Math. Commun. **16**(2011), 147–155.

Fixed point theorems for contractive mapping in cone metric spaces

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Received June 24, 2009; accepted July 10, 2010

Abstract. The aim of this paper is to prove some fixed point theorems on generalized metric spaces for maps satisfying general contractive type conditions. Also we consider some fixed point theorems for certain composition mappings.

AMS subject classifications: 54C60, 54H25

Key words: Cone metric space, fixed point, contractive mappings

1. Introduction and preliminary

The study of fixed points of mappings satisfying certain contractive conditions has been a very active area of research. For a survey of fixed point theory, some applications, comparison of different contractive conditions and related results see [1, 2, 8, 9, 11, 16, 17, 18]. Recently Long-Guang and Xian [10] generalized the concept of a metric space, by introducing cone metric spaces, and obtained some fixed point theorems for mappings satisfying certain contractive conditions. Other papers on cone metric spaces are [3, 4, 7, 10, 14]. The purpose of this paper is to prove some fixed point theorems on cone metric spaces under fairly general contractive conditions. Also we obtain fixed point theorems for certain composition mappings.

Let E be a real Banach space. A nonempty convex closed subset $P \subset E$ is called a cone in E if it satisfies:

(i) P is closed, nonempty and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$,

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‡Supported by Grant No. 174025 of the Ministry of Science, Technology and Development, Republic of Serbia.

(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subset E$; that is, $x \leq y$ if and only if $y - x \in P$. Also we write $x \ll y$ if $y - x \in \text{int } P$, where int P denotes the interior of P.
A cone P is called normal if there exists a constant $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.
In the following we always suppose that E is a real Banach space, P is a cone in E and $\leq$ is partial ordering with respect to P.

Definition 1 (see [10]). *Let X be a nonempty set. Assume that the mapping $d : X \times X \to E$ satisfies*

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 2. *Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then*

(i) $\{x_n\}$ is said to be convergent to $x \in X$ whenever for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$, that is, $\lim_{n\to\infty} x_n = x$.

(ii) $\{x_n\}$ is called a Cauchy sequence in X whenever for every $c \in E$ with $0 \ll c$ there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Remark 1 (see [7]). *Let $P \subset E$ be a normal cone with constant K. Suppose for $a, b, c \in E$ and nonzero $u \in \{a, b, c\}$, $a \leq tu$ for every $t \geq 0$ and $t \neq 1$. Then $u \in \{b, c\}$.*

2. Main results

Given a cone metric space X, a self-map T of X and $u \in X$, the orbit of u, $O(u)$ is defined by $O(u) := \{u, Tu, T^2u, \ldots\}$. The closure of the orbit of u will be denoted by $\overline{O(u)}$. We begin with a general theorem.

Theorem 1. *Let (X, d) be a cone metric space, $P \subset E$ a cone, and T a self-map of X. If there exists a point $u \in X$ and a $\lambda \in [0, 1)$ such that $\overline{O(u)}$ is complete and*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1}$$

for any $x, y \in O(u)$, then $\{T^n u\}$ converges to some $p \in X$, and

$$d(T^n u, p) \leq \frac{\lambda^n}{1 - \lambda} d(u, Tu) \quad for \quad n \geq 1. \tag{2}$$

If (1) holds for any $x, y \in \overline{O(u)}$, then p is a fixed point of T.

Proof. From (1) it follows that $d(T^n u, T^{n+1}u) \leq \lambda d(T^{n-1}u, T^n u)$. Hence

$$d(T^n u, T^{n+1}u) \leq \lambda^n d(u, Tu)$$

for $n > 1$.

Thus, for any $m, n \geq 1$,

$$\begin{aligned} d(T^n u, T^{n+m}u) &\leq d(T^n u, T^{n+1}u) + \cdots + d(T^{n+m-1}u, T^{n+m}u) \\ &\leq (1 + \lambda + \cdots + \lambda^{m-1}) d(T^n u, T^{n+1}u) \\ &= \frac{1-\lambda^m}{1-\lambda} d(T^n u, T^{n+1}u) \leq \frac{1}{1-\lambda} d(T^n u, T^{n+1}u) \\ &\leq \frac{\lambda^n}{1-\lambda} d(u, Tu). \end{aligned}$$

Consequently $\{T^n u\}$ is Cauchy and, since $\overline{O(u)}$ is complete, it converges to a point $p \in X$. Now

$$\begin{aligned} d(T^n u, p) &\leq d(T^n u, T^{n+m}u) + d(T^{n+m}u, p) \\ &\leq \frac{\lambda^n}{1-\lambda} d(u, Tu) + d(T^{n+m}u, p). \end{aligned}$$

Let $0 \ll c$ be given. For each positive integer m choose N_m so that $d(T^{n+m}u, p) \ll \frac{c}{m}$ for $n \geq N_m$. Then

$$d(T^n u, p) - \frac{\lambda^n}{1-\lambda} d(u, Tu) \ll \frac{c}{m},$$

for all $m \geq 1$, which implies that

$$\frac{c}{m} - d(T^n u, p) + \frac{\lambda^n}{1-\lambda} d(u, Tu) \in P$$

for all $m \geq 1$. Since $\lim \frac{c}{m} = 0$ and P is closed, we obtain

$$-d(T^n u, p) + \frac{\lambda^n}{1-\lambda} d(u, Tu) \in P,$$

which is equivalent to (2).

If (1) holds for any $x, y \in \overline{O(u)}$, then

$$d(T^{n+1}u, Tp) \leq \lambda d(T^n u, p).$$

But

$$d(p, Tp) \leq d(T^{n+1}u, p) + d(T^{n+1}u, Tp) \leq d(T^{n+1}u, p) + \lambda d(T^n u, p).$$

If $\lambda \neq 0$, then for any $0 \ll c$ choose N so that, for $n \geq N$, $d(T^{n+1}u, p) \ll \frac{c}{2}$ and $d(T^n u, p) \ll \frac{c}{2\lambda}$. Then we have $d(p, Tp) \ll c$, which implies that $p = Tp$. The case $\lambda = 0$ is trivial. $\square$

Theorem 1 is the cone metric version of Theorem 2 of [13].

Corollary 1. *Let (X,d) be a complete cone metric space, T a self-map of X satisfying*

$$d(Tx,Ty) \leq q(x,y)d(x,y) + r(x,y)d(x,Tx) + s(x,y)d(y,Ty) + t(x,y)(d(x,Ty) + d(y,Tx)) \tag{3}$$

for all $x,y \in X$, where $q,r,s,$ and t are nonnegative numbers satisfying

$$\sup_{x,y\in X} \{q(x,y) + r(x,y) + s(x,y) + 2t(x,y)\} = \lambda < 1. \tag{4}$$

Then T has a unique fixed point in X.

Proof. Setting $y = Tx$ in (3) yields, suppressing x and Tx in the expressions q,r,s,t,

$$d(Tx,T^2x) \leq qd(x,Tx) + rd(x,Tx) + sd(Tx,T^2x) + td(x,T^2x),$$

which implies, that

$$d(Tx,T^2x) \leq \frac{q+r+t}{1-s-t} d(x,Tx) \leq \lambda d(x,Tx).$$

The above inequality implies condition (1), and T has a fixed point by Theorem 1. That the fixed point is unique follows from (3). □

Corollary 1 is the cone metric version of Theorem 2.5(i) of [5].

Corollary 2 (see [15], Theorem 2.6)**.** *Let (X,d) be a complete cone metric space, $P \subset E$ a cone, and T a self-map of X satisfying*

$$d(Tx,Ty) \leq k[d(Tx,x) + d(Ty,y)] \tag{5}$$

for all $x,y \in X$, where k is a constant, $0 < k < 1/2$. Then T has a unique fixed point in X and, for each x in X, $\{T^n x\}$ converges to the fixed point.

Proof. Inequality (5) is a special case of (1), and the result follows from Corollary 1. □

Corollary 3. *Let (X,d) be a complete cone metric space, $P \subset E$ a cone, and T a self-map of X satisfying*

$$d(Tx,Ty) \leq c[d(x,T^m z) + d(y,T^m z)], \tag{6}$$

for some $m \in \mathbb{N}$ and all $x,y,z \in X$, where c is a constant, $0 < c < 1$. Then T has a unique fixed point in X, and, for each $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Let $u \in X$. In (6) set $x = T^{m-1}u, y = T^m u, z = u$. Then we have

$$d(T^m u, T^{m+1} u) \leq cd(T^{m-1}u, T^m u), \tag{7}$$

which is a special case of (1). Thus T has a fixed point p. Condition (6) implies uniqueness. □

Corollary 4. *Let (X, d) be a complete cone metric space, $P \subset E$ a cone, and T a self-map of X satisfying*

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(y, Tx) + fd(x, Ty), \qquad (8)$$

where a, b, c, d, f, e are positive real numbers satisfying $a + b + c + e + f < 1$. Then T has a unique fixed point.

Proof. By [12, Theorem 2.1] we may assume that $b = c$ and $e = f$. In (8) set $y = Tx$ to obtain

$$d(Tx, T^2x) \leq \frac{a + b + f}{1 - c - f} d(x, Tx), \qquad (9)$$

and by symmetry, we may exchange b, c and f with e in 9 to obtain

$$d(Tx, T^2x) \leq \frac{a + c + e}{1 - b - e} d(x, Tx). \qquad (10)$$

Put $k := \min\{\frac{a+b+f}{1-c-f}, \frac{a+c+e}{1-b-e}\}$ so $k < 1$ and

$$d(Tx, T^2x) \leq kd(x, Tx). \qquad (11)$$

This inequality implies (1), and T has a fixed point. Uniqueness follows from (8). $\square$

For any $u \in X$, define $O_{ST}(u) := \{u, STu, (ST)^2u, \ldots, (ST)^nu, \ldots\}$ and

$$O_{T(ST)^n}(u) := \{Tu, T(ST)u, \ldots, T(ST)^nu, \ldots\}.$$

Theorem 2. *Let $(X, d), (Y, D)$ be cone metric spaces, $T : X \to Y, S : Y \to X$. Suppose that there exists a point $u \in X$ such that*

$$D(Tx, TSTx') \leq \lambda d(x, STx') \qquad (12)$$

and

$$d(Sy, STSy') \leq \lambda D(y, TSy') \qquad (13)$$

for each $x, x' \in O_{ST}(u), y, y' \in O_{T(ST)^n}(u)$, where λ is a real number satisfying $0 \leq \lambda < 1$. If $\overline{O_{ST}(u)}$ and $\overline{O_{T(ST)^n}}$ are complete, then $\{(ST)^nu\}$ converges to a point $p \in X$ and

$$d((ST)^nu, p) \leq \frac{\lambda^{2n}}{1 - \lambda} d(u, STu). \qquad (14)$$

Also, $\{T(ST)^nu\}$ converges to a point $q \in Y$ and

$$D(T(ST)^nu, q) \leq \frac{\lambda^{2n}}{1 - \lambda} D(Tu, TSu). \qquad (15)$$

Moreover, if (12) and (13) hold for all $x, x' \in \overline{O_{ST}(u)}, y, y' \in \overline{O_{T(ST)^n}(u)}$, then p is a fixed point of ST and q is a fixed point of TS.

Proof. For any positive integer n, using (12) and (13),

$$\begin{aligned} D(T(ST)^n u, T(ST)^{n+1}u) &\leq \lambda d((ST)^n u, (ST)^{n+1}u) \\ &= \lambda d(S(T(ST)^{n-1}u), S(T(ST)^n u)) \\ &\leq \lambda^2 D(T(ST)^{n-1}u, T(ST)^n u) \leq \cdots \\ &\leq \lambda^{2n} D(Tu, TSTu). \end{aligned}$$

Thus, for any positive integers m, n,

$$\begin{aligned} D(T(ST)^n u, T(ST)^{m+n}u) &\leq D(T(ST)^n u, T(ST)^{n+1}u) + \cdots \\ &\quad + D(T(ST)^{m+n-1}u, T(ST)^{m+n}u) \\ &\leq (\lambda^{2n} + \cdots + \lambda^{2n+m-1}) D(Tu, TSTu) \\ &\leq \frac{\lambda^{2n}}{1-\lambda} D(Tu, TSTu), \end{aligned} \tag{16}$$

and $\{T(ST)^n u\}$ is Cauchy. Since $\overline{O_{T(ST)^n}(u)}$ is complete, the sequence converges to a point $q \in Y$.

Using (16),

$$\begin{aligned} D(T(ST)^n u, q) &\leq D(T(ST)^n u, T(ST)^{n+m}u) + D(T(ST)^{n+m}u, q) \\ &\leq \frac{\lambda^{2n}}{1-\lambda} D(Tu, TSu) + D(T(ST)^{n+m}u, q). \end{aligned}$$

Let $c \in E, c \gg 0$. Then, for each integer m there exists a positive integer M_n such that, for all $m \geq M_m, D(T(St)^{n+m}u, q) \ll \frac{c}{m}$. We then have

$$D(T(ST)^n u, q) - \frac{\lambda^{2n}}{1-\lambda} D(Tu, TSu) \ll \frac{c}{m},$$

which implies that

$$\frac{c}{m} + \frac{\lambda^{2n}}{1-\lambda} D(Tu, TSu) - D(T(ST)^n u, q) \in P$$

for each m. Since $\lim \frac{c}{m} = 0$ and P is closed, one obtains

$$\frac{\lambda^{2n}}{1-\lambda} D(Tu, TSu) - D(T(ST)^n u, q) \in P,$$

which is equivalent to (15).

In a similar manner it can be shown that

$$d((ST)^n u, (ST)^{n+1}u) \leq \lambda^{2n} d(u, STu),$$

and hence $\{(ST)^n u\}$ converges to a point $p \in X$, and one obtains (14).

If (12) is true for all $x, x' \in \overline{O_{ST}(u)}$, then

$$D(Tp, T(ST)^n u) \leq \lambda d(p, (ST)^n u).$$

Therefore

$$\begin{aligned} D(Tp,q) &\leq D(Tp,T(ST)^n u) + D(T(ST)^n u,q) \\ &\leq \lambda d(p,(ST)^n u) + D(T(ST)^n u,q). \end{aligned}$$

If $\lambda \neq 0$, then for any $c \in E$, $c \gg 0$, choose N so that, for all $n \geq N$, $d(p,(ST)^n u) \ll \frac{c}{2\lambda}$ and $D(T(ST)^n u,q) \ll \frac{c}{2}$. Then $D(Tp,q) \ll c$, which implies that $Tp = q$. The case $\lambda = 0$ is trivial.

Similarly, if (13) is true for all $y, y' \in \overline{O_{T(ST)^n}(u)}$, then it can be shown that $Sq = p$.

Therefore $STp = Sq = p$ and p is a fixed point of ST. Also, $TSq = Tp = q$, and q is a fixed point of TS. $\square$

Corollary 5. *Let (X,d) and (Y,D) be complete cone metric spaces with $T : X \to Y, S : Y \to X$ mappings satisfying*

$$D(Tx,TSy) \leq c(d(x,Sy) + D(y,Tx) + D(y,TSy)) \tag{17}$$

and

$$d(Sy,STx) \leq c(D(y,Tx) + d(x,Sy) + d(x,STx)) \tag{18}$$

for all $x \in X, y \in Y$ and, for some $0 \leq c < 1/2$. Then ST has a unique fixed point $p \in X$ and TS has a unique fixed point $q \in Y$ such that $Tp = q$ and $Sq = p$.

Proof. Set $y = Tx$ in (17) to get

$$D(Tx,TSTx) \leq c(d(x,STx) + 0 + D(Tx,TSTx)),$$

which implies that

$$D(Tx,TSTx) \leq \frac{c}{1-c} d(x,STx),$$

and (12) of Theorem 2 is satisfied

Similarly, substituting $x = Sy$ into (18) yields

$$d(Sy,STSy) \leq \frac{c}{1-c} D(y,TSy),$$

and (13) of Theorem 2 is satisfied. Therefore there is a point $p \in X$ which is a fixed point of ST and a point $q \in Y$ which is a fixed point of TS. Using condition (17) gives uniqueness for p, and condition (18) gives uniqueness for q. $\square$

Corollary 6. *Let (X,d) and (Y,D) be complete cone metric spaces, $P \subset E$ with $T : X \to Y, S : Y \to X$ mappings and a constant $c \in [0,1)$ such that, for every $x \in X$ and $y \in Y$ there are nonzero elements $u \in \{d(x,Sy), D(y,Tx), D(y,TSy)\}$ and $v \in \{D(y,Ts), d(s,Sy), d(x,STx)\}$ such that*

$$D(Tx,TSy) \leq cu \tag{19}$$

and

$$d(Sy,STx) \leq cv. \tag{20}$$

Then ST has a unique fixed point $p \in X$ and TS has a unique fixed point $q \in Y$ such that $Tp = q$ and $Sq = p$.

Proof. In (19) set $y = Tx$ to obtain

$$D(Tx, TSTx) \leq cu, \tag{21}$$

where $u \in \{d(x, STx), 0, D(Sx, TSTx)\}$. If $D(Tx, TSTx) = 0$, then Tx is a fixed point of TS. If $D(Tx, TSTx) \neq 0$, then (21) implies that $D(Tx, TSx) \leq cd(x, STx)$, which implies (12).

In (20) set $x = Sy$ to get

$$d(Sy, STSy) \leq cv, \tag{22}$$

where $v \in \{D(y, TSy), 0, d(Sy, STSy)\}$. If $d(Sy, STSy) = 0$, then Sy is a fixed point of ST. Otherwise, (22) implies that $d(Sy, STSy) \leq cD(y, TSy)$, which implies (13).

By Theorem 2 there exists a fixed point p of ST in X and a fixed point q in Y of TS. The uniqueness of p and q follow from (19) and (20), respectively. □

Corollary 6 is the cone metric version of Theorem 1 of [6].

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