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Uniform density u and $\mathcal{I}_u$-convergence on a big set*

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Abstract. We point out that $\mathcal{I}_u$-convergence (where $\mathcal{I}_u$ stands for the ideal of uniform density zero sets) is not equivalent to the convergence on a set from the dual filter. Moreover, we show that there are bounded sequences which are not $\mathcal{I}_u$-convergent on any set with a positive uniform density, i.e. $\mathcal{I}_u$ does not have the Bolzano-Weierstrass property. We also study relationship between the Bolzano-Weierstrass property and nonatomic submeasures. The Borel complexity of the ideal $\mathcal{I}_u$ is determined in the last section.
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1. Preliminaries

An ideal on $\mathbb{N}$ (by $\mathbb{N}$ we mean the set of all natural numbers) is a family $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ (where $\mathcal{P}(\mathbb{N})$ denotes the power set of $\mathbb{N}$) which is closed under taking subsets and finite unions. We assume that all considered ideals are proper ($\neq \mathcal{P}(\mathbb{N})$) and contain all finite sets unless something opposite is explicitly said.

Let $A \subset \mathbb{N}$ and $n \in \mathbb{N}$. Put (the cardinality of a set X is denoted by $|X|$)

$$s_n(A) = \min_{m\in\mathbb{N}} |A \cap \{m, m+1, \ldots, m+n-1\}|,$$
$$S_n(A) = \max_{m\in\mathbb{N}} |A \cap \{m, m+1, \ldots, m+n-1\}|.$$

It is known that the following limits exist

$$\underline{u}(A) = \lim_{n\to\infty} \frac{s_n(A)}{n}, \qquad \overline{u}(A) = \lim_{n\to\infty} \frac{S_n(A)}{n}.$$

They are called *lower* and *upper uniform density* of A. If $\underline{u}(A) = \overline{u}(A)$, then $u(A) = \underline{u}(A) = \overline{u}(A)$ is called the *uniform density* of A. The family

$$\mathcal{I}_u = \{A \subset \mathbb{N} \ : \ \overline{u}(A) = 0\}$$

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forms an ideal which is called the *ideal of uniform density zero sets*.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ of reals is *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if for every $\varepsilon > 0$

$$\{n \in \mathbb{N} \,:\, |x_n - x| \geq \varepsilon\} \in \mathcal{I}.$$

2. Correction

Recall that an ideal $\mathcal{I}$ is a *P-ideal* if for every sequence $(A_n)_{n\in\mathbb{N}}$ of sets from $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n.

In [2, Theorem 3] authors say that a sequence $x = (x_n)_{n\in\mathbb{N}}$ is $\mathcal{I}_u$-convergent to x if and only if there exists a set $K \subset \mathbb{N}$ with $u(K) = 1$ such that the subsequence $(x_n)_{n\in K}$ is convergent to x. By [9, Theorem 3.2] this is equivalent to the fact that $\mathcal{I}_u$ is a P-ideal. However, in [7, p. 299] it is shown that $\mathcal{I}_u$ is not a P-ideal (the authors of [7] use the equivalent notion of the AP0 property of submeasure $\overline{u}$.) So, Theorem 3 from [2] is not true (we know that this fact was pointed out earlier by several other mathematicians.)

In the next proposition we consider another property of $\mathcal{I}$-convergence which is equivalent to the statement that $\mathcal{I}$ is a P-ideal.

Proposition 1. *An ideal $\mathcal{I}$ is a P-ideal if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ which is $\mathcal{I}$-convergent to x there exists a sequence $(y_n)_{n\in\mathbb{N}}$ convergent to x and a sequence $(z_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-convergent to* 0*, such that $x_n = y_n + z_n$ for all $n \in \mathbb{N}$ and $\{n \in \mathbb{N} \,:\, z_n \neq 0\} \in \mathcal{I}$.*

Proof. ($\Leftarrow$). Take any sequence $(x_n)_{n\in\mathbb{N}}$ which is $\mathcal{I}$-convergent to x. There exists a sequence $(y_n)_{n\in\mathbb{N}}$ convergent to x and a sequence $(z_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-convergent to 0, such that $x_n = y_n + z_n$ for all $n \in \mathbb{N}$ and $\{n \in \mathbb{N} \,:\, z_n \neq 0\} \in \mathcal{I}$. Put $K = \{n \in \mathbb{N} \,:\, z_n = 0\}$. Obviously $\mathbb{N} \setminus K \in \mathcal{I}$ and $x_n = y_n$ for all $n \in K$. Then a subsequence $(x_n)_{n\in K}$ is convergent to x. Thus $\mathcal{I}$ is a P-ideal (by [9, Theorem 3.2]).

($\Rightarrow$). Take any sequence $(x_n)_{n\in\mathbb{N}}$ which is $\mathcal{I}$-convergent to x. By [9, Theorem 3.2] there exists a set $K \subset \mathbb{N}$ such that $\mathbb{N} \setminus K \in \mathcal{I}$ and $(x_n)_{n\in K}$ is convergent to x. Let $y_n = x_n$ and $z_n = 0$ for $n \in K$ and $y_n = x$ and $z_n = x_n - x$ for $n \notin K$. Obviously $x_n = y_n + z_n$ for all $n \in \mathbb{N}$ and $\{n \in \mathbb{N} \,:\, z_n \neq 0\} = \mathbb{N} \setminus K \in \mathcal{I}$. Moreover, $(y_n)_{n\in\mathbb{N}}$ is convergent to x and $(z_n)_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to 0. $\square$

3. Bolzano-Weierstrass property and nonatomic submeasures

We say that an ideal $\mathcal{I}$ on $\mathbb{N}$ has:

1. *the Fin-BW property* if for any bounded sequence $(x_n)_{n\in\mathbb{N}}$ of reals there is $A \notin \mathcal{I}$ such that $(x_n)_{n\in A}$ is convergent;

2. *the BW property* if for any bounded sequence $(x_n)_{n\in\mathbb{N}}$ of reals there is $A \notin \mathcal{I}$ such that $(x_n)_{n\in A}$ is $\mathcal{I}$-convergent.

In the first case we say that $\mathcal{I}$ has the *finite Bolzano-Weierstrass property*, in the second case we say that $\mathcal{I}$ has the *Bolzano-Weierstrass property*. By the well-known

Bolzano-Weierstrass theorem the ideal Fin, of all finite sets, has the Fin-BW property. Clearly, every ideal which has the Fin-BW property also has the BW property. For the discussion and applications of these properties see [6], where both the BW and Fin-BW properties are examined.

By $2^{\mathbb{N}}$ ($2^{<\mathbb{N}}$, 2^n) we mean the set of all infinite sequences of zeros and ones (the set of all finite sequences of zeros and ones, the set of all sequences of zeros and ones of length n, respectively.) If $s \in 2^n$ and $i \in \mathbb{N}$, then by $s\,\hat{}\,i$ we mean the sequence of length $n+1$ which extends s by i. If $x \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, then $x \restriction n = (x(0), x(1), \ldots, x(n-1))$.

In this section we will use the following characterizations of the BW and Fin-BW properties.

Proposition 2 (see [6, Proposition 3.3])**.** *An ideal $\mathcal{I}$ has the BW property (the Fin-BW property) if and only if for every family of sets $\{A_s : s \in 2^{<\mathbb{N}}\}$ fulfilling the following conditions*

(S1) $A_\emptyset = \mathbb{N}$,

(S2) $A_s = A_{s\,\hat{}\,0} \cup A_{s\,\hat{}\,1}$,

(S3) $A_{s\,\hat{}\,0} \cap A_{s\,\hat{}\,1} = \emptyset$,

there exist $x \in 2^{\mathbb{N}}$ and $B \subset \mathbb{N}$, $B \notin \mathcal{I}$ such that $B \setminus A_{x\restriction n} \in \mathcal{I}$ ($B \setminus A_{x\restriction n}$ is finite, respectively) for all n.

An ideal $\mathcal{I}$ of subsets of naturals is called *nonatomic* if there exists a sequence $(\mathcal{P}_n)$ of finite partitions of $\mathbb{N}$ such that each $\mathcal{P}_n$ is refined by $\mathcal{P}_{n+1}$, and whenever (A_n) is a decreasing sequence with $A_n \in \mathcal{P}_n$ for each n, and a set $Z \subset \mathbb{N}$ is such that $Z \setminus A_n$ is finite for each n, then $Z \in \mathcal{I}$.

A function $\phi : \mathcal{P}(\mathbb{N}) \to [0, +\infty)$ is called a *submeasure* if $\phi(\emptyset) = 0$, ϕ is monotone (i.e. $A \subset B \Rightarrow \phi(A) \le \phi(B)$) and ϕ is subadditive (i.e. $\phi(A \cup B) \le \phi(A) + \phi(B)$).

A submeasure ϕ is called *strongly nonatomic* if for every $\varepsilon > 0$ there exists a finite partition $A_1, \ldots, A_k$ of $\mathbb{N}$ such that $\phi(A_i) < \varepsilon$ for every i.

In [4] authors showed that the ideal $\mathcal{Z}(\phi) = \{A \subset \mathbb{N} \,:\, \phi(A) = 0\}$ is nonatomic for every strongly nonatomic submeasure ϕ. And they also showed that the converse does not hold.

For more information on strongly nonatomic submeasure and nonatomic ideals see e.g. [3, 4, 10, 1, 5].

Proposition 3. *An ideal $\mathcal{I}$ does not have the Fin-BW property if and only if it is nonatomic.*

Proof. ($\Rightarrow$). It is obvious by Proposition 2.

($\Leftarrow$). Let $(\mathcal{P}_n)$ be a sequence of finite partitions of $\mathbb{N}$ which shows that $\mathcal{I}$ is nonatomic. Let $\mathcal{P}_n = \{P_i^n : i < k_n\}$.

Let $\bigcup_{n\in\mathbb{N}} \mathcal{P}_n = \{Q_n : n \in \mathbb{N}\}$ be an enumeration such that if $Q_n = P_i^k$, $Q_m = P_j^l$, and $n < m$, then either $k < l$, or $k = l$ and $i < j$.

Let $A_\emptyset = \mathbb{N}$. Suppose that we have already defined sets A_s for $s \in 2^k$ for $k \le n$. For every $s \in 2^n$ we define $A_{s\,\hat{}\,0} = A_s \cap Q_n$ and $A_{s\,\hat{}\,1} = A_s \setminus Q_n$.

It is not difficult to see that the family $\{A_s : s \in 2^{<\mathbb{N}}\}$ fulfills the conditions $(S1)$, $(S2)$, and $(S3)$ of Proposition 2. On the other hand, there is no $x \in 2^{\mathbb{N}}$ and $B \subset \mathbb{N}$, $B \notin \mathcal{I}$ such that $B \setminus A_{x\restriction n}$ is finite for all n (for $\{A_s : s \in 2^{k_0+\cdots+k_n}\} \subset \mathcal{P}_n \cup \{\emptyset\}$ for every $n \in \mathbb{N}$). Thus $\mathcal{I}$ does not have the Fin-BW property (by Proposition 2). $\square$

Theorem 1. *If a submeasure ϕ is strongly nonatomic, then $\mathcal{Z}(\phi)$ does not have the BW property.*

Proof. For every $n \in \mathbb{N}$ we have a partition $\mathcal{P}_n = \{P_i^n : i < k_n\}$ such that $\phi(P_i^n) < \frac{1}{n}$ for every $i, n \in \mathbb{N}$. We can assume that each $\mathcal{P}_n$ is refined by $\mathcal{P}_{n+1}$.

Let a family $\{A_s : s \in 2^{<\mathbb{N}}\}$ be defined as in the proof of Proposition 3. Then this family fulfills the conditions $(S1)$, $(S2)$, and $(S3)$ of Proposition 2.

Suppose that there is $x \in 2^{\mathbb{N}}$ and $B \subset \mathbb{N}$, $B \notin \mathcal{Z}(\phi)$ such that $B \setminus A_{x\restriction n} \in \mathcal{Z}(\phi)$ for all n. Then

$$\phi(B) \leq \phi(B \cap A_{x\restriction(k_0+\cdots+k_n)}) + \phi(B \setminus A_{x\restriction(k_0+\cdots+k_n)}) < \frac{1}{n} + 0$$

for every $n \in \mathbb{N}$ (for $\{A_s : s \in 2^{k_0+\cdots+k_n}\} \subset \mathcal{P}_n \cup \{\emptyset\}$ for every $n \in \mathbb{N}$). Thus $\phi(B) = 0$, hence $B \in \mathcal{Z}(\phi)$, a contradiction. So $\mathcal{Z}(\phi)$ does not have the BW property (by Proposition 2). $\square$

In [10] authors showed that $\overline{u}$ is a strongly nonatomic submeasure. Thus by Theorem 1 we get the following corollary.

Corollary 1. *The ideal $\mathcal{I}_u$ does not have the BW property.*

We cannot prove the converse of Theorem 1. Indeed, let ϕ be a submeasure defined by $\phi(A) = 0$ if $\overline{u}(A) = 0$ and $\phi(A) = 1$ otherwise. Then it is easy to see that ϕ is not strongly nonatomic. On the other hand, $\mathcal{Z}(\phi) = \mathcal{I}_u$ and it does not have the BW property (by Corollary 1).

Below, we show that for some subclass of submeasures Theorem 1 can be conversed.

A submeasure ϕ is called *lower semi-continuous* (or *lsc*, in short) if $\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0, 1, \ldots, n\})$ for every $A \subset \mathbb{N}$. We say that ϕ is the $\limsup$ of lsc submeasures if there is a sequence of lsc submeasures $(\phi_n)_{n\in\mathbb{N}}$, such that $\phi(A) = \limsup_{n\to\infty} \phi_n(A)$ for every $A \subset \mathbb{N}$.

Proposition 4. *Let ϕ be the* $\limsup$ *of lsc submeasures such that $\mathcal{Z}(\phi)$ contains all finite sets. A submeasure ϕ is strongly nonatomic if and only if the ideal $\mathcal{Z}(\phi)$ does not have the BW property.*

Proof. By Theorem 1 we only have to prove $(\Leftarrow)$.

By Proposition 2 there is a family $\{A_s : s \in 2^{<\mathbb{N}}\}$ fulfilling conditions $(S1)$, $(S2)$, and $(S3)$ of Proposition 2 such that there is no $x \in 2^{\mathbb{N}}$ and $B \subset \mathbb{N}$, $B \notin \mathcal{Z}(\phi)$ with $B \setminus A_{x\restriction n} \in \mathcal{Z}(\phi)$ for all n.

We will show that for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ with $\phi(A_s) < \varepsilon$ for every $s \in 2^n$ (this means that ϕ is strongly nonatomic.)

For the sake of contradiction, suppose that there is $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there is $s_n \in 2^n$ with $\phi(A_{s_n}) > \varepsilon$. Since $A_s \supset A_t$ for $s \subset t$, using König Lemma we can find $x \in 2^{\mathbb{N}}$ with $\phi(A_{x\restriction n}) > \varepsilon$ for every $n \in \mathbb{N}$.

Since ϕ is the $\limsup$ of submeasures ϕ_n and each ϕ_n is lsc, we can construct a sequence $k_0 < k_1 < \dots$ of natural numbers and a sequence $F_0, F_1, \dots$ of finite sets such that $F_n \subset A_{x\restriction n}$ and $\phi_{k_n}(F_n) > \varepsilon$.

Let $F = \bigcup_{n\in\mathbb{N}} F_n$. Then $\phi(F) = \limsup_{n\to\infty} \phi_n(F) \geq \limsup_{k\to\infty} \phi_{n_k}(F) \geq \varepsilon > 0$, hence $F \notin \mathcal{Z}(\phi)$. On the other hand, $F \setminus A_{x\restriction n} \subset F_0 \cup \dots \cup F_{n-1}$ is finite (hence in $\mathcal{Z}(\phi)$) for every $n \in \mathbb{N}$, a contradiction. $\square$

Remark 1. *It is not difficult to show that $\overline{u}$ is the* $\limsup$ *of lsc submeasures.*

4. Borel class of $\mathcal{I}_u$

By identifying sets of naturals with their characteristic functions, we equip $\mathcal{P}(\mathbb{N})$ with the Cantor-space topology, and therefore we can assign a topological complexity to ideals of sets of integers.

In the proof of the next proposition we will use the *ideal of asymptotic (statistical) density zero sets*

$$\mathcal{I}_d = \left\{ A \subset \mathbb{N} \ : \limsup_{n\to\infty} d_n(A) = 0 \right\},$$

where $d_n(A) = \frac{|A\cap\{0,1,\dots,n-1\}|}{n}$.

Theorem 2. *$\mathcal{I}_u$ is a proper $F_{\sigma\delta}$ subset of $\mathcal{P}(\mathbb{N})$, i.e. $\mathcal{I}_u \in F_{\sigma\delta} \setminus G_{\delta\sigma}$.*

Proof. First we show that $\mathcal{I}_u$ is $F_{\sigma\delta}$. By definition

$$\begin{aligned}\mathcal{I}_u &= \left\{ A \in \mathcal{P}(\mathbb{N}) : \forall_k \exists_N \forall_{n>N} \ \frac{S_n(A)}{n} < \frac{1}{k} \right\} \\ &= \left\{ A \in \mathcal{P}(\mathbb{N}) : \forall_k \exists_N \forall_{n>N} \forall_m \ |A \cap \{m, m+1, \dots, m+n-1\}| < \frac{n}{k} \right\}.\end{aligned}$$

Let $F_{k,n,m} = \left\{ A \in \mathcal{P}(\mathbb{N}) : |A \cap \{m, m+1, \dots, m+n-1\}| < \frac{n}{k} \right\}$. Then

$$F_{k,n,m} = \bigcup_{I\in\mathcal{A}_{k,n,m}} \{ A \in \mathcal{P}(\mathbb{N}) : A \cap \{m, m+1, \dots, m+n-1\} = I \},$$

where $\mathcal{A}_{k,n,m} = \{ I \subset \{m, m+1, \dots, m+n-1\} : |I| < \frac{n}{k} \}$. Observe that $F_{k,n,m}$ is a finite union of basic closed sets, therefore a closed set in $\mathcal{P}(\mathbb{N})$. Consequently, $\mathcal{I}_u = \bigcap_k \bigcup_N \bigcap_{n>N} \bigcap_m F_{k,n,m}$, so $\mathcal{I}_u \in F_{\sigma\delta}$.

It is enough to show that $\mathcal{I}_u$ is not $G_{\delta\sigma}$. In [8] it was shown that $\mathcal{I}_d$ is not a $G_{\delta\sigma}$ subset of $\mathcal{P}(\mathbb{N})$. Let us describe a continuous function $f\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $f^{-1}(\mathcal{I}_u) = \mathcal{I}_d$. This implies that $\mathcal{I}_u$ also cannot be a $G_{\delta\sigma}$ subset of $\mathcal{P}(\mathbb{N})$.

Let $B_{n,k} = \{ i : 2^n \leq i < 2^{n+1} \wedge k | i - 2^n + 1 \}$. For $A \in \mathcal{P}(\mathbb{N})$ let $E_n(A) = [1/d_n(A)]$, where $[r]$ denotes the integer part of r.

We define $f\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by the following formula

$$f(A) = \bigcup_{n=0}^{\infty} B_{n,E_n(A)}$$

Observe that the initial segment of $f(A)$ is determined by the initial segment of A, so the function f is continuous.

We show that $f^{-1}(\mathcal{I}_u) = \mathcal{I}_d$. Suppose that $A \in \mathcal{I}_d$ and choose $\varepsilon > 0$. Since $\lim_{n\to\infty} d_n(A) = 0$, we can find N such that $E_n(A) > \frac{1}{\varepsilon}$ for each $n > N$. Hence for $n > N$

$$S_n(f(A)) < 2^{N+1} + n\varepsilon + 1.$$

This implies that $\lim_{n\to\infty} \frac{S_n(f(A))}{n} \leq \varepsilon$. Hence $f(A) \in \mathcal{I}_u$.

Suppose that $A \notin \mathcal{I}_d$. Hence there exists ε_0 such that $d_n(A) > \varepsilon_0$ for infinitely many n. For each such n

$$S_{2^{n+1}}(f(A)) > 2^n\varepsilon_0 - 1.$$

Consequently, $\overline{u}(f(A)) \geq \frac{\varepsilon_0}{2}$ and $f(A) \notin \mathcal{I}_u$. □

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