q^{6m})_\infty(-q^m;q^{6m})_\infty}{(q^{5m};q^{6m})_\infty(q^m;q^{6m})_\infty}\\
&=(q^{6m};q^{6m})^2_\infty(q^{6m};q^{12m})^2_\infty\frac{(-q^m;q^{2m})_\infty(q^{3m};q^{6m})_\infty}{(q^m;q^{2m})_\infty(-q^{3m};q^{6m})_\infty}\\
&=\frac{(-q^m;q^{2m})_\infty(q^{3m};q^{6m})_\infty(q^{6m};q^{6m})^2_\infty}{(q^m;q^{2m})_\infty(-q^{3m};q^{6m})_\infty(-q^{6m};q^{6m})^2_\infty}.
\end{aligned} \tag{12}$$

Now we show that $\theta_3(q^m)\theta_3(q^{3m})\frac{\theta_4^3(q^{3m})}{\theta_4(q^m)}$ is the square of the right-hand side of (12). Thus

$$\begin{aligned}
\theta_3(q^m)\theta_3(q^{3m})\frac{\theta_4^3(q^{3m})}{\theta_4(q^m)} &= (q^{2m};q^{2m})_\infty(-q^m;q^{2m})^2_\infty(q^{6m};q^{6m})_\infty\\
&\quad\times(-q^{3m};q^{6m})^2_\infty\frac{(q^{6m};q^{6m})^3_\infty(q^{3m};q^{6m})^6_\infty}{(q^{2m};q^{2m})_\infty(q^m;q^{2m})^2_\infty}\\
&=\frac{(-q^m;q^{2m})^2_\infty(q^{3m};q^{6m})^2_\infty(q^{6m};q^{6m})^4_\infty}{(q^m;q^{2m})^2_\infty(-q^{3m};q^{6m})^2_\infty(-q^{6m};q^{6m})^4_\infty}.
\end{aligned} \tag{13}$$

Hence by (11), (12) and (13), we have

$$\left[\theta_3(q^m)\theta_3(q^{3m})\frac{\theta_4^3(q^{3m})}{\theta_4(q^m)}\right]^{\frac{1}{2}}=1+2\sum_{n=1}^{\infty}\frac{q^{5mn}+q^{mn}}{1+q^{6mn}}. \tag{14}$$

Writing q^m and $q^{3m}$for q in (4) and $z=0$, respectively,

$$\theta_3^2(q^m)=1+4\sum_{n=1}^{\infty}\frac{q^{mn}}{1+q^{2mn}}$$

and

$$\theta_3^2(q^{3m})=1+4\sum_{n=1}^{\infty}\frac{q^{3mn}}{1+q^{6mn}}.$$

So

$$\begin{aligned}\theta_3^2(q^m)+\theta_3^2(q^{3m}) &= 2+4\sum_{n=1}^{\infty}\left(\frac{q^{mn}}{1+q^{2mn}}+\frac{q^{3mn}}{1+q^{6mn}}\right)\\ &= 2+4\sum_{n=1}^{\infty}\left(\frac{q^{mn}(1-q^{2mn}+q^{4mn})+q^{3mn}}{1+q^{6mn}}\right)\\ &= 2\left(1+2\sum_{n=1}^{\infty}\left(\frac{q^{5mn}+q^{mn}}{1+q^{6mn}}\right)\right). \qquad (15)\end{aligned}$$

Hence by (14) and (15), we have

$$\left[\theta_3(q^m)\theta_3(q^{3m})\frac{\theta_4^3(q^{3m})}{\theta_4(q^m)}\right]^{\frac{1}{2}}=\frac{1}{2}\left(\theta_3^2(q^m)+\theta_3^2(q^{3m})\right). \qquad (16)$$

It can also be easily seen

$$\theta_3(q^{3m})\theta_4(q^{3m})\frac{\theta_3(\frac{\pi}{6},q^m)}{\theta_4(\frac{\pi}{6},q^m)}=\frac{(-q^m;q^{2m})_\infty(q^{3m};q^{6m})_\infty(q^{6m};q^{6m})_\infty^2}{(q^m;q^{2m})_\infty(-q^{3m};q^{6m})_\infty(-q^{6m};q^{6m})_\infty^2}. \qquad (17)$$

Hence from (13), (16) and (17), we have

$$\theta_3(q^{3m})\theta_4(q^{3m})\frac{\theta_3(\frac{\pi}{6},q^m)}{\theta_4(\frac{\pi}{6},q^m)}=\frac{1}{2}\left(\theta_3^2(q^m)+\theta_3^2(q^{3m})\right). \qquad (18)$$

For different values of m we have different modular identities.

5. Lambert series as special cases that are essential for modular equations

Case 1:

Take $b=0$, $a=\frac{1}{4}$ and $m=\frac{3}{2}$ in (6), to get

$$\frac{\theta_4^3(q^3)}{\theta_4(q)}=1+2\sum_{n=1}^{\infty}\frac{q^{2n}+q^n}{1+q^{3n}}=1+2\sum_{n=0}^{\infty}(-1)^n\left(\frac{q^{3n+1}}{1-q^{3n+1}}+\frac{q^{3n+2}}{1-q^{3n+2}}\right), \qquad (19)$$

which is (2.2) of Shen [4].

Replacing q by $-q$ in (19), we get

$$\frac{\theta_3^3(q^3)}{\theta_3(q)}=1-2\sum_{n=1}^{\infty}\left(\frac{q^{6n+1}}{1+q^{6n+1}}-\frac{q^{6n+2}}{1-q^{6n+2}}+\frac{q^{6n+4}}{1-q^{6n+4}}-\frac{q^{6n+5}}{1+q^{6n+5}}\right), \qquad (20)$$

which is (2.3) of Shen [4]. Here we used $\theta_4(-q)=\theta_3(q)$.

Case II:

Take $m=\frac{3}{2}$, $a=\frac{1}{4}$ and $b=\frac{1}{2}$ in (6), to get

$$\theta_4(q)\theta_4(q^3)=1-2\sum_{n=0}^{\infty}(-1)^n\left(\frac{q^{3n+1}}{1+q^{3n+1}}+\frac{q^{3n+2}}{1+q^{3n+2}}\right), \qquad (21)$$

which is (2.4) of Shen [4]. Replacing q by $-q$ in (21), we get

$$\theta_3(q)\theta_3(q^3) = 1 + 2\sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1-q^{6n+1}} - \frac{q^{6n+2}}{1+q^{6n+2}} + \frac{q^{6n+4}}{1+q^{6n+4}} - \frac{q^{6n+5}}{1-q^{6n+5}}\right), \quad (22)$$

which is (2.5) of Shen [4].

Case III:

Take $m = 1$, $a = 0$, and $b = \frac{1}{6}$ in (6), to get

$$\begin{aligned}&\frac{(q;q)_\infty(-q^2;q^2)_\infty(-qe^{\frac{i\pi}{3}};q^2)_\infty(-qe^{-\frac{i\pi}{3}};q^2)_\infty}{(-q;q)_\infty(q^2;q^2)_\infty(qe^{\frac{i\pi}{3}};q^2)_\infty(qe^{-\frac{i\pi}{3}};q^2)_\infty}\frac{\theta_4^3(q^2)}{\theta_4(q)}\\ &= 1 + 2\sum_{n=1}^{\infty} q^n\left(\frac{e^{\frac{in\pi}{3}} + e^{-\frac{in\pi}{3}}}{1+q^{2n}}\right). \end{aligned} \quad (23)$$

The left-hand side of (23) is equal to

$$\begin{aligned}&\frac{(q;q)_\infty(-q^2;q^2)_\infty(q^3;q^6)_\infty(-q;q)_\infty(q^4;q^4)_\infty^3(q^2;q^4)_\infty^6}{(-q;q)_\infty(q^2;q^2)_\infty(-q^3;q^6)_\infty(q;q)_\infty(q^2;q^2)_\infty(q;q^2)_\infty^2}\\ &= \frac{(q^2;q^2)_\infty(-q;q^2)_\infty(q^3;q^6)_\infty}{(-q^2;q^2)_\infty(q;q^2)_\infty(-q^3;q^6)_\infty}\\ &= \left(\frac{\theta_3^3(q)}{\theta_3(q^3)}\right)^{\frac{1}{2}}\left(\theta_4(q)\theta_4(q^3)\right)^{\frac{1}{2}}, \end{aligned} \quad (24)$$

and the right-hand side of (23) is equal to

$$1 + 4\sum_{n=1}^{\infty}\frac{q^n cos\frac{n\pi}{3}}{1+q^{2n}}. \quad (25)$$

Taking $n = 3n$, $3n+1$,$3n+2$ in the right-hand side of (25) we get

$$1 + 2\sum_{n=0}^{\infty}(-1)^n\left(\frac{q^{3n+1}}{1+q^{6n+2}} - \frac{q^{3n+2}}{1+q^{6n+4}} + \frac{2q^{3n+3}}{1+q^{6n+6}}\right), \quad (26)$$

From (24) and (26), we have

$$\begin{aligned}\left(\theta_4(q)\theta_4(q^3)\right)^{\frac{1}{2}}\left(\frac{\theta_3^3(q)}{\theta_3(q^3)}\right)^{\frac{1}{2}} &= 1 + 2\sum_{n=0}^{\infty}(-1)^n\\ &\times\left(\frac{q^{3n+1}}{1+q^{6n+2}} - \frac{q^{3n+2}}{1+q^{6n+4}} + \frac{2q^{3n+3}}{1+q^{6n+6}}\right), \end{aligned} \quad (27)$$

which is (2.7) of Shen [4].

6. Specializing Theorem 1

We write our Theorem in a more convenient form. Take $a = \frac{m-1}{2}$ and $b = 0$ in (6), to get

$$\begin{aligned} &\frac{(q;q)_\infty(-q^{2m};q^{2m})_\infty(-q^{2m-1};q^{2m})_\infty(-q;q^{2m})_\infty}{(-q;q)_\infty(q^{2m};q^{2m})_\infty(q^{2m-1};q^{2m})_\infty(q;q^{2m})_\infty}\frac{\theta_4^3(q^{2m})}{\theta_4(q)} \\ &= 1+2\sum_{n=1}^{\infty}\frac{q^{n(2m-1)}+q^n}{1+q^{2mn}} \\ &= 1+2\sum_{n=0}^{\infty}(-1)^n\left(\frac{q^{2mn+2m-1}}{1-q^{2mn+2m-1}}+\frac{q^{2mn+1}}{1-q^{2mn+1}}\right). \end{aligned} \tag{28}$$

6.1. Special cases

(i)

$$\frac{\theta_4^3(q^3)}{\theta_4(q)} = 1+2\sum_{n=0}^{\infty}(-1)^n\left(\frac{q^{3n+1}}{1-q^{3n+1}}+\frac{q^{3n+2}}{1-q^{3n+2}}\right), \tag{29}$$

($m = 3/2$ in (28), Shen[4, eq.(2.2)]).

(ii)

$$\frac{(q^2;q^4)_\infty}{(-q^2;q^4)_\infty}\frac{\theta_4^3(q^4)}{\theta_4(q)} = 1+2\sum_{n=0}^{\infty}(-1)^n\left(\frac{q^{4n+1}}{1-q^{4n+1}}+\frac{q^{4n+3}}{1-q^{4n+3}}\right), \tag{30}$$

($m = 2$ in (28))

(iii)

$$\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(-q^2;q^5)_\infty(-q^3;q^5)_\infty}\frac{\theta_4^3(q^5)}{\theta_4(q)} = 1+2\sum_{n=0}^{\infty}(-1)^n\left(\frac{q^{5n+1}}{1-q^{5n+1}}+\frac{q^{5n+4}}{1-q^{5n+4}}\right), \tag{31}$$

($m = 5/2$ in (28)).

(iv)

$$\begin{aligned} &\frac{(q^2;q^6)_\infty(q^3;q^6)_\infty(q^4;q^6)_\infty}{(-q^2;q^6)_\infty(-q^3;q^6)_\infty(-q^4;q^6)_\infty}\frac{\theta_4^3(q^6)}{\theta_4(q)} \\ &= 1+2\sum_{n=0}^{\infty}(-1)^n\left(\frac{q^{6n+1}}{1-q^{6n+1}}+\frac{q^{6n+5}}{1-q^{6n+5}}\right), \end{aligned} \tag{32}$$

($m = 3$ in (28)).

7. Consequence of Entry 33(iii) of Ramanujan

We prove a consequence of Entry 33(iii) of Chapter 16 of Ramanujan.

Theorem 2.

$$\varphi^4(-q^2) = \left(1 + 4\sum_{n=1}^{\infty} \frac{q^n cos2nz}{1+q^{2n}}\right)\left(1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^n cos2nz}{1+q^{2n}}\right) \tag{33}$$

Proof. Writing $-q$ for q in (4), we have

$$\theta_3(-q)\theta_4(-q)\frac{\theta_3(z,-q)}{\theta_4(z,-q)} = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^n cos2nz}{1+q^{2n}}. \tag{34}$$

Multiplying (4) by (34), we obtain

$$\theta_3(q)\theta_3(-q)\theta_4(q)\theta_4(-q)\frac{\theta_3(z,q)\theta_3(z,-q)}{\theta_4(z,q)\theta_4(z,-q)} = \left(1 + 4\sum_{n=1}^{\infty} \frac{q^n cos2nz}{1+q^{2n}}\right) \times \left(1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^n cos2nz}{1+q^{2n}}\right). \tag{35}$$

Since $\theta_3(-q) = \theta_4(q)$, the left-hand side of (35)

$$\theta_3^2(q)\theta_4^2(q) = (q^2;q^2)_\infty^4 (q^2;q^4)_\infty^4 = \frac{(q^2;q^2)_\infty^4}{(-q^2;q^2)_\infty^4} = \varphi^4(-q^2),$$

which proves Theorem 2. □

Since the result is interesting, I state the proof given by Berndt [2, eq. 33.5, p. 54]:

From the product of the two series on the right-hand side and then integrate both sides with respect to z, over the interval$[-\pi,\pi]$. Since the set of functions $\{cos2nk\}$, $0 \le n \le \infty$ is orthogonal on$[-\pi,\pi]$, we have a result of Jacobi

$$\varphi^4(-q^2) = 1 + 8\sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{(1+q^{2n})^2}. \tag{36}$$

Concluding remarks

We have also obtained modular equations in earlier papers. In the earlier papers we used a different method. In [5], we obtain Hecke type modular series by the Bailey pair method. In [6], we gave a modular transformation for tenth order mock theta functions by using modular group generators. In [7], we proved a simple theta functions identities on base four from which modular equations can be obtained. In [8], from a theta function identity of McCullough and Shen [3] we obtained Eisenstein series related to modular equations.

In this paper we have used the Fourier series of the theta functions to obtain Lambert series identities which are essential for modular equations. This approach is quite different.

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