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Univalence criteria for general integral operator

Basem Aref Frasin[1,*]

[1] *Department of Mathematics, Al al–Bayt University, P. O. Box 130 095, Mafraq, Jordan*

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Abstract. Let $\mathcal{A}$ be the class of all analytic functions which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and

$$G_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, \quad z \in \mathcal{U} \right\}.$$

In this paper, we derive sufficient conditions for the integral operator

$$I_\gamma^{\alpha_i}\ (f_1, ..., f_n)(z) = \left\{ \gamma \int_0^z t^{\gamma-1} \left(f_1'(t)\right)^{\alpha_1} \left(\frac{f_1(t)}{t}\right)^{1-\alpha_1} ... \left(f_n'(t)\right)^{\alpha_n} \left(\frac{f_n(t)}{t}\right)^{1-\alpha_n} dt \right\}^{\frac{1}{\gamma}}$$

to be analytic and univalent in the open unit disc $\mathcal{U}$, when $f_i \in G_{b_i}$ for all $i = 1, \ldots, n$.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

In [16], Silverman investigated an expression involving the quotient of the analytic representation of convex and starlike functions. Precisely, for $0 < b \leq 1$ he considered the class

$$G_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, \quad z \in \mathcal{U} \right\}. \tag{2}$$

Let $\alpha_i \in \mathbb{C}$ for all $i = 1, \ldots, n$, $n \in \mathbb{N}$, $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$. We let $I_\gamma^{\alpha_i} : \mathcal{A}^n \to \mathcal{A}$ be the integral operator defined by

*Corresponding author. *Email address:* `bafrasin@yahoo.com` (B. A. Frasin)

$$I_\gamma^{\alpha_i}\ (f_1,...,f_n)(z) = \left\{ \gamma \int\limits_0^z t^{\gamma-1} \left(f_1'(t)\right)^{\alpha_1} \left(\frac{f_1(t)}{t}\right)^{1-\alpha_1} ... \left(f_n'(t)\right)^{\alpha_n} \left(\frac{f_n(t)}{t}\right)^{1-\alpha_n} dt \right\}^{\frac{1}{\gamma}} \quad (3)$$

Here and throughout the sequel every many-valued function is taken with the principal branch. The integral operator $I_\gamma^{\alpha_i}\ (f_1,...,f_n)(z)$ was introduced and studied by Frasin [7] and this integral operator is a generalization of the integral operator

$$H(z) = \left\{ \gamma \int\limits_0^z t^{\gamma-1} \left(\frac{f(t)}{t}\right)^{\beta} \left(f'(t)\right)^{\delta} dt \right\}^{\frac{1}{\gamma}}$$

introduced by Ovesea in [10].

Many authors studied the problem of integral operators which preserve the class $\mathcal{S}$ (see, for example, [1, 2, 3, 4, 5, 6, 9, 14, 15]).

In the present paper, we derive a sufficient condition for the integral operator $I_\gamma^{\alpha_i}\ (f_1,...,f_n)(z)$ to be analytic and univalent in $\mathcal{U}$, when $f_i \in G_{b_i}$ for all $i = 1, \ldots, n$.

In order to derive our main results, we have to recall here the following univalence criteria.

Lemma 1 (see [11]). *Let $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2Re(\gamma)}}{Re(\gamma)} \left| \frac{zf''(z)}{f'(z)} \right| \le 1,$$

for all $z \in \mathcal{U}$, then the integral operator

$$F_\gamma(z) = \left\{ \gamma \int\limits_0^z t^{\gamma-1} f'(t) dt \right\}^{\frac{1}{\gamma}}$$

is in the class $\mathcal{S}$.

Lemma 2 (see [12]). *Let $\delta \in \mathbb{C}$ with $Re(\delta) > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2Re(\delta)}}{Re(\delta)} \left| \frac{zf''(z)}{f'(z)} \right| \le 1,$$

for all $z \in \mathcal{U}$, then, for any complex number γ with $Re(\gamma) \ge Re(\delta)$, the integral operator

$$F_\gamma(z) = \left\{ \gamma \int\limits_0^z t^{\gamma-1} f'(t) dt \right\}^{\frac{1}{\gamma}}$$

is in the class $\mathcal{S}$.

Lemma 3 (see [13]). *Let $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$, $c \in \mathbb{C}$ with $|c| \leq 1$, $c \neq -1$. If $f \in \mathcal{A}$ satisfies*

$$\left| c\,|z|^{2\gamma} + (1 - |z|^{2\gamma}) \frac{zf''(z)}{\gamma f'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$, then the integral operator

$$F_\gamma(z) = \left\{ \gamma \int_0^z t^{\gamma-1} f'(t)dt \right\}^{\frac{1}{\gamma}}$$

is in the class $\mathcal{S}$.

Further, we need the following general Schwarz Lemma.

Lemma 4 (see [8]). *Let the function f be regular in the disc $\mathcal{U}_R = \{z : |z| < R\}$, with $|f(z)| < M$ for fixed M. If $f(z)$ has one zero with multiplicity order greater than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \qquad (z \in \mathcal{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} (M/R^m) z^m,$$

where θ is constant.

2. Univalence conditions for $I_\gamma^{\alpha_i}(f_1, ..., f_n)(z)$

We first prove

Theorem 1. *Let $\gamma \in \mathbb{C}$ and $\alpha_i \in \mathbb{C}$ for all $i = 1, \ldots, n$ with*

$$Re(\gamma) \geq \sum_{i=1}^{n} (2\,|\alpha_i|\, b_i + 1). \tag{4}$$

If for all $i = 1, \ldots, n$, $f_i \in \mathcal{G}_{b_i}$; $0 < b_i \leq 1$ and

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| < 1 \qquad (z \in \mathcal{U}), \tag{5}$$

then the integral operator $I_\gamma^{\alpha_i}(f_1, ..., f_n)(z)$ defined by (3) is analytic and univalent in $\mathcal{U}$.

Proof. Define

$$h(z) = \int_0^z \prod_{i=1}^{n} \left[(f_i'(t))^{\alpha_i} \left(\frac{f_i(t)}{t} \right)^{1-\alpha_i} \right] dt,$$

so that, obviously

$$h'(z) = \prod_{i=1}^{n} \left[(f_i'(z))^{\alpha_i} \left(\frac{f_i(z)}{z} \right)^{1-\alpha_i} \right], \tag{6}$$

Differentiating both sides of (6) logarithmically, we obtain

$$\frac{zh''(z)}{h'(z)}=\sum_{i=1}^{n}\left[\alpha_i\left(1+\frac{zf_i''(z)}{f_i'(z)}-\frac{zf_i'(z)}{f_i(z)}\right)+\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right] \tag{7}$$

Since $f_i\in\mathcal{G}_{b_i};\,0<b_i\leq 1$ for all $i=1,\ldots,n$, from (2) and (5), we obtain

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n}\left[|\alpha_i|\left|1+\frac{zf_i''(z)}{f_i'(z)}-\frac{zf_i'(z)}{f_i(z)}\right|+\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\right]\\ &\leq \sum_{i=1}^{n}\left[|\alpha_i|\,b_i\left|\frac{zf_i'(z)}{f_i(z)}\right|+\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\right]\\ &\leq \sum_{i=1}^{n}\left[|\alpha_i|\,b_i\left[\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+1\right]+\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\right]\\ &= \sum_{i=1}^{n}\left[|\alpha_i|\,b_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+|\alpha_i|\,b_i+\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\right]\\ &= \sum_{i=1}^{n}\left[\left((|\alpha_i|\,b_i+1)\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\right)+|\alpha_i|\,b_i\right]\\ &\leq \sum_{i=1}^{n}(2\,|\alpha_i|\,b_i+1), \end{aligned} \tag{8}$$

which readily shows that

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\gamma)}}{\mathrm{Re}(\gamma)}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \frac{1-|z|^{2\mathrm{Re}(\gamma)}}{\mathrm{Re}(\gamma)}\left(\sum_{i=1}^{n}(2\,|\alpha_i|\,b_i+1)\right)\\ &\leq \frac{1}{\mathrm{Re}(\gamma)}\left(\sum_{i=1}^{n}(2\,|\alpha_i|\,b_i+1)\right)\\ &\leq 1.\end{aligned}$$

Applying Lemma 1 for the function $h(z)$, we prove that $I_{\gamma}^{\alpha_i}\,(f_1,...,f_n)(z)\in\mathcal{S}$. □

Let $\alpha_i=1$ for all $i=1,\ldots,n$ in Theorem 1,we have

Corollary 1. *Let $\gamma\in\mathbb{C}$ with*

$$Re(\gamma)\geq\sum_{i=1}^{n}(2b_i+1). \tag{9}$$

If for all $i=1,\ldots,n, f_i\in\mathcal{G}_{b_i};\,0<b_i\leq 1$ and

$$\left|\frac{zf_i'(z)}{f_i(z)}-1\right|<1 \qquad (z\in\mathcal{U}), \tag{10}$$

then the integral operator

$$I_\gamma(f_1, ..., f_n)(z) = \left\{ \gamma \int_0^z t^{\gamma-1} \prod_{i=1}^n (f_i'(t))\, dt \right\}^{\frac{1}{\gamma}} \tag{11}$$

is analytic and univalent in $\mathcal{U}$.

Let $n = 1$, $\alpha_1 = \alpha$, $b_1 = b$ and $f_1 = f$ in Theorem 1, we have

Corollary 2. *Let* $\gamma \in \mathbb{C}$ *and* $\alpha \in \mathbb{C}$ *with*

$$Re(\gamma) \geq 2\,|\alpha|\, b + 1. \tag{12}$$

If $f \in \mathcal{G}_b;\ 0 < b \leq 1$ *and*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \qquad (z \in \mathcal{U}), \tag{13}$$

then the integral operator defined by

$$I_\gamma^\alpha\ (f)(z) = \left\{ \gamma \int_0^z t^{\gamma+\alpha-2} \left(\frac{f'(t)}{f(t)} \right)^\alpha f(t) dt \right\}^{\frac{1}{\gamma}} \tag{14}$$

is analytic and univalent in $\mathcal{U}$.

Making use of Lemma 2 and Schwarz Lemma, we prove

Theorem 2. *Let* $\alpha_i \in \mathbb{C}$, $M_i \geq 1$ *for all* $i = 1, \ldots, n$ *and* $\delta \in \mathbb{C}$ *with*

$$Re(\delta) \geq \sum_{i=1}^n [(|\alpha_i|\, b_i + 1)\,(2M_i + 1) + |\alpha_i|\, b_i]. \tag{15}$$

If for all $i = 1, \ldots, n$, $f_i \in \mathcal{G}_{b_i};\ 0 < b_i \leq 1$ *satisfy*

$$\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| < 1 \qquad (z \in \mathcal{U}) \tag{16}$$

and

$$|f_i(z)| \leq M_i\ \ (z \in \mathcal{U}, i = 1, \ldots, n),$$

then for any complex number γ *with* $Re(\gamma) \geq Re(\delta)$, *the integral operator* $I_\gamma^{\alpha_i}\ (f_1, \ldots, f_n)(z)$ *defined by (3) is analytic and univalent in* $\mathcal{U}$.

Proof. From the proof of Theorem 1, we have

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^n \left[\left((|\alpha_i|\, b_i + 1) \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \right) + |\alpha_i|\, b_i \right]. \tag{17}$$

Thus, we obtain

$$\begin{aligned}
&\frac{1-|z|^{2\mathrm{Re}(\delta)}}{\mathrm{Re}(\delta)}\left|\frac{zh''(z)}{h'(z)}\right| \\
&\leq \frac{1-|z|^{2\mathrm{Re}(\delta)}}{\mathrm{Re}(\delta)}\sum_{i=1}^{n}\left[\left((|\alpha_i|\,b_i+1)\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\right)+|\alpha_i|\,b_i\right] \\
&\leq \frac{1-|z|^{2\mathrm{Re}(\delta)}}{\mathrm{Re}(\delta)}\sum_{i=1}^{n}\left[\left((|\alpha_i|\,b_i+1)\left|\frac{z^2f_i'(z)}{[f_i(z)]^2}\right|\left|\frac{f_i(z)}{z}\right|+1\right)+|\alpha_i|\,b_i\right].
\end{aligned}$$

Since $|f_i(z)| \leq M_i \ (z \in \mathcal{U},\ i = 1, \ldots, n)$, and each f_i satisfies condition (16) for all $i = 1, \ldots, n$, then we have

$$\begin{aligned}
&\frac{1-|z|^{2\mathrm{Re}(\delta)}}{\mathrm{Re}(\delta)}\left|\frac{zh''(z)}{h'(z)}\right| \\
&\leq \frac{1-|z|^{2\mathrm{Re}(\delta)}}{\mathrm{Re}(\delta)}\sum_{i=1}^{n}\left[(|\alpha_i|\,b_i+1)\left(\left|\frac{z^2f_i'(z)}{[f_i(z)]^2}-1\right|M_i+M_i+1\right)+|\alpha_i|\,b_i\right] \\
&\leq \frac{1}{\mathrm{Re}(\delta)}\sum_{i=1}^{n}[(|\alpha_i|\,b_i+1)\,(2M_i+1)+|\alpha_i|\,b_i] \quad (z \in \mathcal{U}),
\end{aligned}$$

which, in the light of hypothesis (15), yields

$$\frac{1-|z|^{2\mathrm{Re}(\delta)}}{\mathrm{Re}(\delta)}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1 \quad (z \in \mathcal{U}).$$

Applying Lemma 2 for the function $h(z)$, we prove that $I_\gamma^{\alpha_i}\ (f_1, ..., f_n)(z) \in \mathcal{S}$. □

Let $\alpha_i = 1$ for all $i = 1, \ldots, n$ in Theorem 2, we have

Corollary 3. *Let $M_i \geq 1$ for all $i = 1, \ldots, n$ and $\delta \in \mathbb{C}$ with*

$$Re(\delta) \geq \sum_{i=1}^{n}[(b_i+1)\,(2M_i+1)+b_i]. \tag{18}$$

If for all $i = 1, \ldots, n$, $f_i \in \mathcal{G}_{b_i}$; $0 < b_i \leq 1$ satisfy

$$\left|\frac{z^2f_i'(z)}{[f_i(z)]^2}-1\right| < 1 \qquad (z \in \mathcal{U}) \tag{19}$$

and

$$|f_i(z)| \leq M_i \ \ (z \in \mathcal{U}, i = 1, \ldots, n),$$

then for any complex number γ with $Re(\gamma) \geq Re(\delta)$, the integral operator $I_\gamma(f_1, \ldots, f_n)(z)$ defined by (11) is analytic and univalent in $\mathcal{U}$.

Let $n = 1,\ \alpha_1 = \alpha,\ b_1 = b,\ M_1 = M$ and $f_1 = f$ in Theorem 2, we have

Corollary 4. *Let $\alpha \in \mathbb{C}$, $M \geq 1$ and $\delta \in \mathbb{C}$ with*

$$Re(\delta) \geq [(|\alpha|\, b + 1)\,(2M + 1) + |\alpha|\, b]. \tag{20}$$

If $f \in \mathcal{G}_b$; $0 < b \leq 1$ satisfies

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| < 1 \qquad (z \in \mathcal{U}) \tag{21}$$

and

$$|f(z)| \leq M \qquad (z \in \mathcal{U}),$$

then for any complex number γ with $Re(\gamma) \geq Re(\delta)$, the integral operator $I_\gamma^\alpha\,(f)(z)$ defined by (14) is analytic and univalent in $\mathcal{U}$.

Next, we prove

Theorem 3. *Let $\alpha_i \in \mathbb{C}$ for all $i = 1, \ldots, n$ and $\gamma \in \mathbb{C}$ with*

$$Re(\gamma) \geq \sum_{i=1}^{n} (2\,|\alpha_i|\, b_i + 1)$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re(\gamma)} \sum_{i=1}^{n} (2\,|\alpha_i|\, b_i + 1).$$

If for all $i = 1, \ldots, n$, $f_i \in \mathcal{G}_{b_i}$; $0 < b_i \leq 1$, and

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| < 1 \qquad (z \in \mathcal{U}), \tag{22}$$

then the integral operator $I_\gamma^{\alpha_i}\,(f_1, ..., f_n)(z)$ defined by (3) is analytic and univalent in $\mathcal{U}$.

Proof. From (8), we deduce that

$$\begin{aligned}
\left| c\,|z|^{2\gamma} + (1 - |z|^{2\gamma}) \frac{z h''(z)}{\gamma h'(z)} \right| &\leq |c| + \left| \frac{1 - |z|^{2\gamma}}{\gamma} \right| \left| \frac{z h''(z)}{h'(z)} \right| \\
&\leq |c| + \left| \frac{1 - |z|^{2\gamma}}{\gamma} \right| \sum_{i=1}^{n} (2\,|\alpha_i|\, b_i + 1) \\
&\leq |c| + \frac{1}{|\gamma|} \sum_{i=1}^{n} (2\,|\alpha_i|\, b_i + 1) \\
&\leq |c| + \frac{1}{\mathrm{Re}(\gamma)} \sum_{i=1}^{n} (2\,|\alpha_i|\, b_i + 1) \\
&\leq 1.
\end{aligned}$$

Finally, by applying Lemma 3, we conclude that $I_\gamma^{\alpha_i}\,(f_1, ..., f_n) \in \mathcal{S}$. □

Let $\alpha_i = 1$ for all $i = 1, \dots, n$ in Theorem 3, we have

Corollary 5. *Let* $\gamma \in \mathbb{C}$ *with*

$$Re(\gamma) \geq \sum_{i=1}^{n} (2b_i + 1)$$

and let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{Re(\gamma)} \sum_{i=1}^{n} (2b_i + 1).$$

If for all $i = 1, \dots, n,\ f_i \in \mathcal{G}_{b_i};\ 0 < b_i \leq 1$ *, and*

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| < 1 \qquad (z \in \mathcal{U}), \tag{23}$$

then the integral operator $I_\gamma(f_1, ..., f_n)(z)$ *defined by (11) is analytic and univalent in* $\mathcal{U}$.

Let $n = 1,\ \alpha_1 = \alpha,\ b_1 = b$ and $f_1 = f$ in Theorem 3, then we have

Corollary 6. *Let* $\alpha \in \mathbb{C}$ *and* $\gamma \in \mathbb{C}$ *with*

$$Re(\gamma) \geq (2\,|\alpha|\, b + 1)$$

and let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{Re(\gamma)} (2\,|\alpha|\, b + 1).$$

If $f \in \mathcal{G}_b;\ 0 < b \leq 1$ *satisfies*

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < 1 \qquad (z \in \mathcal{U}), \tag{24}$$

then the integral operator $I_\gamma^\alpha\ (f)(z)$ *defined by (14) is analytic and univalent in* $\mathcal{U}$.

Finally, we prove

Theorem 4. *Let* $\alpha_i \in \mathbb{C}$ *,* $M_i \geq 1$ *for all* $i = 1, \dots, n$ *and* $\gamma \in \mathbb{C}$ *with*

$$Re(\gamma) \geq \sum_{i=1}^{n} [(|\alpha_i|\, b_i + 1)\,(2M_i + 1) + |\alpha_i|\, b_i]. \tag{25}$$

and let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{Re(\gamma)} \sum_{i=1}^{n} [(|\alpha_i|\, b_i + 1)\,(2M_i + 1) + |\alpha_i|\, b_i]. \tag{26}$$

If for all $i = 1, \dots, n,\ f_i \in \mathcal{G}_{b_i};\ 0 < b_i \leq 1$ *satisfy*

$$\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| < 1 \qquad (z \in \mathcal{U}), \tag{27}$$

then the integral operator $I_\gamma^{\alpha_i}\ (f_1, ..., f_n)(z)$ *defined by (3) is analytic and univalent in* $\mathcal{U}$ *.*

Proof. From the proof of Theorem 2, we have

$$\left| c\,|z|^{2\gamma} + (1-|z|^{2\gamma})\frac{zh''(z)}{\gamma h'(z)} \right| \le |c| + \frac{1}{\mathrm{Re}(\gamma)}\sum_{i=1}^{n}[(|\alpha_i|\,b_i+1)\,(2M_i+1)+|\alpha_i|\,b_i] \quad (z\in\mathcal{U}),$$

which, in the light of hypothesis (26), yields

$$\left| c\,|z|^{2\gamma} + (1-|z|^{2\gamma})\frac{zh''(z)}{\gamma h'(z)} \right| \le 1 \quad (z\in\mathcal{U}).$$

Applying Lemma 3 for the function $h(z)$, we prove that $I_\gamma^{\alpha_i}\,(f_1,...,f_n)(z)\in\mathcal{S}$. □

Let $\alpha_i = 1$ for all $i = 1,\dots,n$ in Theorem 4, we have

Corollary 7. *Let $M_i \ge 1$ for all $i = 1,\dots,n$ and $\gamma\in\mathbb{C}$ with*

$$Re(\gamma) \ge \sum_{i=1}^{n}[(b_i+1)\,(2M_i+1)+b_i]. \tag{28}$$

and let $c\in\mathbb{C}$ be such that

$$|c| \le 1 - \frac{1}{Re(\gamma)}\sum_{i=1}^{n}[(b_i+1)\,(2M_i+1)+b_i]. \tag{29}$$

If for all $i = 1,\dots,n$, $f_i\in\mathcal{G}_{b_i}$; $0<b_i\le 1$ satisfy

$$\left|\frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1\right| < 1 \qquad (z\in\mathcal{U}), \tag{30}$$

then the integral operator $I_\gamma(f_1,...,f_n)(z)$ defined by (11) is analytic and univalent in $\mathcal{U}$.

Let $n=1$, $\alpha_1=\alpha$, $b_1=b$, $M_1=M$ and $f_1=f$ in Theorem 4, then we have

Corollary 8. *Let $\alpha\in\mathbb{C}$, $M\ge 1$ and $\gamma\in\mathbb{C}$ with*

$$Re(\gamma) \ge [(|\alpha|\,b+1)\,(2M+1)+|\alpha|\,b]. \tag{31}$$

and let $c\in\mathbb{C}$ be such that

$$|c| \le 1 - \frac{1}{Re(\gamma)}[(|\alpha|\,b+1)\,(2M+1)+|\alpha|\,b]. \tag{32}$$

If $f\in\mathcal{G}_b$; $0<b\le 1$ satisfies

$$\left|\frac{z^2 f'(z)}{[f(z)]^2} - 1\right| < 1 \qquad (z\in\mathcal{U}) \tag{33}$$

then the integral operator $I_\gamma^{\alpha}\,(f)(z)$ defined by (14) is analytic and univalent in $\mathcal{U}$.

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