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# Improving Stirling's formula

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**Abstract.** We calculated the optimal values of the real parameters $a$ and $b$ in such a way that the asymptotic formula

$$n! \sim e^{-a}\left(\frac{n+a}{e}\right)^n \sqrt{2\pi(n+b)} \ \ (as \ \ n \to \infty)$$

gives the best accurate values for $n!$. Our estimations improve the classical Stirling and Burnside's formulas and their several recent improvements due to the author and C. Mortici. Apart from their simplicities and beauties our formulas give very accurate values for factorial $n$. Also, our results lead to new upper and lower bounds for the gamma function and recover some published inequalities for the gamma function.

**AMS subject classifications**: 33B15, 40A25

**Key words**: Stirling formula, Burnside's formula, gamma function, digamma function, inequalities

## 1. Introduction

The gamma function defined by the improper integral

$$\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}du \ (z > 0)$$

has numerous applications in mathematics and sciences and it occurs e.g. in the expression of various mathematical constants. For instance, the formula for the volume of the unit ball in $\mathbb{R}^n$ involves the gamma function, see the recent papers [2, 18, 7] and its monotonicity properties have been studies in [7, 9]. It is also notable that the author used the gamma function to evaluate the sum of some series involving reciprocals of binomial coefficients, see [5, 6]. The gamma function and the factorials are related with $\Gamma(n+1) = n!$, $n \in \mathbb{N}$. The logarithmic derivative of the gamma function is called the digamma (or psi) function and denoted by $\psi$. The derivatives $\psi'$, $\psi''$, $\psi'''$... are called polygamma functions in the literature. The Stirling's formula

$$n! \approx n^n e^{-n}\sqrt{2\pi n} = \alpha_n \tag{1}$$

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is used to estimate factorial $n$ and has many applications in statistical physics, probability theory and number theory. Actually it was discovered by A. De Moivre (1667-1754) in the form

$$n! \approx C \cdot n^n e^{-n} \sqrt{n},$$

and Stirling (1692-1770) identified the constant $C$ precisely $\sqrt{2\pi}$. After the Stirling's formula has been published, many formulas to approximate $n!$ have appeared in the literature but most of them are complicated and do not have a simple form. E. A. Karatsuba [11] answered a question posed by S. Ponnusamy and M. Vuorinen [15, Conjecture 2.35, p.293] motivated by a notebook of S. Ramanujan (1887-1920). She proved in [11] the following asymptotic expansion (for additional terms see [11, Eq. (5.5)]

$$\Gamma(x+1) \approx \sqrt{\pi} x^x e^{-x} \sqrt[6]{8x^3 + 4x^2 + x + \frac{1}{30} - \frac{11}{240x} + \cdots} = \phi_x.$$

The accuracy of this and some other asymptotic expansions for the gamma function have been recently investigated by G. Nemes [14]. Undoubtedly in the literature there exist much better approximation formulas to approximate the gamma function or $n!$ like this than (1), but in this work we are interested only in formulas having a simple form. The most well known simple estimation for $n!$ after the Stirling's formula is

$$n! \approx \sqrt{2\pi} \left( \frac{n+1/2}{e} \right)^{n+\frac{1}{2}} = \beta_n, \tag{2}$$

which was published by Burnside [8] in 1917. It is known from [8] that this formula is better than (1). Very recently the author [3] and C. Mortici [12] published the following simple estimations for $n!$:

$$n! \approx \sqrt{\frac{2\pi}{e}} \left( \frac{n+1}{e} \right)^{n+\frac{1}{2}} = \gamma_n, \quad \text{(C. Mortici [12])}, \tag{3}$$

$$n! \approx \sqrt{2\pi e} \cdot e^{-\omega} \left( \frac{n+\omega}{e} \right)^{n+\frac{1}{2}} = \delta_n, \quad \text{(C. Mortici [12])}, \tag{4}$$

where $\omega = (3-\sqrt{3})/6$,

$$n! \approx \sqrt{2\pi} \cdot n^n e^{-n} \sqrt{n + \frac{1}{6}} = \theta_n, \quad \text{(N. Batir [3])}. \tag{5}$$

The aim of this work is to improve all these known estimations keeping their simplicities. If we carefully look at these formulas, we see that all are special cases of the following asymptotic formula: For real numbers $a, b \geq 0$

$$n! \sim e^{-a} \left( \frac{n+a}{e} \right)^n \sqrt{2\pi(n+b)}, \quad (as \quad n \to \infty). \tag{6}$$

Indeed, (1) is obtained from (6) for $(a,b) = (0,0)$; (2) is obtained for $(a,b) = (1/2, 1/2)$; (3) is obtained for $(a,b) = (1,1)$, (4) is obtained for $(a,b) = (\omega, \omega)$ $(\omega = (3-\sqrt{3})/6)$, and finally (5) is obtained from (6) for $(a,b) = (0, 1/6)$. Hence it is natural to look for the best possible constant pairs $(a, b)$ such that formula (6) gives the best accurate values for factorial $n$. We prove that the best approximations are obtained for the constant pairs $(a, b) = (a_1, b_1)$ and $(a, b) = (a_2, b_2)$, where

$$a_1 = \frac{1}{3} + \frac{\lambda}{6} - \frac{1}{6}\sqrt{6 - \lambda^2 + 4/\lambda} = 0.540319070367..., \tag{7}$$

and

$$a_2 = \frac{1}{3} + \frac{\lambda}{6} + \frac{1}{6}\sqrt{6 - \lambda^2 + 4/\lambda} = 0.950108831378..., \tag{8}$$

which are the real roots of the quartic equation $3a^4 - 4a^3 + a^2 + \frac{1}{12} = 0$, and

$$b_1 = a_1^2 + \frac{1}{6} = 0.458611364468... \quad \text{and} \quad b_2 = a_2^2 + \frac{1}{6} = 1.069373458129..., \tag{9}$$

where $\lambda = \sqrt{2 + 2^{2/3} + 2^{4/3}}$. The following lemmas are key in our proofs. The first lemma was proved in [13].

**Lemma 1.** *If $(\omega_n)_{n \geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n \to \infty} n^k \cdot (\omega_n - \omega_{n+1}) = c \in \mathbb{R},$$

*with $k > 1$, then there exists the limit*

$$\lim_{n \to \infty} n^{k-1} \cdot \omega_n = \frac{c}{k-1}.$$

This lemma, despite its simple appearance, is a strong tool to accelerate and measure the speed of convergence of some sequences having limit equal to zero. It is evident from this lemma that the speed of convergence of the sequence $(\omega_n)$ is as higher as the value of $k$ is greater. The next lemma, as far as we know, was first used in [10] (without proof) to establish some monotonicity results for the gamma function and later used by the author [4] and F. Qi and B-N Guo [16, 17] to prove some monotonicity and complete monotonicity properties of the polygamma functions.

**Lemma 2.** *Let $f$ be a function defined on an interval $I$ and $\lim_{x \to \infty} f(x) = 0$. If $f(x+1) - f(x) > 0$ for all $x \in I$, then $f(x) < 0$ . If $f(x+1) - f(x) < 0$, then $f(x) > 0$.*

**Proof**. Let $f(x+1) - f(x) > 0$ for all $x \in I$. By mathematical induction we have $f(x) < f(x+n)$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get $f(x) < 0$. The proof of the second part of the lemma follows from the same argument. □

The numerical and algebraic computations have been carried out with the computer program MATHEMATICA 5.

## 2. Main results

For real numbers $a, b \geq 0$ and $n \in \mathbb{N}$, we define

$$\sigma_n(a,b) = \log(n!) - \frac{1}{2}\log(2\pi) + n + a - n\log(n+a) - \frac{1}{2}\log(n+b). \tag{10}$$

Using the Stirling's formula (1) we see that

$$\lim_{n\to\infty} \sigma_n = 0. \tag{11}$$

Now we are ready to state and prove our main results.

**Theorem 1.**

*(i) If $b \neq a^2 + \frac{1}{6}$, then the speed of convergence of $(\sigma_n)$ is $n^{-1}$.*

*(ii) If $a \neq a_1, a_2$ and $b = a^2 + \frac{1}{6}$, then the speed of convergence of $(\sigma_n)$ is $n^{-2}$.*

*(iii) If $(a,b) = (a_1, b_1)$ or $(a,b) = (a_2, b_2)$, then the speed of convergence of $(\sigma_n)$ is $n^{-3}$.*

Here $a_i, b_i (i = 1, 2)$ are as given in (7), (8) and (9).

**Proof**. For real numbers $x \geq 0$, we define

$$H_{a,b}(x) = \log\Gamma(x+1) - \frac{1}{2}\log(2\pi) + x + a - x\log(x+a) - \frac{1}{2}\log(x+b).$$

Differentiation gives

$$\begin{aligned} H'_{a,b}(x) &= \psi(x+1) - \log(x+a) + \frac{a}{x+a} - \frac{1}{2(x+b)}, \\ H''_{a,b}(x) &= \psi'(x+1) - \frac{1}{x+a} - \frac{a}{(x+a)^2} + \frac{1}{2(x+b)^2}, \end{aligned}$$

and

$$H''_{a,b}(x+1) - H''_{a,b}(x) = \frac{P_{a,b}(x)}{Q_{a,b}(x)}, \tag{12}$$

where we have used the functional relation $\psi'(x+1)-\psi'(x)=-1/x^2$, and

$$\begin{aligned}
P_{a,b}(x) = {} & (-2-12a^2+12b)\mathbf{x}^6+(-8-8a-48a^2-16a^3+40b \\
& +48ab-48a^2b+24b^2)\mathbf{x}^5+(-12-28a-88a^2-56a^3-4a^4 \\
& +48b+136ab-96a^2b-64a^3b+72b^2+96ab^2-72a^2b^2 \\
& +16b^3)\mathbf{x}^4+(-8-36a-96a^2-80a^3-12a^4+24b+136ab \\
& -64a^2b-144a^3b-16a^4b+76b^2+240ab^2-72a^2b^2-96a^3b^2 \\
& +40b^3+64ab^3-48a^2b^3+4b^4)\mathbf{x}^3+(-2-20a-62a^2-60a^3 \\
& -14a^4+4b+56ab-28a^2b-120a^3b-28a^4b+32b^2+208ab^2 \\
& +56ab+36a^2b^2-144a^3b^2-24a^4b^2+32b^3+128ab^3-24a^2b^3 \\
& -64a^3b^3+8b^4+16ab^4+12a^2b^4)\mathbf{x}^2+(-4a-20a^2-24a^3-8a^4 \\
& +8ab-16a^2b-48a^3b-16a^4b+4b^2+72ab^2+48a^2b^2-64a^3b^2 \\
& -24a^4b^2+8b^3+24ab^4-16a^3b^4)\mathbf{x}-2a^2-4a^3-2a^4-4a^2b \\
& -8a^3b-4a^4b+8ab^2+8a^2b^2-8a^3b^2-4a^4b^2+16ab^3 \\
& +16a^2b^3-16a^3b^3-8a^4b^3+8ab^4+8a^2b^4-8a^3b^4-4a^4b^4,
\end{aligned} \tag{13}$$

and $Q(x)=2(x+1)^2(x+a)^2(x+b)^2$. Utilizing Stirling's formula and the the following asymptotic formulas

$$\begin{aligned}
\psi(x) &\sim \log x-\frac{1}{2x}-\frac{1}{12x^2}+\cdots. \ (as\ x\to\infty), \\
\psi'(x) &\sim \frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}+\cdots \ (as\ x\to\infty),
\end{aligned}$$

(see [1, p.259]), we obtain

$$\lim_{x\to\infty} H_{a,b}(x)=\lim_{x\to\infty} H'_{a,b}(x)=\lim_{x\to\infty} H''_{a,b}(x)=0. \tag{14}$$

(i) If $b\neq a^2+\frac{1}{6}$, using (12), (14) and l'Hospital rule, we obtain

$$\begin{aligned}
\lim_{n\to\infty} n^2\cdot(\sigma_n-\sigma_{n+1}) &= \lim_{n\to\infty}\frac{H_{a,b}(n)-H_{a,b}(n+1)}{1/n^2} \\
&= \lim_{n\to\infty}\frac{H'_{a,b}(n)-H'_{a,b}(n+1)}{-2/n^3} \\
&= -\frac{1}{6}\lim_{n\to\infty} n^4[H''_{a,b}(n+1)-H''_{a,b}(n)] \\
&= -\frac{1}{6}\lim_{n\to\infty}\frac{n^4P_{a,b}(n)}{Q_{a,b}(n)} \\
&= a^2+\frac{1}{6}-b.
\end{aligned} \tag{15}$$

Since (11) holds, an application of Lemma 1 yields

$$\lim_{n\to\infty} n\cdot\sigma_n=a^2+\frac{1}{6}-b\neq 0 \tag{16}$$

This proves (i). (ii) Let $b = a^2 + \frac{1}{6}$ and $a \neq a_1, a_2$. Then, the coefficient of $x^6$ in (13) vanishes, so using Stirling's formula we get

$$\begin{aligned}
\lim_{n\to\infty} n^3 \cdot (\sigma_n - \sigma_{n+1}) &= \lim_{n\to\infty} \frac{H_{a,b}(n) - H_{a,b}(n+1)}{1/n^3} \\
&= \lim_{n\to\infty} \frac{H'_{a,b}(n) - H'_{a,b}(n+1)}{-3/n^4} \\
&= -\frac{1}{12} \lim_{n\to\infty} n^5 [H''_{a,b}(n+1) - H''_{a,b}(n)] \\
&= -\frac{1}{12} \lim_{n\to\infty} \frac{n^5 P_{a,b}(n)}{Q_{a,b}(n)} \\
&= \frac{1}{3}(1 + a + 6a^2 + 2a^3 - 5b - 6ab + 6a^2 b - 3b^2). \qquad (17)
\end{aligned}$$

If we set $b = a^2 + \frac{1}{6}$ in the last line, we get

$$\lim_{n\to\infty} n^3 \cdot (\sigma_n - \sigma_{n+1}) = \frac{1}{3}\left(3a^4 - 4a^3 + a^2 + \frac{1}{12}\right).$$

By virtue of Lemma 1, we find that

$$\lim_{n\to\infty} n^2 \cdot \sigma_n = \frac{1}{6}\left(3a^4 - 4a^3 + a^2 - \frac{1}{12}\right) \neq 0, \qquad (18)$$

since $a \neq a_1, a_2$, this proves (ii). Now we shall prove (iii). Let $(a, b) = (a_1, b_1)$. In this case the coefficients of $x^6$ and $x^5$ in (13) vanish, thus by Stirling's formula we obtain

$$\begin{aligned}
\lim_{n\to\infty} n^4 \cdot (\sigma_n - \sigma_{n+1}) &= \lim_{n\to\infty} \frac{H_{a,b}(n) - H_{a_1,b_1}(n+1)}{1/n^4} \\
&= \lim_{n\to\infty} \frac{H'_{a_1,b_1}(n) - H'_{a_1,b_1}(n+1)}{-4/n^5} \\
&= -\frac{1}{20} \lim_{n\to\infty} n^5 [H''_{a_1,b_1}(n+1) - H''_{a_1,b_1}(n)] \\
&= -\frac{1}{20} \lim_{n\to\infty} \frac{n^6 P_{a,b}(n)}{Q_{a,b}(n)} \\
&= -0.014723740642427497..., \qquad (19)
\end{aligned}$$

which implies by Lemma 1

$$\lim_{n\to\infty} n^3 \cdot \sigma_n(a_1, b_1) = -0.004907913547475832. \qquad (20)$$

In the same way we find that

$$\lim_{n\to\infty} n^3 \cdot \sigma_n(a_2, b_2) = 0.005746538709721112. \qquad (21)$$

□

If we set $(a, b) = (a_1, b_1)$ and $(a, b) = (a_2, b_2)$ in (13), respectively, we obtain

$$\begin{aligned} P_{a_1, b_1}(x) = & -0.393886 - 1.93247x - 3.31813x^2 - 2.35329x^3 \\ & -0.58895x^4 < 0 \end{aligned} \tag{22}$$

and

$$\begin{aligned} P_{a_2, b_2}(x) = & \ 1.47415 + 5.06985x + 6.40706x^2 + 3.50099x^3 \\ & +0.689585x^4 > 0, \end{aligned} \tag{23}$$

and $H''_{a_1, b_1}(x+1) - H''_{a_1, b_1}(x) < 0$ and $H''_{a_2, b_2}(x+1) - H''_{a_2, b_2}(x) > 0$ by (12) for all $x \geq 0$. Utilizing (14) and Lemma 2 we see that $H''_{a_1, b_1}(x) > 0$ and $H''_{a_2, b_2}(x) < 0$ for all $x \geq 0$. From (14) it results that $H_{a_1, b_1}$ is strictly decreasing and $H_{a_2, b_2}$ is strictly increasing on $[0, \infty)$, thus we have

$$0 = \lim_{x \to \infty} H_{a_1, b_1}(x) < H_{a_1, b_1}(x) \leq H_{a_1, b_1}(0) = -\frac{1}{2} \log(2\pi) + a_1 - \frac{1}{2} \log b_1$$

and

$$-\frac{1}{2} \log(2\pi) + a_2 - \frac{1}{2} \log b_2 = H_{a_2, b_2}(0) \leq H_{a_2, b_2}(x) < \lim_{x \to \infty} H_{a_2, b_2}(x) = 0.$$

These can be written equivalently as follows:

**Theorem 2.** *For all reals $x \geq 0$ the following double inequalities hold:*

$$\alpha \cdot e^{-x-a_1}(x + a_1)^x \sqrt{x + b_1} < \Gamma(x + 1) \leq \beta \cdot e^{-x-a_1}(x + a_1)^x \sqrt{x + b_1} \tag{24}$$

*where the constants*

$$\alpha = \sqrt{2\pi} = 2.5066282746\ldots \quad \textit{and} \quad \beta = \frac{e^{a_1}}{\sqrt{b_1}} = 2.5347503081133245\ldots$$

*are the best possible constants; and*

$$\alpha^* \cdot e^{-x-a_2}(x + a_2)^x \sqrt{x + b_2} < \Gamma(x + 1) \leq \beta^* \cdot (x + a_2)^x e^{-x-a_2} \sqrt{x + b_2} \tag{25}$$

*where the constants*

$$\alpha^* = \frac{e^{a_2}}{\sqrt{b_2}} = 2.5007041931433194\ldots \quad \textit{and} \quad \beta^* = \sqrt{2\pi} = 2.506628274\ldots$$

*are the best possible constants, and $a_i, b_i$(i=1,2) are as given in (7), (8) and (9).*

**Remark 1.** *If we set $(a, b) = (\omega, \omega)(\omega = \frac{3-\sqrt{3}}{6})$ and $(a, b) = (\varsigma, \varsigma)(\varsigma = \frac{3+\sqrt{3}}{6})$ in (13), respectively, we get*

$$\begin{aligned} P_{\omega, \omega}(x) = & -0.05772 - 0.69205x - 2.79458x^2 - 4.45859x^3 - 3.07122x^4 \\ & -0.7698x^5 < 0 \end{aligned}$$

*and*

$$P_{\varsigma,\varsigma}(x) = 1.31081 + 6.32168x + 11.8316x^2 + 10.6808x^3 \\ +4.62678x^4 + 0.7698x^5 > 0,$$

*that is, $H_{\omega,\omega}(x+1) - H_{\omega,\omega}(x) < 0$ and $H_{\varsigma,\varsigma}(x+1) - H_{\varsigma,\varsigma}(x) > 0$ by (12) for all $x \geq 0$. Hence by Lemma 2 and (14), $H_{\omega,\omega}$ is strictly decreasing and $H_{\varsigma,\varsigma}$ is strictly increasing on $[0, \infty)$. It results that*

$$\lim_{x\to\infty} H_{\omega,\omega}(x) = 0 \leq H_{\omega,\omega}(x) < -\frac{1}{2}\log(2\omega\pi) + \omega = H_{\omega,\omega}(0),$$

*and*

$$H_{\varsigma,\varsigma}(0) = -\frac{1}{2}\log(2\varsigma\pi) + \varsigma < H_{\varsigma,\varsigma}(x) \leq 0 = \lim_{x\to\infty} H_{\varsigma,\varsigma}(x).$$

*These lead to the following double inequalities:*

$$\sqrt{2\pi e}\cdot e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}} < \Gamma(x+1) < \alpha\cdot\sqrt{2\pi e}\cdot e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}},$$

*where $\alpha = \frac{e^{\omega}}{\sqrt{2\pi\omega}}$, and*

$$\beta\cdot\sqrt{2\pi e}\cdot e^{-\varsigma}\left(\frac{x+\varsigma}{e}\right)^{x+\frac{1}{2}} < \Gamma(x+1) < \sqrt{2\pi e}\cdot e^{-\varsigma}\left(\frac{x+\varsigma}{e}\right)^{x+\frac{1}{2}},$$

*where $\beta = \frac{e^{\varsigma}}{\sqrt{2\pi\varsigma}}$. Precisely, these are the main results obtained in [12].*

**Remark 2.** *Equations (15) and (18) enable us to measure the speed of convergence of the approximations $n! \approx \alpha_n$, $n! \approx \beta_n$, $n! \approx \gamma_n$, $n! \approx \delta_n$ and $n! \approx \theta_n$ from (1)-(5): we obtain from (16)*

$$\begin{aligned}
\lim_{n\to\infty} n\cdot\log(n!/\alpha_n) &= \lim_{n\to\infty} n\cdot\sigma_n(0,0) = \frac{1}{6},\\
\lim_{n\to\infty} n\cdot\log(n!/\beta_n) &= \lim_{n\to\infty} n\cdot\sigma_n(1/2,1/2) = \frac{1}{12},\\
\lim_{n\to\infty} n\cdot\log(n!/\gamma_n) &= \lim_{n\to\infty} n\cdot\sigma_n(1,1) = \frac{1}{6},
\end{aligned}$$

*and from (19) for $a = 0$*

$$\lim_{n\to\infty} n^2\cdot\log(n!/\theta_n) = \lim_{n\to\infty} n^2\cdot\sigma_n(0,1/6) = \frac{1}{72} = 0.0138888\ldots,$$

*and finally*

$$\lim_{n\to\infty} n^2\cdot\log(n!/\delta_n) = \lim_{n\to\infty} n^2\cdot\sigma_n(\omega,\omega) = 0.01603750\ldots.$$

*We let*

$$\rho_n = e^{-a_1}\left(\frac{n+a_1}{e}\right)^n\sqrt{2\pi(n+b_1)}$$

*and*

$$\tau_n = e^{-a_2}\left(\frac{n+a_2}{e}\right)^n \sqrt{2\pi(n+b_2)}.$$

*Then from (20) and (21) we obtain*

$$\lim_{n\to\infty} n^3 \cdot \log(n!/\rho_n) = -0.0049079\ldots \quad and \quad \lim_{n\to\infty} n^3 \cdot \log(n!/\tau_n) = 0.005746538\ldots.$$

*These prove theoretically the great superiority of our estimations* $n! \approx \rho_n$ *and* $n! \approx \tau_n$ *over* $n! \approx \theta_n$*, which is the best of (1)-(5). We conclude our paper with a numerical comparison between our estimations* $n! \approx \tau_n$*,* $n! \approx \rho_n$ *and* $n! \approx \theta_n$*,* $n! \approx \phi_n$*, where* $\phi_n$ *is Karatsuba's asymptotic formula for* $\Gamma(n+1)$ *given on the second page.*

| n | $\lvert\theta_n - n!\rvert$ | $\lvert\tau_n - n!\rvert$ | $\lvert\rho_n - n!\rvert$ | $\lvert\phi_n - n!\rvert$ |
|---|---|---|---|---|
| 1 | 0.00397819 | **0.0003378** | **0.0006483** | 0.0003037 |
| 2 | 0.0026369 | **0.00020675** | **0.00029994** | 0.000026 |
| 10 | 239.175 | **7.81549** | **7.67069** | 0.017993 |
| 25 | $1.688\times10^{20}$ | $\mathbf{2.534\times10^{18}}$ | $\mathbf{2.294\times10^{18}}$ | $7.841\times10^{14}$ |
| 50 | $8.36204\times10^{58}$ | $\mathbf{6.587\times10^{56}}$ | $\mathbf{5.794\times10^{56}}$ | $4.789\times10^{52}$ |
| 100 | $4.33\times10^{151}$ | $\mathbf{2.602\times10^{149}}$ | $\mathbf{2.256\times10^{149}}$ | $4.585\times10^{144}$ |

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