MATHEMATICAL COMMUNICATIONS
Math. Commun. **16**(2011), 95–104.

On an iterative algorithm for variational inequalities in Banach spaces

YONGHONG YAO[1], MUHAMMAD ASLAM NOOR[2,*], KHALIDA INAYAT NOOR[3] AND YEONG-CHENG LIOU[4]

[1] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300 160, P.R. China*
[2] *Mathematics Department, College of Science, King Saud University, Riyadh 11 451, Saudi Arabia*
[3] *Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan*
[4] *Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan*

Received April 26, 2009; accepted June 7, 2010

Abstract. In this paper, we suggest and analyze a new iterative method for solving some variational inequality involving an accretive operator in Banach spaces. We prove the strong convergence of the proposed iterative method under certain conditions. As a special of the proposed algorithm, we proved that the algorithm converges strongly to the minimum norm solution of some variational inequality.

AMS subject classifications: 49J30, 47H05, 47H10

Key words: iterative algorithm, accretive mapping, variational inequality, strong convergence, Banach space

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a nonlinear operator. The variational inequality problem is formulated as finding a point $x^* \in C$ such that

$$\langle x - x^*, Ax^* \rangle \geq 0, \forall x \in C. \tag{1}$$

It is well known that variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [1]-[3],[6]-[15],[18]-[22] and the references therein.

*Corresponding author. *Email addresses:* `yaoyonghong@yahoo.cn` (Y. Yao), `noormaslam@hotmail.com` (M. A. Noor), `khalidanoor@hotmail.com` (K. I. Noor), `simplex_liou@hotmail.com` (Y. C. Liou)

Recall that a mapping $A : C \to H$ is said to be α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C.$$

Some iterative algorithms for solving the variational inequality (1) involving an inverse strongly monotone operator A have been extensively studied in the literature. See, for example, [9], [19], [21] and the references therein. Especially, Iiduka, Takahashi and Toyoda [10] introduced the following iterative algorithm

$$x_{n+1} = P(\alpha_n x_n + (1 - \alpha_n)P(x_n - \lambda_n Ax_n)), n \geq 0. \tag{2}$$

They proved that the sequence $\{x_n\}$ generated by (2) converges weakly to x^* which solves the variational inequality (1).

We now consider the problem of finding $x^* \in C, \quad j(x-x^*) \in J(x-y)$ such that

$$\langle Ax^*, j(x - x^*)\rangle \geq 0, \forall x \in C, \tag{3}$$

which is called the generalized variational inequality in Banach spaces. The set of solutions of the variational inequality (3) is denoted by $S(C, A)$, that is,

$$S(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0,\ x \in C\}.$$

In order to find a solution of the variational inequality (3), Aoyama, Iiduka and Takahashi [2] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q(x_n - \lambda_n Ax_n), n \geq 0, \tag{4}$$

where Q is a sunny nonexpansive retraction from E onto C. However, they only obtained the weak convergence of the proposed scheme (4). Hence, it is an interesting topic to construct an iterative algorithm which strongly converges to the solution of variational inequality (3). In this respect, Aoyama, Iiduka and Takahashi [3] further introduced another iterative algorithm which has strong convergence.

On the other hand, we note that in many problems, it is needed to find a solution with minimum norm (see [16]). It is interesting to find the minimum-norm solution of the variational inequality (3).

Motivated and inspired by the research going on in this field, we suggest and analyze a new iterative method for solving the variational inequality (3) in Banach spaces, which is the main motivation of this paper. We prove the strong convergence of the proposed iterative method under certain conditions. As a special of the proposed algorithm, we proved that the algorithm converges strongly to the minimum norm solution of the variational inequality (3).

2. Preliminaries

Let E be a Banach space, E^* the dual space of E and let $\langle\cdot,\cdot\rangle$ denote the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}, \forall x \in E.$$

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in U$

$$\|x - y\| \geq \epsilon \text{ implies } \left\|\frac{x + y}{2}\right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\},$$

where $\rho : [0, \infty) \to [0, \infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau \to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

Let C be a nonempty closed convex subset of a Banach space E. An operator A of C into E is said to be α-inverse strongly accretive if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, J(x - y)\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C.$$

Let D be a subset of C and Q a mapping of C into D. Then Q is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. We also need the following results.

Lemma 1 (see [20]). *Let C be a closed convex subset of a smooth Banach space E, D a nonempty subset of C and Q a retraction from C onto D. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu)\rangle \leq 0$$

for all $u \in C$ and $y \in D$.

Lemma 2 (see [4, 5]). *Let q be a given real number with $1 < q \leq 2$ and E a q-uniformly smooth Banach space. Then*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q$$

for all $x, y \in E$, where K is the q-uniformly smoothness constant of E.

Lemma 3 (see [2]). *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then, for all $\lambda > 0$,*

$$S(C, A) = F(Q(I - \lambda A)).$$

Lemma 4 (see [17]). *Let $\{x_n\}$, $\{z_n\}$ be bounded sequences in a Banach space E and $\{\alpha_n\}$ a sequence in $[0,1]$ which satisfies the following condition: $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose that $x_{n+1} = \alpha_n x_n + (1-\alpha_n)z_n$ for all $n = 0, 1, 3, \cdots$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 5 (see [20]). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\sigma_n, \ n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ are satisfied that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) either $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then $\{a_n\}_{n=0}^{\infty}$ converges to 0.

3. Main results

In this section, we will introduce our algorithm and prove that the proposed algorithm converges strongly to a solution of variational inequality (3) in a uniformly convex and 2-uniformly smooth Banach space.

Algorithm 1. *For fixed $u \in E$ and given $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Q[\beta_n u + (1-\beta_n)Q(x_n - \lambda_n A x_n)], n \geq 0, \tag{5}$$

where Q is a sunny nonexpansive retraction from E onto C, $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$ and $\{\lambda_n\}$ is a sequence of real numbers.

In particular, if we take $u = 0$, then (5) reduces to

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Q[(1-\beta_n)Q(x_n - \lambda_n A x_n)], n \geq 0. \tag{6}$$

We now state and prove our main result, which is the main motivation of our next result.

Theorem 1. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping. Let $A : C \to E$ be an α-inverse strongly accretive operator such that $S(C, A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0,1)$ and $\{\lambda_n\}$ a real number sequence in $[a, \frac{\alpha}{K^2}]$. Suppose the following conditions are satisfied:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n \leq 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ generated by (5) converges strongly to $Q'u$, where Q' is a sunny nonexpansive retraction of E onto $S(C,A)$.

In particular, if we take $u = 0$, then the sequence $\{x_n\}$ generated by (6) converges strongly to the minimum norm element in $S(C,A)$.

Proof. For all $x, y \in C$ and $\lambda_n \in \left(0, \frac{\alpha}{K^2}\right]$, it is known [2] that $I - \lambda_n A$ is nonexpansive.

Take $p \in S(C,A)$. From Lemma 3, we have $p = Q(p - \lambda_n Ap)$. Setting $y_n = Q[\beta_n u + (1-\beta_n)Q(x_n - \lambda_n Ax_n)], n \geq 0$. Note that $p = Q(p)$, then we can write

$$p = Q[\beta_n p + (1-\beta_n)Q(p - \lambda_n Ap)].$$

Hence, from (5), we get

$$\begin{aligned}
\|y_n - p\| &= \|Q[\beta_n u + (1-\beta_n)Q(x_n - \lambda_n Ax_n)] \\
&\quad -Q[\beta_n p + (1-\beta_n)Q(p - \lambda_n Ap)]\| \\
&\leq \|\beta_n u + (1-\beta_n)Q(x_n - \lambda_n Ax_n) - [\beta_n p + (1-\beta_n)Q(p - \lambda_n Ap)]\| \\
&\leq \beta_n\|u - p\| + (1-\beta_n)\|Q(x_n - \lambda_n Ax_n) - Q(p - \lambda_n Ap)\| \\
&\leq \beta_n\|u - p\| + (1-\beta_n)\|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\| \\
&\leq \beta_n\|u - p\| + (1-\beta_n)\|x_n - p\|.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(x_n - p) + (1-\alpha_n)(y_n - p)\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)[\beta_n\|u - p\| + (1-\beta_n)\|x_n - p\|] \\
&= (1-\alpha_n)\beta_n\|u - p\| + [1 - (1-\alpha_n)\beta_n]\|x_n - p\| \\
&\leq \max\{\|u - p\|, \|x_0 - p\|\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded and so is $\{y_n\}$.

From (5), we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|Q[\beta_{n+1}u + (1-\beta_{n+1})Q(x_{n+1} - \lambda_{n+1}Ax_{n+1})] \\
&\quad -Q[\beta_n u + (1-\beta_n)Q(x_n - \lambda_n Ax_n)]\| \\
&\leq \|(\beta_{n+1} - \beta_n)u + Q(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - Q(x_n - \lambda_n Ax_n) \\
&\quad +\beta_n Q(x_n - \lambda_n Ax_n) - \beta_{n+1}Q(x_{n+1} - \lambda_{n+1}Ax_{n+1})\| \\
&\leq \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_n Ax_n)\| \\
&\quad +\beta_n(\|u\| + \|Q(x_n - \lambda_n Ax_n)\|) \\
&\quad +\beta_{n+1}(\|u\| + \|Q(x_{n+1} - \lambda_{n+1}Ax_{n+1})\|) \\
&= \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n) + (\lambda_n - \lambda_{n+1})Ax_n\| \\
&\quad +\beta_n(\|u\| + \|Q(x_n - \lambda_n Ax_n)\|) \\
&\quad +\beta_{n+1}(\|u\| + \|Q(x_{n+1} - \lambda_{n+1}Ax_{n+1})\|) \\
&\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \beta_n(\|u\| + \|Q(x_n - \lambda_n Ax_n)\|) \\
&\quad +\beta_{n+1}(\|u\| + \|Q(x_{n+1} - \lambda_{n+1}Ax_{n+1})\|),
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Therefore, from Lemma 4, we obtain

$$\lim_{n\to\infty}\|y_n-x_n\|=0.$$

At the same time, we observe that

$$\begin{aligned}\|y_n-Q(x_n-\lambda_nAx_n)\| &= \|Q[\beta_nu+(1-\beta_n)Q(x_n-\lambda_nAx_n)]-Q[Q(x_n-\lambda_nAx_n)]\|\\ &\le \beta_n\|u-Q(x_n-\lambda_nAx_n)\|\\ &\to 0.\end{aligned}$$

Next, we show that

$$\limsup_{n\to\infty}\langle u-Q'u, j(x_n-Q'u)\rangle\le 0 \tag{7}$$

where Q' is a sunny nonexpansive retraction of E onto $S(C,A)$

To prove (7), we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ to converge weakly to z such that

$$\limsup_{n\to\infty}\langle u-Q'u, j(x_n-Q'u)\rangle=\limsup_{i\to\infty}\langle u-Q'u, j(x_{n_i}-Q'u)\rangle. \tag{8}$$

We first prove $z\in S(C,A)$. Since λ_n is in $[a,\frac{\alpha}{K^2}]$ for some $a>0$, it follows that $\{\lambda_{n_i}\}$ is bounded and so there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ which converges to $\lambda_0\in[a,\frac{\alpha}{K^2}]$. We may assume, without loss of generality, that $\lambda_{n_i}\to\lambda_0$. Since Q is nonexpansive, we have

$$\begin{aligned}\|Q(x_{n_i}-\lambda_0Ax_{n_i})-x_{n_i}\| &\le \|Q(x_{n_i}-\lambda_0Ax_{n_i})-y_{n_i}\|+\|y_{n_i}-x_{n_i}\|\\ &= \|Q[Q(x_{n_i}-\lambda_0Ax_{n_i})]-Q[\beta_nu+(1-\beta_n)\\ &\quad \times Q(x_{n_i}-\lambda_{n_i}Ax_{n_i})]\|+\|y_{n_i}-x_{n_i}\|\\ &\le \beta_n\|u-Q(x_{n_i}-\lambda_0Ax_{n_i})\|+\|y_{n_i}-x_{n_i}\|\\ &\quad +(1-\beta_n)\|Q(x_{n_i}-\lambda_0Ax_{n_i})-Q(x_{n_i}-\lambda_{n_i}Ax_{n_i})\|\\ &\le \beta_n\|u-Q(x_{n_i}-\lambda_0Ax_{n_i})\|+\|y_{n_i}-x_{n_i}\|\\ &\quad +(1-\beta_n)\|(x_{n_i}-\lambda_0Ax_{n_i})-(x_{n_i}-\lambda_{n_i}Ax_{n_i})\|\\ &\le \beta_n\|u-Q(x_{n_i}-\lambda_0Ax_{n_i})\|+\|y_{n_i}-x_{n_i}\|\\ &\quad +|\lambda_{n_i}-\lambda_0|\|Ax_{n_i}\|,\end{aligned}$$

which implies that

$$\lim_{i\to\infty}\|Q(I-\lambda_0A)x_{n_i}-x_{n_i}\|=0. \tag{9}$$

By the demiclosedness principle for nonexpansive mappings and (9), we have $z\in F(Q(I-\lambda_0A))$. It follows from Lemma 3 that $z\in S(C,A)$.

From (8) and Lemma 1, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u-Q'u, j(x_n-Q'u)\rangle &= \limsup_{i\to\infty}\langle u-Q'u, j(x_{n_i}-Q'u)\rangle \\ &= \langle u-Q'u, j(z-Q'u)\rangle \quad \leq 0. \end{aligned} \tag{10}$$

Setting $z_n=\beta_n u+(1-\beta_n)(Q(x_n-\lambda_n Ax_n))$ for all $n\geq 0$. We note that

$$\begin{aligned}\|z_n-x_n\| &\leq \beta_n\|u-x_n\|+(1-\beta_n)\|Q(x_n-\lambda_n Ax_n)-x_n\| \\ &\leq \beta_n\|u-x_n\|+(1-\beta_n)\|Q(x_n-\lambda_n Ax_n)-y_n\| \\ &\quad +\|y_n-x_n\| \\ &\to 0.\end{aligned}$$

This together with (10) implies that

$$\limsup_{n\to\infty}\langle u-Q'u, j(z_n-Q'u)\rangle \leq 0. \tag{11}$$

Next we prove $x_n\to Q'u$. From (5), we have

$$\begin{aligned}\|y_n-Q'u\|^2 &= \|Q[\beta_n u+(1-\beta_n)Q(x_n-\lambda_n Ax_n)]-Q[Q'u]\|^2 \\ &\leq \|\beta_n(u-Q'u)+(1-\beta_n)(Q(x_n-\lambda_n Ax_n)-Q'u)\|^2 \\ &\leq (1-\beta_n)^2\|Q(x_n-\lambda_n Ax_n)-Q'u\|^2 \\ &\quad +2\beta_n\langle u-Q'u, j(z_n-Q'u)\rangle \\ &\leq (1-\beta_n)\|x_n-Q'u\|^2+2\beta_n\langle u-Q'u, j(z_n-Q'u)\rangle.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|x_{n+1}-Q'u\|^2 &\leq \alpha_n\|x_n-Q'u\|^2+(1-\alpha_n)\|y_n-Q'u\|^2 \\ &\leq \alpha_n\|x_n-Q'u\|^2+(1-\alpha_n)(1-\beta_n)\|x_n-Q'u\|^2 \\ &\quad +2(1-\alpha_n)\beta_n\langle u-Q'u, j(z_n-Q'u)\rangle \\ &= [1-(1-\alpha_n)\beta_n]\|x_n-Q'u\|^2 \\ &\quad +2(1-\alpha_n)\beta_n\langle u-Q'u, j(z_n-Q'u)\rangle.\end{aligned} \tag{12}$$

By Lemma 5, (11) and (12) we conclude that $\{x_n\}$ converges strongly to $Q'u$.

In particular, if $u=0$, then the sequence $\{x_n\}$ generated by (6) converges strongly to $Q'(0)$ which is the minimum norm element in $S(C,A)$. This completes the proof. □

Remark 1. *(1) From [2], we know that $Q(I-\lambda_n A)$ is nonexpansive.*
(2) If $S(C,A)\neq\emptyset$, it follows from Lemma 3 that there exists a sunny nonexpansive retraction Q' of E onto $F(Q(I-\lambda_n A))=S(C,A)$.

It is easy to obtain the following corollary.

Corollary 1. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping. Let $A: C\to E$ be an α-inverse strongly accretive operator such that $S(C,A)\neq\emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0,1)$ and $\{\lambda_n\}$ a real number sequence in $[a,\frac{\alpha}{K^2}]$. Suppose the following conditions are satisfied:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n \le 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)[\beta_n u + (1-\beta_n)Q(x_n - \lambda_n A x_n)], n \ge 0. \tag{13}$$

Then the sequence $\{x_n\}$ defined by (13) converges strongly to $Q'u$, where Q' is a sunny nonexpansive retraction of C onto $S(C, A)$.

In particular, if we take $u = 0$, then the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)[(1-\beta_n)Q(x_n - \lambda_n A x_n)], n \ge 0,$$

converges strongly to the minimum norm element in $S(C, A)$.

4. Applications

In this section, we consider some applications of our main result. To be more precise, let C be a nonempty closed convex subset of a Hilbert space H. One method of finding a point $x^* \in VI(C, A)$ is to use the projection algorithm $\{x_n\}$ defined by

$$x_{n+1} = P_C(x_n - \lambda A x_n)$$

for all $n \ge 0$, where P_C is the metric projection from H onto C, A is a monotone (accretive) operator of C into H and λ is a positive real number. It is well known that if A is an α-strongly accretive and L-Lipschitz continuous operator of C into H and $\lambda \in (0, \frac{2\alpha}{L^2})$, then the operator $P_C(I - \lambda A)$ is a contraction of C into itself.

We now prove a strong convergence theorem for a strongly accretive operator.

Theorem 2. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping. Let Q be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A an α-strongly accretive and L-Lipschitz continuous operator of C into E with $S(C, A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0, 1)$ and $\{\lambda_n\}$ a real number sequence in $[a, \frac{\alpha}{K^2L^2}]$ for some $a > 0$. Suppose the following conditions are satisfied:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n \le 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ defined by (5) converges strongly to $Q'u$, where Q' is a sunny nonexpansive retraction of E onto $S(C, A)$.

In particular, if we take $u = 0$, then the sequence $\{x_n\}$ generated by (6) converges strongly to the minimum norm element in $S(C, A)$.

Proof. Since A is an α-strongly accretive and L-Lipschitz continuous operator of C into E, we have

$$\langle Ax - Ay, J(x-y)\rangle \geq \alpha\|x-y\|^2 \geq \frac{\alpha}{L^2}\|Ax - Ay\|^2$$

for all $x, y \in C$. Therefore, A is $\frac{\alpha}{L^2}$-inverse strongly accretive. Using Theorem 1, we can obtain that $\{x_n\}$ converges strongly to $Q'u$. This completes the proof. □

Corollary 2. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping. Let Q be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A an α-strongly accretive and L-Lipschitz continuous operator of C into E with $S(C,A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0,1)$ and $\{\lambda_n\}$ a real number sequence in $[a, \frac{\alpha}{K^2L^2}]$ for some $a > 0$. Suppose the following conditions are satisfied:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n \leq 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)[\beta_n u + (1-\beta_n)Q(x_n - \lambda_n A x_n)], n \geq 0. \quad (14)$$

Then the sequence $\{x_n\}$ defined by (14) converges strongly to $Q'u$, where Q' is a sunny nonexpansive retraction of C onto $S(C,A)$.

Remark 2. *We would like to emphasize that our iterative methods suggested and anlyzed in this paper are different from those in [2],[11] and [3]. Especially, we obtain the minimum norm solution of the variational inequality (3).*

Acknowledgement

The authors are extremely grateful to the referee for his/her useful comments and suggestions which helped to improve this paper. The first author was supported in part by Colleges and Universities Science and Technology Development Foundation (20091003) of Tianjin, China. The second author was supported by the Visiting Professor Program of King Saud University, Riyadh, Saudi Arabia under Grant No. KSU.VPP.108.

References

[1] Q. H. Ansari, L. C. Ceng, J. C. Yao, *Mann Type Steepest-Descent and Modified Hybrid Steepest-Descent Methods for Variational Inequalities in Banach Spaces*, Numerical Funct. Anal. Optim. **29**(2008), 987–1033.

[2] K. Aoyama, H. Iiduka, W. Takahashi, *Weak convergence of an iterative sequence for accretive operators in Banach spaces*, Fixed Point Theory Appl. **2006**(2006), Art. ID 35390.

[3] K. Aoyama, H. Iiduka, W. Takahashi, *Strong convergence of Halpern's sequence for accretive operators in a Banach space*, Panamer. Math. J. **17**(2007), 75–89.

[4] K. Ball, E. A. Carlen, E. H. Lieb, *Sharp uniform convexity and smoothness inequalities for trace norms*, Invent. Math. **115**(1994), 463–482.

[5] B. Beauzamy, *Introduction to Banach spaces and their geometry*, 2nd ed., North-Holland Mathematics Studies, vol. 68, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 86.

[6] L. C. Ceng, Q. H. Ansari, J. C. Yao, *On relaxed viscosity iterative methods for variational inequalities in Banach spaces*, J. Comput. Appl. Math. **230**(2009), 813–822.

[7] L. C. Ceng, J. C. Yao, *An extragradient-Like approximation method for variational inequality problems and fixed point problems*, Appl. Math. Comput. **190**(2007), 205–215.

[8] E. G. Golśhtein, N. V. Tretýakov, *Modified Lagrangians in convex programming and their generalizations*, Math. Programming Study **10**(1979), 86–97.

[9] H. Iiduka, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal. **61**(2005) 341–350.

[10] H. Iiduka, W. Takahashi, M. Toyoda, *Approximation of solutions of variational inequalities for monotone mappings*, PanAmer. Math. J. **14**(2004), 49–61.

[11] S. Kamimura, W. Takahashi, *Weak and strong convergence of solutions to accretive operator inclusions and applications*, Set-Valued Anal. **8**(2000), 361–374.

[12] L. J. Lin, M. F. Yang, Q. H. Ansari, G. Kassay, *Existence results for Stampacchia and Minty type implicit variational inequalities with multivalued maps*, Nonlinear Anal. **61**(2005), 1–19.

[13] F. Liu, M. Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal. **6**(1998), 313–344.

[14] M. A. Noor, *Some development in general variational inequalities*, Appl. Math. Comput. **152**(2004), 199–277.

[15] M. A. Noor, *Princiles of Variational Inequalities*, Lap-Lambertr Academic Publishing AG & Co. Saarbrucken, Germany, 2009.

[16] A. Sabharwal, L. C. Potter, *Convexly constrained linear inverse problems: iterative least-squares and regularization*, IEEE Trans. Signal Process. **46**(1998), 2345–2352.

[17] T. Suzuki, *A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. **135**(2007), 99–106.

[18] Y. Takahashi, K. Hashimoto, M. Kato, *On sharp uniform convexity, smoothness, and strong type, co-type inequalities*, J. Nonlinear and Convex Anal. **3**(2002), 267–281.

[19] W. Takahashi, M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118**(2003), 417–428.

[20] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298**(2004), 279–291.

[21] Y. Yao, J. C. Yao, *On modified iterative method for nonexpansive mappings and monotone mappings*, Appl. Math. Comput. **186**(2007) 1551–1558.

[22] L. C. Zeng, Q. H. Ansari, S. Y. Wu, *Strong convergence theorems of relaxed hybrid steepest-descent methods for variational inequalities*, Taiwanese J. Math. **10**(2006), 13–29.