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The stability of the cubic functional equation in various spaces

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Abstract. We prove the stability of the cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

in the setting of various spaces.
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1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [34] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [11]. Subsequently, the result of Hyers was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [25] for linear mappings by considering an unbounded Cauchy difference. The paper [25] of Th. M. Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. We refer the interested readers for more information on such problems to the papers [3, 6, 12, 14, 23, 24, 26, 27, 33].

The functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1}$$

is said to be the *cubic functional equation.* In particular, every solution of a cubic functional equation is said to be a cubic mapping. The stability problem for a cubic type functional equation was proved by K. W. Jun and H. M. Kim [13] for mappings $f : X \longrightarrow Y$, where X is a real normed space and Y is a Banach space.

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Recently some authors, see [4], [15]-[19], [21, 22, 30] and [35], investigated the stability of some functional equations in the settings of fuzzy, probabilistic and random normed spaces.

In this paper, we investigate the stability of the cubic functional equation (1) at a non-Archimedean random normed space and intuitionistic Random normed Spaces.

2. Preliminaries

In this section we recall some definitions and results which will be used later on in the paper.

A *triangular norm* (shorter *t-norm*) is a binary operation on the unit interval $[0,1]$, i.e., a function $T : [0,1] \times [0,1] \to [0,1]$ such that for all $a, b, c \in [0,1]$ the following four axioms are satisfied:

(i) $T(a,b) = T(b,a)$ (commutativity);

(ii) $T(a,(T(b,c))) = T(T(a,b),c)$ (associativity);

(iii) $T(a,1) = a$ (boundary condition);

(iv) $T(a,b) \le T(a,c)$ whenever $b \le c$ (monotonicity).

Basic examples are the Łukasiewicz t-norm T_L, $T_L(a,b) = \max(a+b-1,0)\ \forall a,b \in [0,1]$ and the t-norms T_P, T_M, T_D, where $T_P(a,b) := ab$, $T_M(a,b) := \min\{a,b\}$,

$$T_D(a,b) := \begin{cases} \min(a,b), & \text{if } \max(a,b)=1; \\ 0, & \text{otherwise.} \end{cases}$$

If T is a t-norm, then $x_T^{(n)}$ is defined for every $x \in [0,1]$ and $n \in N \cup \{0\}$ by 1, if $n = 0$ and $T(x_T^{(n-1)}, x)$, if $n \ge 1$. A t-norm T is said to be *of Hadžić-type* (denoted by $T \in \mathcal{H}$) if the family $(x_T^{(n)})_{n \in N}$ is equicontinuous at $x = 1$ (cf. [8]).

Other important triangular norms are (see [9]):

-the *Sugeno-Weber family* $\{T_\lambda^{SW}\}_{\lambda \in [-1,\infty]}$, defined by $T_{-1}^{SW} = T_D$, $T_\infty^{SW} = T_P$ and

$$T_\lambda^{SW}(x,y) = \max\left(0, \frac{x+y-1+\lambda xy}{1+\lambda}\right)$$

if $\lambda \in (-1,\infty)$.

-the *Domby family* $\{T_\lambda^{D}\}_{\lambda \in [0,\infty]}$, defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^{D}(x,y) = \frac{1}{1 + ((\frac{1-x}{x})^\lambda + (\frac{1-y}{y})^\lambda)^{1/\lambda}}$$

if $\lambda \in (0,\infty)$.

-the *Aczel-Alsina family* $\{T_\lambda^{AA}\}_{\lambda \in [0,\infty]}$, defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^{AA}(x,y) = e^{-(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}}$$

if $\lambda \in (0,\infty)$.

A t-norm T can be extended (by associativity) in a unique way to an n-array operation taking for $(x_1,...,x_n) \in [0,1]^n$ the value $T(x_1,...,x_n)$ defined by

$$\mathrm{T}_{i=1}^{0} x_i = 1, \mathrm{T}_{i=1}^{n} x_i = T(\mathrm{T}_{i=1}^{n-1} x_i, x_n) = T(x_1,...,x_n).$$

T can also be extended to a countable operation taking for any sequence $(x_n)_{n\in N}$ in $[0,1]$ the value

$$\mathrm{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathrm{T}_{i=1}^{n} x_i. \tag{2}$$

The limit on the right-hand side of (2) exists since the sequence $(\mathrm{T}_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below.

Proposition 1 (see [9]).

(i) For $T \geq T_L$ the following implication holds:

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

(ii) If T is of Hadžić-type, then

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1$$

for every sequence $(x_n)_{n\in N}$ in $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$.

(iii) If $T \in \{T_\lambda^{AA}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{D}\}_{\lambda\in(0,\infty)}$, then

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n)^{\alpha} < \infty.$$

(iv) If $T \in \{T_\lambda^{SW}\}_{\lambda\in[-1,\infty)}$, then

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

In the sequel we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [5, 15, 16, 31, 32]. Throughout this paper, Δ^+ is the space of all measure (probability) distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow [0,1]$, such that F is left-continuous, nondecreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^-F(+\infty) = 1$, where $l^-f(x)$ denotes the left limit of the function f at the point x, that is, $l^-f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, \text{ if } t \leq 0, \\ 1, \text{ if } t > 0. \end{cases}$$

Definition 1 (see [32]). *A random normed space (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:*

(RN1) $\mu_x(t) = \varepsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$*;*

(RN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ *for all* $x \in X$*,* $\alpha \neq 0$*;*

(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ *for all* $x, y, z \in X$ *and* $t, s \geq 0$*.*

Definition 2. *Let* (X, μ, T) *be an RN-space.*

(1) A sequence $\{x_n\}$ *in* X *is said to be* convergent *to* x *in* X *if for every* $\epsilon > 0$ *and* $\lambda > 0$ *there exists a positive integer* N *such that* $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ *whenever* $n \geq N$*.*

(2) A sequence $\{x_n\}$ *in* X *is called* Cauchy *if for every* $\epsilon > 0$ *and* $\lambda > 0$ *there exists a positive integer* N *such that* $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ *whenever* $n \geq m \geq N$*.*

(3) An RN-space (X, μ, T) *is said to be* complete *if every Cauchy sequence in* X *is convergent to a point in* X*.*

Theorem 1 (see [31])**.** *If* (X, μ, T) *is an RN-space and* $\{x_n\}$ *is a sequence such that* $x_n \to x$*, then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ *almost everywhere.*

3. Non-Archimedean random normed spaces

By a *non-Archimedean field* we mean a field $\mathcal{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathcal{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r|\,|s|$, and $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathcal{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. Let X be a vector space over a field $\mathcal{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$, i.e., that there is an $a_0 \in \mathcal{K}$ such that $|a_0|$ is not in $\{0, 1\}$.

The most important examples of non-Archimedean spaces are p-adic numbers. In 1897, Hensel [10] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

A function $\|\cdot\| : X \to [0, \infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) for any $r \in \mathcal{K}, x \in X$, $\|rx\| = |r|\|x\|$;

(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\} \qquad (n > m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

Definition 3. *A non-Archimedean random normed space (briefly, non-Archimedean RN-space) is a triple* (X, μ, T)*, where* X *is a linear space over a non-Archimedean field* $\mathcal{K}$*,* T *is a continuous* t*–norm, and* μ *is a mapping from* X *into* D^+ *such that the following conditions hold:*

(NA-RN1) $\mu_x(t) = \varepsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$*;*

(NA-RN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ *for all* $x \in X$*,* $t > 0$*,* $\alpha \neq 0$*;*

(NA-RN3) $\mu_{x+y}(\max\{t, s\}) \geq T(\mu_x(t), \mu_y(s))$ *for all* $x, y, z \in X$ *and* $t, s \geq 0$*.*

It is easy to see that, if (NA-RN3) holds, then so is

(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$*.*

As a classical example, if $(X, \|.\|)$ is a non-Archimedean normed linear space, then the triple (X, μ, T_M), where

$$\mu_x(t) = \begin{cases} 0 & t \leq \|x\| \\ 1 & t > \|x\| \end{cases}$$

is a non-Archimedean RN-space.

Example 1. *Let* $(X, \|\cdot\|)$ *be a non-Archimedean normed linear space. Define*

$$\mu_x(t) = \frac{t}{t + \|x\|}, \quad \forall x \in X, \ t > 0.$$

Then (X, μ, T_M) *is a non-Archimedean RN-space.*

Definition 4. *Let* (X, μ, T) *be a non-Archimedean RN-space. Let* $\{x_n\}$ *be a sequence in* X*. Then* $\{x_n\}$ *is said to be convergent if there exists* $x \in X$ *such that*

$$\lim_{n \to \infty} \mu_{x_n - x}(t) = 1$$

for all $t > 0$*. In that case,* x *is called the limit of the sequence* $\{x_n\}$*.*

A sequence $\{x_n\}$ *in* X *is called Cauchy if for each* $\varepsilon > 0$ *and each* $t > 0$ *there exists* n_0 *such that for all* $n \geq n_0$ *and all* $p > 0$ *we have* $\mu_{x_{n+p} - x_n}(t) > 1 - \varepsilon$*.*

Remark 1 (see [20])**.** *Let* (X, μ, T_M) *be a non-Archimedean RN-space, then*

$$\mu_{x_{n+p} - x_n}(t) \geq \min\{\mu_{x_{n+j+1} - x_{n+j}}(t) : j = 0, 1, 2, ...p - 1\}.$$

So, the sequence $\{x_n\}$ *is Cauchy if for each* $\varepsilon > 0$ *and* $t > 0$ *there exists* n_0 *such that for all* $n \geq n_0$ *we have*

$$\mu_{x_{n+1} - x_n}(t) > 1 - \varepsilon.$$

If each Cauchy sequence is convergent, then the random normed space is said to be *complete* and the non-Archimedean RN-space is called a non-Archimedean *random Banach space.*

4. Random Hyers–Ulam–Rassias stability for cubic functional equation

Let $\mathcal{K}$ be a non-Archimedean field, X a vector space over $\mathcal{K}$ and (Y, μ, T) a non-Archimedean random Banach space over $\mathcal{K}$.

We investigate the stability of the cubic functional equation (1) where f is a mapping from X to Y and $f(0) = 0$. It is known that a mapping f satisfies the above functional equation if and only if it is cubic (see [13]).

Next we define a random approximately cubic mapping. Let Ψ be a distribution function on $X \times X \times [0, \infty)$ such that $\Psi(x, y, \cdot)$ is nondecreasing and

$$\Psi(cx, cx, t) \geq \Psi\left(x, x, \frac{t}{|c|}\right) \quad (x \in X, c \neq 0).$$

Definition 5. *A mapping $f : X \to Y$ is said to be* Ψ-approximately cubic *if*

$$\mu_{f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)}(t) \geq \Psi(x, y, t) \quad (x, y \in X, t > 0). \tag{3}$$

In this section, we assume that $2 \neq 0$ in $\mathcal{K}$ (i.e. characteristic of $\mathcal{K}$ is not 2). Our main result in this section is the following:

Theorem 2. *Let $\mathcal{K}$ be a non-Archimedean field, X a vector space over $\mathcal{K}$ and (Y, μ, T) a non-Archimedean random Banach space over $\mathcal{K}$. Let $f : X \to Y$ be a Ψ-approximately cubic function. If $|2| \neq 1$ and for some $\alpha \in \mathbb{R}, \alpha > 0$, and some integer $k, k \geq 2$ with $|2^k| < \alpha$,*

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq \Psi(x, y, \alpha t) \quad (x \in X, t > 0) \tag{4}$$

and

$$\lim_{n \to \infty} \mathrm{T}_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1 \quad (x \in X, t > 0) \tag{5}$$

then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mathrm{T}_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|2|^{ki}}\right), \tag{6}$$

for all $x \in X$ and $t > 0$, where

$$M(x, t) := T(\Psi(x, 0, t), \Psi(2x, 0, t), \ldots, \Psi(2^{k-1}x, 0, t)) \quad (x \in X, t > 0).$$

Proof. First, we show by induction on j that for each $x \in X$, $t > 0$ and $j \geq 1$,

$$\mu_{f(2^j x)-8^j f(x)}(t) \geq M_j(x, t) := T(\Psi(x, 0, t), \ldots, \Psi(2^{j-1}x, 0, t)). \tag{7}$$

Put $y = 0$ in (3) to obtain

$$\mu_{2f(2x)-16f(x)}(t) \geq \Psi(x, 0, t) \qquad (x \in X, t > 0),$$

$$\mu_{f(2x)-8f(x)}(t) \geq \Psi(x, 0, 2t) \geq \Psi(x, 0, t) \qquad (x \in X, t > 0).$$

This proves (7) for $j=1$. Let (7) hold for some $j>1$. Replacing y by 0 and x by $2^j x$ in (3), we get

$$\mu_{f(2^{j+1}x)-8f(2^jx)}(t) \geq \Psi(2^j x, 0, t) \quad (x \in X, t>0).$$

Since $|8| \leq 1$, then

$$\begin{aligned}
\mu_{f(2^{j+1}x)-8^{j+1}f(x)}(t) &\geq T\Big(\mu_{f(2^{j+1}x)-8f(2^jx)}(t), \mu_{8f(2^jx)-8^{j+1}f(x)}(t)\Big) \\
&= T\Big(\mu_{f(2^{j+1}x)-8f(2^jx)}(t), \mu_{f(2^jx)-8^jf(x)}\left(\frac{t}{|8|}\right)\Big) \\
&\geq T\Big(\mu_{f(2^{j+1}x)-8f(2^jx)}(t), \mu_{f(2^jx)-8^jf(x)}(t)\Big) \\
&\geq T(\Psi(2^jx,0,t), M_j(x,t)) \\
&= M_{j+1}(x,t)
\end{aligned}$$

for each $x \in X$. Thus (7) holds for all $j \geq 1$. In particular

$$\mu_{f(2^kx)-8^kf(x)}(t) \geq M(x,t) \quad (x \in X, t>0). \tag{8}$$

Replacing x by $2^{-(kn+k)}x$ in (8) and using inequality (4) we obtain

$$\begin{aligned}
\mu_{f\left(\frac{x}{2^{kn}}\right)-8^kf\left(\frac{x}{2^{kn+k}}\right)}(t) &\geq M\left(\frac{x}{2^{kn+k}}, t\right) \\
&\geq M(x, \alpha^{n+1}t) \quad (x \in X, t>0, n=0,1,2,\dots).
\end{aligned} \tag{9}$$

Then,

$$\begin{aligned}
\mu_{(2^{3k})^n f\left(\frac{x}{(2^k)^n}\right)-(2^{3k})^{n+1}f\left(\frac{x}{(2^k)^{n+1}}\right)}(t) &\geq M\left(x, \frac{\alpha^{n+1}}{|(2^{3k})^n|}t\right) \\
&\geq M\left(x, \frac{\alpha^{n+1}}{|(2^k)^n|}t\right)
\end{aligned} \tag{10}$$

for $x \in X, t>0, n=0,1,2,\dots$. Hence,

$$\begin{aligned}
\mu_{(2^{3k})^n f\left(\frac{x}{(2^k)^n}\right)-(2^{3k})^{n+p}f\left(\frac{x}{(2^k)^{n+p}}\right)}(t) &\geq \mathrm{T}_{j=n}^{n+p}\left(\mu_{(2^{3k})^j f\left(\frac{x}{(2^k)^j}\right)-(2^{3k})^{j+p}f\left(\frac{x}{(2^k)^{j+p}}\right)}(t)\right) \\
&\geq \mathrm{T}_{j=n}^{n+p} M\left(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t\right)
\end{aligned}$$

for $x \in X, t>0, n=0,1,2,\dots$. Since,

$$\lim_{n\to\infty} \mathrm{T}_{j=n}^{\infty} M\left(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t\right) = 1 \quad (x \in X, t>0),$$

$\left\{(2^{3k})^n f\left(\frac{x}{(2^k)^n}\right)\right\}_{n\in N}$ is a Cauchy sequence in the non-Archimedean random Banach space (Y, μ, T). Hence, we can define a mapping $C: X \to Y$ such that

$$\lim_{n\to\infty} \mu_{(2^{3k})^n f\left(\frac{x}{(2^k)^n}\right)-C(x)}(t) = 1 \quad (x \in X, t>0). \tag{11}$$

Next, for each $n \geq 1$, $x \in X$ and $t > 0$,

$$\begin{aligned}
\mu_{f(x)-(2^{3k})^n f\left(\frac{x}{(2^k)^n}\right)}(t) &= \mu_{\sum_{i=0}^{n-1}(2^{3k})^i f\left(\frac{x}{(2^k)^i}\right)-(2^{3k})^{i+1} f\left(\frac{x}{(2^k)^{i+1}}\right)}(t) \\
&\geq \mathrm{T}_{i=0}^{n-1}\left(\mu_{(2^{3k})^i f\left(\frac{x}{(2^k)^i}\right)-(2^{3k})^{i+1} f\left(\frac{x}{(2^k)^{i+1}}\right)}(t)\right) \\
&\geq \mathrm{T}_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}t}{|2^{3k}|^i}\right) \\
&\geq \mathrm{T}_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}t}{|2^{k}|^i}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mu_{f(x)-C(x)}(t) &\geq T\left(\mu_{f(x)-(2^{3k})^n f\left(\frac{x}{(2^k)^n}\right)}(t), \mu_{(2^{3k})^n f\left(\frac{x}{(2^k)^n}\right)-C(x)}(t)\right) \\
&\geq T\left(\mathrm{T}_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}t}{|2^k|^i}\right), \mu_{(2^{3k})^n f\left(\frac{x}{(2^k)^n}\right)-C(x)}(t)\right).
\end{aligned}$$

By letting $n \to \infty$ we obtain

$$\mu_{f(x)-C(x)}(t) \geq \mathrm{T}_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1}t}{|2^k|^i}\right).$$

This proves (6).

As T is continuous, from a well known result in the probabilistic metric space (see e.g., [31], Chapter 12) it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mu_{(2^{3k})^n f(2^{-kn}(2x+y))+(2^{3k})^n f(2^{-kn}(2x-y))} \\
&\qquad {}_{-2(2^{3k})^n f(2^{-kn}(x+y))-2(2^{3k})^n f(2^{-kn}(x-y))-12(2^{3k})^n f(3^{-kn}x)}(t) \\
&= \mu_{C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x)}(t)
\end{aligned}$$

for almost all $t > 0$. On the other hand, replace x, y by $2^{-kn}x, 2^{-kn}y$ in (3) and use (NA-RN2) and (4) to get

$$\begin{aligned}
&\mu_{(2^{3k})^n f(2^{-kn}(2x+y))+(2^{3k})^n f(2^{-kn}(2x-y))} \\
&\qquad {}_{-2(2^{3k})^n f(2^{-kn}(x+y))-2(2^{3k})^n f(2^{-kn}(x-y))-12(2^{3k})^n f(2^{-kn}x)}(t) \\
&\geq \Psi\left(2^{-kn}x, 2^{-kn}y, \frac{t}{|2^k|^n}\right) \\
&\geq \Psi\left(x, y, \frac{\alpha^n t}{|2^k|^n}\right)
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Since $\lim_{n\to\infty} \Psi\left(x, y, \frac{\alpha^n t}{|2^k|^n}\right) = 1$, we infer that C is a cubic mapping.

If $C': X \to Y$ is another cubic mapping such that $\mu_{C'(x)-f(x)}(t) \geq M(x,t)$ for all $x \in X$ and $t > 0$, then for each $n \in N$, $x \in X$ and $t > 0$,

$$\mu_{C(x)-C'(x)}(t) \geq T\left(\mu_{C(x)-(2^{3k})^n f\left(\frac{x}{(2^k)^n}\right)}(t), \mu_{(2^{3k})^n f\left(\frac{x}{(2^k)^n}\right)-C'(x)}(t)\right).$$

Thanks to (11), we conclude that $C = C'$. $\square$

Corollary 1. *Let $\mathcal{K}$ be a non-Archimedean field, X a vector space over $\mathcal{K}$ and (Y, μ, T) a non-Archimedean random Banach space over $\mathcal{K}$ under a t-norm $T \in \mathcal{H}$. Let $f : X \to Y$ be a Ψ-approximately cubic function. If for some $\alpha \in \mathbb{R}, \alpha > 0$, $|2| \neq 1$ and some integer $k, k \geq 2$ with $|2^k| < \alpha$,*

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq \Psi(x, y, \alpha t) \quad (x \in X, t > 0)$$

then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mathrm{T}_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1}t}{|2|^{ki}}\right),$$

for all $x \in X$ and $t > 0$, where

$$M(x,t) := T(\Psi(x,0,t), \Psi(2x,0,t), \ldots, \Psi(2^{k-1}x,0,t)) \quad (x \in X, t > 0).$$

Proof. Since

$$\lim_{n\to\infty} M\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1 \quad (x \in X, t > 0)$$

and T is of Hadžić type, from Proposition 1 it follows that

$$\lim_{n\to\infty} \mathrm{T}_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1 \quad (x \in X, t > 0).$$

Now we can apply Theorem 2. $\square$

5. Intuitionistic random normed spaces

Definition 6. *A non-measure distribution function is a function $\nu : \mathbb{R} \to [0,1]$ which is right continuous on $\mathbb{R}$, non-increasing and* $\inf_{t\in\mathbb{R}} \nu(t) = 0$, $\sup_{t\in\mathbb{R}} \nu(t) = 1$.

We will denote by B the family of all non-measure distribution functions and by G a special element of B defined by

$$G(t) = \begin{cases} 1 \text{ if } t \leq 0, \\ 0 \text{ if } t > 0. \end{cases}$$

If X is a nonempty set, then $\nu : X \longrightarrow B$ is called a *probabilistic non-measure* on X and $\nu(x)$ is denoted by ν_x.

Lemma 1 (see [2, 7])**.** *Consider the set L^* and operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1,\ x_2 \geq y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a *triangular norm* $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \longrightarrow [0,1]$ satisfying $T(1,x) = 1 * x = x$ for all $x \in [0,1]$. A *triangular conorm* $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \longrightarrow [0,1]$ satisfying $S(0,x) = 0 \diamond x = x$ for all $x \in [0,1]$.

Using the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

Definition 7 (see [7]). *A triangular norm (t–norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:*

(a) $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$ *(boundary condition);*

(b) $(\forall (x,y) \in (L^*)^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$ *(commutativity);*

(c) $(\forall (x,y,z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y),z))$ *(associativity);*

(d) $(\forall (x,x',y,y') \in (L^*)^4)(x \leq_{L^*} x'$ *and* $y \leq_{L^*} y' \Longrightarrow \mathcal{T}(x,y) \leq_{L^*} \mathcal{T}(x',y'))$ *(monotonicity).*

Let $\{x_n\}$ be a sequence in L^* converging to $x \in L^*$ (equipped order topology). The t-norm $\mathcal{T}$ is said to be a *continuous t–norm* if

$$\lim_{n\to\infty} \mathcal{T}(x_n, y) = \mathcal{T}(x,y),$$

for each $y \in L^*$.

Definition 8 (see [7]). *A continuous t–norm $\mathcal{T}$ on L^* is said to be continuous t–representable if there exist a continuous t–norm $*$ and a continuous t–conorm $\diamond$ on $[0,1]$ such that for all $x = (x_1,x_2), y = (y_1,y_2) \in L^*$,*

$$\mathcal{T}(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\mathcal{T}(a,b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

and

$$\mathbf{M}(a,b) = (\min\{a_1,b_1\}, \max\{a_2,b_2\})$$

for all $a = (a_1,a_2)$, $b = (b_1,b_2) \in L^*$ are continuous t–representable.

Now, we define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \cdots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \cdots, x^{(n)}), x^{(n+1)}), \quad \forall n \geq 2,\ x^{(i)} \in L^*.$$

Definition 9 ([28, 29]). *A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N : [0,1] \longrightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0,1]$ defined by*

$$N_s(x) = 1 - x, \quad \forall x \in [0,1].$$

Definition 10. *Let μ and ν be a measure and a non-measure distribution function from $X \times (0, +\infty)$ to $[0, 1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and $t > 0$ where X is a real vector space. The triple $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an intuitionistic random normed space (briefly IRN-space) if X is a real vector space, $\mathcal{T}$ a continuous t–representable and $\mathcal{P}_{\mu,\nu}$ a mapping $X \times (0, +\infty) \rightarrow L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,*

(a) $\mathcal{P}_{\mu,\nu}(x, 0) = 0_{L^*}$;

(b) $\mathcal{P}_{\mu,\nu}(x, t) = 1_{L^*}$ *if and only if* $x = 0$;

(c) $\mathcal{P}_{\mu,\nu}(\alpha x, t) = \mathcal{P}_{\mu,\nu}\left(x, \frac{t}{|\alpha|}\right)$ *for all* $\alpha \neq 0$;

(d) $\mathcal{P}_{\mu,\nu}(x + y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{P}_{\mu,\nu}(x, t), \mathcal{P}_{\mu,\nu}(y, s))$.

In this case, $\mathcal{P}_{\mu,\nu}$ is called an intuitionistic random norm. Here,

$$\mathcal{P}_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)).$$

Example 2. *Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$ and μ, ν be measure and non-measure distribution functions defined by*

$$\mathcal{P}_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)) = \Big(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\Big), \quad \forall t \in \mathbb{R}^+.$$

Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an IRN-space.

Definition 11. *(1) A sequence $\{x_n\}$ in an IRN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is called a Cauchy sequence if for any $\varepsilon > 0$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$\mathcal{P}_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon), \quad \forall n, m \geq n_0,$$

where N_s is the standard negator.

(2) The sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ (denoted by $x_n \xrightarrow{\mathcal{P}_{\mu,\nu}} x$) if $\mathcal{P}_{\mu,\nu}(x_n - x, t) \longrightarrow 1_{L^*}$ as $n \longrightarrow \infty$ for every $t > 0$.

(3) An IRN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be complete if every Cauchy sequence in X is convergent to a point $x \in X$.

6. A stability result in intuitionistic random normed spaces

Theorem 3. *Let X be a real linear space and $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ a complete IRN-space. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ for which there are $\xi, \zeta : X^2 \longrightarrow D^+$ ($\xi(x, y)$ is denoted by $\xi_{x,y}$ and $\zeta(x, y)$ is denoted by $\zeta_{x,y}$, further, $(\xi_{x,y}(t), \zeta_{x,y}(t))$ is denoted by $Q_{\xi,\zeta}(x, y, t)$) with the property:*

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), t)\\&\quad \geq_{L^*} Q_{\xi,\zeta}(x, y, t). \end{aligned} \tag{12}$$

If

$$\mathcal{T}_{i=1}^{\infty}(Q_{\xi,\zeta}(2^{n+i-1}x, 0, 2^{3n+2i+1}t)) = 1_{L^*} \tag{13}$$

and

$$\lim_{n\to\infty} Q_{\xi,\zeta}(2^n x, 2^n y, 2^{3n} t) = 1_{L^*} \tag{14}$$

for every $x, y \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - C(x), t) \geq_{L^*} \mathcal{T}_{i=1}^{\infty} \left(Q_{\xi,\zeta}\left(2^{i-1}x, 0, 2^{2i+1}t\right)\right). \tag{15}$$

Proof. Putting $y = 0$ in (12), we have

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2x)}{8} - f(x), t\right) \geq_{L^*} Q_{\xi,\zeta}\left(x, 0, 2^4 t\right) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 2^3 t). \tag{16}$$

Therefore, it follows that

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{3(k+1)}} - \frac{f(2^k x)}{2^{3k}}, \frac{t}{2^{3k}}\right) \geq_{L^*} Q_{\xi,\zeta}\left(2^k x, 0, 2^3 t\right), \tag{17}$$

which implies that

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{3(k+1)}} - \frac{f(2^k x)}{2^{3k}}, t\right) \geq_{L^*} Q_{\xi,\zeta}\left(2^k x, 0, 2^{3(k+1)} t\right), \tag{18}$$

that is,

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{3(k+1)}} - \frac{f(2^k x)}{2^{3k}}, \frac{t}{2^{k+1}}\right) \geq_{L^*} Q_{\xi,\zeta}(2^k x, 0, 2^{2(k+1)} t) \tag{19}$$

for all $k \in \mathbf{N}$ and $t > 0$. As $1 > 1/2 + ... + 1/2^n$, by the triangle inequality it follows

$$\begin{aligned}\mathcal{P}_{\mu,\nu}\Big(\frac{f(2^n x)}{8^n} - f(x), t\Big) &\geq_{L^*} \mathcal{T}_{k=0}^{n-1}\left(\mathcal{P}_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{3(k+1)}} - \frac{f(2^k x)}{2^{3k}}, \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} t\right)\right) \\ &\geq_{L^*} \mathcal{T}_{i=1}^{n}(Q_{\xi,\zeta}(2^{i-1}x, 0, 2^{2i+1}t)).\end{aligned} \tag{20}$$

In order to prove convergence of the sequence $\{\frac{f(2^n x)}{8^n}\}$, we replace x by $2^m x$ in (20) to find that for $m, n > 0$

$$\mathcal{P}_{\mu,\nu}\Big(\frac{f(2^{n+m}x)}{8^{(n+m)}} - \frac{f(2^m x)}{8^m}, t\Big) \geq_{L^*} \mathcal{T}_{i=1}^{n}(Q_{\xi,\zeta}(2^{i+m-1}x, 2^{i+m-1}x, 2^{2i+m+1}t)). \tag{21}$$

Since the right hand-side of the inequality tends to 1_{L^*} as m tends to infinity, the sequence $\{\frac{f(2^n x)}{2^{3n}}\}$ is a Cauchy sequence. Therefore, we may define $C(x) = \lim_{n\longrightarrow\infty} \frac{f(2^n x)}{2^{3n}}$ for all $x \in X$.

Now, we show that C is a cubic map. Replacing x, y by $2^n x$ and $2^n y$, respectively in (12) then it follows that

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}\Big(\frac{f(2^{n+1}x + 2^n y)}{2^{3n}} + \frac{f(2^{n+1}x - 2^n y)}{2^{3n}} - 2\frac{f(2^n x + 2^n y)}{2^{3n}} \\ &\quad -2\frac{f(2^n x - 2^n y)}{2^{3n}} - 12\frac{f(2^n x)}{2^{3n}}, t\Big) \\ &\geq_{L^*} Q_{\xi,\zeta}(2^n x, 2^n y, 2^{3n} t).\end{aligned} \tag{22}$$

Taking the limit as $n \longrightarrow \infty$, we find that C satisfies (1) for all $x, y \in X$.

To prove (15), take the limit as $n \longrightarrow \infty$ in (20).

To prove the uniqueness of the cubic function C subject to (15), let us assume that there exists a cubic function C' which satisfies (15). Obviously we have $C(2^n x) = 2^{3n}C(x)$ and $C'(2^n x) = 2^{3n}C'(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (15) that

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}\Big(C(x) - C'(x), t\Big)\\
&\quad\geq_{L^*} \mathcal{P}_{\mu,\nu}\Big(C(2^n x) - C'(2^n x), 2^{3n}t\Big)\\
&\quad\geq_{L^*} \mathcal{T}\Big(\mathcal{P}_{\mu,\nu}\Big(C(2^n x) - f(2^n x), 2^{3n-1}t\Big), \mathcal{P}_{\mu,\nu}\Big(f(2^n x) - Q'(2^n x), 2^{3n-1}t\Big)\Big)\\
&\quad\geq_{L^*} \mathcal{T}\Big(\mathcal{T}_{i=1}^{\infty}(Q_{\xi,\zeta}(2^{n+i-1}x, 0, 2^{3n+2i+1}t)), \mathcal{T}_{i=1}^{\infty}(Q_{\xi,\zeta}(2^{n+i-1}x, 0, 2^{3n+2i+1}t)\Big)
\end{aligned}$$

for all $x \in X$. By letting $n \longrightarrow \infty$ in (15), we find that the uniqueness of C. This completes the proof. $\square$

Corollary 2. *Let $(X, \mathcal{P}'_{\mu',\nu'}, \mathcal{T})$ be an IRN-space and $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ a complete IRN-space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), t)\\
&\geq_{L^*} \mathcal{P}'_{\mu',\nu'}(x+y, t)
\end{aligned}$$

for $t > 0$ in which

$$\lim_{n\longrightarrow\infty} \mathcal{T}_{i=1}^{\infty}(\mathcal{P}'_{\mu',\nu'}(x, 2^{2n+i+2}t)) = 1_{L^*}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - C(x), t) \geq_{L^*} \mathcal{T}_{i=1}^{\infty}(\mathcal{P}'_{\mu',\nu'}(2^{i+1}t)).$$

Now, we give one example to validate the main result, Theorem 3, as follows:

Example 3. *Let $(X, \|.\|)$ be a real Banach algebra, $(X, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ a complete IRN-space in which*

$$\mathcal{P}_{\mu,\nu}(x, t) = \Big(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\Big)$$

for all $x \in X$. Define $f : X \longrightarrow X$ by $f(x) = x^3 + x_0$, where x_0 is a unit vector in X. A straightforward computation shows that

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), t)\\
&\quad\geq_{L^*} \mathcal{P}_{\mu,\nu}(x+y, t), \quad \forall t > 0.
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \mathbf{M}_{i=1}^{\infty}(\mathcal{P}_{\mu,\nu}(x, 2^{2n+i+1}t)) &= \lim_{n\longrightarrow\infty} \lim_{m\longrightarrow\infty} \mathbf{M}_{i=1}^{m}(\mathcal{P}_{\mu,\nu}(x, 2^{2n+i+1}t))\\
&= \lim_{n\longrightarrow\infty} \lim_{m\longrightarrow\infty} \mathcal{P}_{\mu,\nu}(x, 2^{2n+2}t)\\
&= \lim_{n\longrightarrow\infty} \mathcal{P}_{\mu,\nu}(x, 2^{2n+2}t)\\
&= 1_{L^*}.
\end{aligned}$$

Therefore, all the conditions of Theorem 3 hold and so there exists a unique cubic mapping $C : X \longrightarrow X$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - C(x), t) \geq_{L^*} \mathcal{P}_{\mu,\nu}(x, 2^2 t).$$

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