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Diagonal triangle of a non-tangential quadrilateral in the isotropic plane

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Abstract. Properties of the non-tangential quadrilateral $\mathcal{ABCD}$ in the isotropic plane concerning its diagonal triangle are given in this paper. A quadrilateral is called standard if a parabola with the equation $x = y^2$ is inscribed in it. Every non-tangential quadrilateral can be represented in the standard position. First, the vertices and the equations of the sides of the diagonal triangle are introduced. It is shown that the midlines of the diagonal triangle touch the inscribed parabola of the quadrilateral. Furthermore, quadrilaterals formed by two diagonals and some two sides of the non-tangential quadrilateral $\mathcal{ABCD}$ are studied and a few theorems on its foci are presented.

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Key words: isotropic plane, non-tangential quadrilateral, diagonal triangle, focus, median

1. Introduction

A projective plane with the absolute (in the sense of Cayley-Klein) consisting of a line ω and a point Ω on ω is called an *isotropic plane.* If $T = (x, y, z)$ denotes any point in the plane presented in homogeneous coordinates, then usually a projective coordinate system where $\Omega = (0, 1, 0)$ and the line ω with the equation $z = 0$ is chosen. The line ω is said to be the *absolute line* and the point Ω the *absolute point.* The points of the absolute line ω are called *isotropic points* and the lines incident with the absolute point Ω are called *isotropic lines.* Two lines are *parallel* if they have the same isotropic point, and two points are *parallel* if they are incident with the same isotropic line.

The conic $\mathcal{C}$ in the isotropic plane is a *circle* if the absolute line ω is tangent to it at the absolute point Ω, and it is a *parabola* if the absolute line ω is tangent to it at an isotropic point different from Ω.

All the notions related to the geometry of the isotropic plane can be found in [6] and [7].

A non-tangential quadrilateral in the isotropic plane has been introduced in [8]. The purpose of this paper is to investigate its properties concerning its diagonal triangle.

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The figure consisting of four lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ (any two of them not being parallel and any three of them not being concurrent) with their six points of intersection is called the *complete quadrilateral* $\mathcal{ABCD}$. Hence, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are the *sides* of the quadrilateral, $\mathcal{A}\cap\mathcal{B}, \mathcal{A}\cap\mathcal{C}, \mathcal{A}\cap\mathcal{D}, \mathcal{B}\cap\mathcal{C}$, $\mathcal{B}\cap\mathcal{D}$, and $\mathcal{C}\cap\mathcal{D}$ are the *vertices* where pairs of vertices $(\mathcal{A}\cap\mathcal{B}, \mathcal{C}\cap\mathcal{D})$; $(\mathcal{A}\cap\mathcal{C}, \mathcal{B}\cap\mathcal{D})$, and $(\mathcal{A}\cap\mathcal{D}, \mathcal{B}\cap\mathcal{C})$ are the *opposite* ones. There is a unique conic $\mathcal{K}$ that touches the lines $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and the absolute line ω. If $\mathcal{K}$ touches ω at the point Ω, then $\mathcal{K}$ is an isotropic circle and we say that $\mathcal{ABCD}$ is a *tangential quadrilateral*. In the case when $\mathcal{K}$ touches ω in the point different from Ω, $\mathcal{K}$ is a parabola and we say that $\mathcal{ABCD}$ is a *non-tangential quadrilateral*. The notion of the focus, the directrix, the axis and the diameters of this parabola are observed in [2].

As we have shown in [8], a non-tangential quadrilateral in the isotropic plane is called standard if a parabola with the equation $y^2 = x$ is inscribed in it. Every non-tangential quadrilateral can be represented in the standard position. Such a standard quadrilateral has sides of the form

$$\begin{aligned} \mathcal{A} \ldots 2ay &= x + a^2, \\ \mathcal{B} \ldots 2by &= x + b^2, \\ \mathcal{C} \ldots 2cy &= x + c^2, \\ \mathcal{D} \ldots 2dy &= x + d^2, \end{aligned} \tag{1}$$

where

$$A = (a^2, a), B = (b^2, b), C = (c^2, c), D = (d^2, d) \tag{2}$$

represent their points of contact with the parabola.

Vertices of such a quadrilateral are given with

$$\begin{aligned} \mathcal{A}\cap\mathcal{B} &= \left(ab, \frac{a+b}{2}\right), \quad \mathcal{C}\cap\mathcal{D} = \left(cd, \frac{c+d}{2}\right), \\ \mathcal{A}\cap\mathcal{C} &= \left(ac, \frac{a+c}{2}\right), \quad \mathcal{B}\cap\mathcal{D} = \left(bd, \frac{b+d}{2}\right), \\ \mathcal{A}\cap\mathcal{D} &= \left(ad, \frac{a+d}{2}\right), \quad \mathcal{B}\cap\mathcal{C} = \left(bc, \frac{b+c}{2}\right). \end{aligned} \tag{3}$$

As was the case in [8], the following symmetric functions of the numbers a, b, c, d will be of great benefit:

$$\begin{aligned} s &= a + b + c + d, \\ q &= ab + ac + ad + bc + bd + cd, \\ r &= abc + abd + acd + bcd, \\ p &= abcd. \end{aligned} \tag{4}$$

Two important notions related to the standard quadrilateral, a focus and a median, were introduced in [8], in the next two theorems:

Theorem 1 (see [8], p. 119)**.** *The midpoints of the line segments connecting the pairs of opposite vertices of the non-tangential quadrilateral lie on the line $\mathcal{M}$, with* *the equation*

$$y = \frac{s}{4}, \tag{5}$$

related to the standard quadrilateral $\mathcal{ABCD}$.

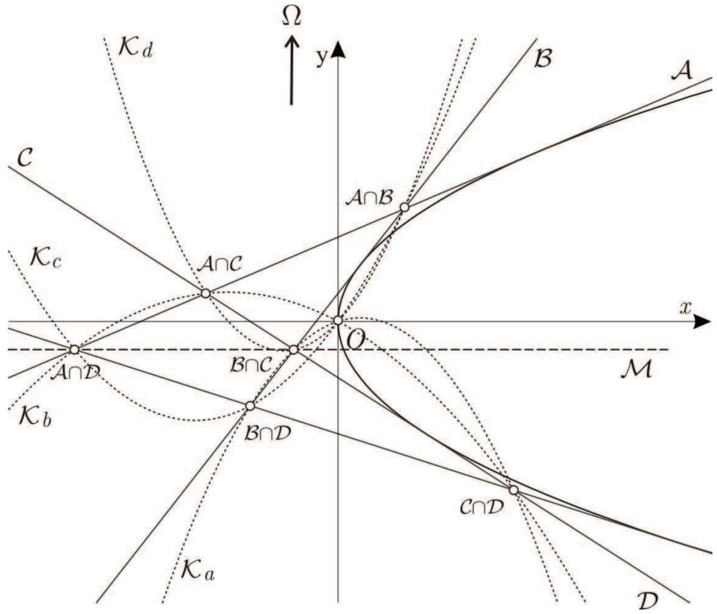

Figure 1: *The focus* O *and the median* $\mathcal{M}$ *and the circles* $\mathcal{K}_a, \mathcal{K}_b, \mathcal{K}_c$, *and* $\mathcal{K}_d$ *circumscribed to the triangles* $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, *and* $\mathcal{ABC}$ *of the standard quadrilateral* $\mathcal{ABCD}$

The line $\mathcal{M}$ from Theorem 1 is the *median* of the quadrilateral $\mathcal{ABCD}$ (see Figure 1).

Theorem 2 (see [8], p. 120)**.** *The circumscribed circles of the four triangles formed by the three sides of the non-tangential quadrilateral are incident with the same point* O, *which coincides with the focus of the parabola inscribed in this quadrilateral.*

The point O is the *focus* of the quadrilateral $\mathcal{ABCD}$ (see Figure 1).

2. Diagonal triangle of the non-tangential quadrilateral

In this section we will study the diagonal triangle of the non-tangential quadrilateral.

The joint lines of the opposite vertices $\mathcal{A}\cap\mathcal{B}, \mathcal{C}\cap\mathcal{D}$; $\mathcal{A}\cap\mathcal{C}, \mathcal{B}\cap\mathcal{D}$, and $\mathcal{A}\cap\mathcal{D}, \mathcal{B}\cap\mathcal{C}$ are the sides of the diagonal triangle and we will denote them by $\mathcal{T}_{ab,cd}$, $\mathcal{T}_{ac,bd}$, and $\mathcal{T}_{ad,bc}$, respectively. Furthermore, the vertices of the diagonal triangle will be represented by $T_{ad,bc} = \mathcal{T}_{ab,cd} \cap \mathcal{T}_{ac,bd}$, $T_{ac,bd} = \mathcal{T}_{ab,cd} \cap \mathcal{T}_{ad,bc}$, and $T_{ab,cd} = \mathcal{T}_{ac,bd} \cap \mathcal{T}_{ad,bc}$ (see Figure 2). The equations of the sides $\mathcal{T}_{ab,cd}$, $\mathcal{T}_{ac,bd}$, and $\mathcal{T}_{ad,bc}$ are given in the theorem that follows.

Theorem 3. *The diagonal triangle of the standard quadrilateral $\mathcal{ABCD}$ has the sides $\mathcal{T}_{ab,cd}$, $\mathcal{T}_{ac,bd}$ and $\mathcal{T}_{ad,bc}$ with the equations*

$$\begin{aligned} \mathcal{T}_{ab,cd}: &\quad 2(ab-cd)y = (a+b-c-d)x + ab(c+d) - cd(a+b), \\ \mathcal{T}_{ac,bd}: &\quad 2(ac-bd)y = (a+c-b-d)x + ac(b+d) - bd(a+c), \\ \mathcal{T}_{ad,bc}: &\quad 2(ad-bc)y = (a+d-b-c)x + ad(b+c) - bc(a+d). \end{aligned} \tag{6}$$

Proof. The equality

$$(ab-cd)(a+b) = (a+b-c-d)\cdot ab + ab(c+d) - cd(a+b)$$

shows that the point $\mathcal{A}\cap\mathcal{B}$ is incident with $\mathcal{T}_{ab,cd}$. □

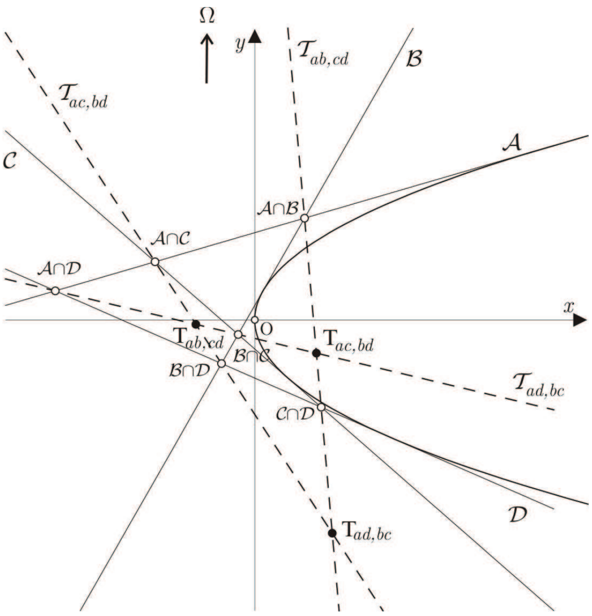

Figure 2: *The diagonal triangle $\mathcal{T}_{ab,cd}\mathcal{T}_{ac,bd}\mathcal{T}_{ad,bc}$ of the standard quadrilateral $\mathcal{ABCD}$*

The coordinates of the vertices $T_{ab,cd}$, $T_{ac,bd}$ and $T_{ad,bc}$ of the diagonal triangle are obtained by means of Theorem 4.

Theorem 4. *The vertices $T_{ab,cd}$, $T_{ac,bd}$ and $T_{ad,bc}$ of the diagonal triangle of the standard quadrilateral $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ have the forms*

$$\begin{aligned} T_{ab,cd} &= \left(\frac{ab(c+d)-cd(a+b)}{a+b-c-d}, \frac{ab-cd}{a+b-c-d}\right), \\ T_{ac,bd} &= \left(\frac{ac(b+d)-bd(a+c)}{a+c-b-d}, \frac{ac-bd}{a+c-b-d}\right), \\ T_{ad,bc} &= \left(\frac{ad(b+c)-bc(a+d)}{a+d-b-c}, \frac{ad-bc}{a+d-b-c}\right). \end{aligned} \tag{7}$$

Proof. From the equality

$$2(ac-bd)(ab-cd) = (a+c-b-d)[ab(c+d)-cd(a+b)]+[ac(b+d)-bd(a+c)](a+b-c-d)$$

it follows that the point $T_{ab,cd}$ from (7) is incident with the line $\mathcal{T}_{ac,bd}$ from (6). $\square$

Theorem 5. *The circle circumscribed to the diagonal triangle has the equation of the form*

$$\begin{aligned} 4(ab-cd)(ac-bd)(ad-bc)y = {} & (a+b-c-d)(a+c-b-d)(a+d-b-c)x^2 \\ & +s(ab-cd)(ac-bd)(ad-bc) \\ & -p(a+b-c-d)(a+c-b-d)(a+d-b-c). \end{aligned} \tag{8}$$

Proof. It can be easily shown that e.g. the point $T_{ad,bc}$ satisfies equation (8). $\square$

Onwards we present the results concerning the diagonal triangle.
For the Euclidean version of the next theorem see [5].

Theorem 6. *Midlines of the diagonal triangle of the quadrilateral $\mathcal{ABCD}$ touch its inscribed parabola with the equation $y^2 = x$.*

Proof. Let us denote the midline of $\mathcal{T}_{ac,bd}$ and $\mathcal{T}_{ad,bc}$ by $\mathcal{M}_{ab,cd}$. The equation of this line is of the form

$$2(ab-cd)(a+b-c-d)y = (a+b-c-d)^2x + (ab-cd)^2.$$

If we construct a line through the vertex $T_{ab,cd}$ parallel to the side $\mathcal{T}_{ab,cd}$, its equation is

$$\begin{aligned} 2(ab-cd)(a+b-c-d)y = {} & (a+b-c-d)^2x + 2(ab-cd)^2 + ab(c+d)^2 \\ & +cd(a+b)^2 - (ab+cd)(a+b)(c+d). \end{aligned} \tag{9}$$

It is easily verified that the coordinates of the point $T_{ab,cd}$ satisfy the equation, as given above, of the line that has the same slope as the side $\mathcal{T}_{ab,cd}$. The first equation in (6) is the equation of $\mathcal{T}_{ab,cd}$. Multiplying this equality by $(a+b-c-d)$ and summing it and (9) up, the equation of the midline $\mathcal{M}_{ab,cd}$ is obtained.

As the equality

$$(a+b-c-d)^2y^2 - 2(ab-cd)(a+b-c-d)y + (ab-cd)^2 = 0$$

holds, the line $\mathcal{M}_{ab,cd}$ touches the inscribed parabola $y^2 = x$. Obviously, the point of contact $M_{ab,cd}$ of the midline $\mathcal{M}_{ab,cd}$ and the parabola $y^2 = x$ is of the form

$$\left(\frac{(ab-cd)^2}{(a+b-c-d)^2}, \frac{ab-cd}{a+b-c-d}\right).$$

$\square$

Corollary 1. *The points $M_{ab,cd}$ and $T_{ab,cd}$ are incident with the line having the equation*

$$y = \frac{ab-cd}{a+b-c-d}.$$

Another interesting feature is given in the next theorem. First, let us recall from [6] the notion of span: for two parallel points $T_1 = (x, y_1)$ and $T_2 = (x, y_2)$, $d(T_1, T_2) := y_2 - y_1$ defines the *span* between them.

Theorem 7. *There is an isotropic line such that the distances from the midpoints of the diagonals of the quadrilateral to this line are proportional to the squares of the spans of these diagonals. The equation of this line is*

$$x = \frac{s^2}{8} - \frac{q}{6}.$$

Proof. If we take e.g. the diagonal $\mathcal{T}_{ab,cd}$, then its midpoint is $P_1 = (\frac{ab+cd}{2}, \frac{s}{2})$. Analogously, the midpoints $P_2 = (\frac{ac+bd}{2}, \frac{s}{2})$ and $P_3 = (\frac{ad+bc}{2}, \frac{s}{2})$ are obtained for another two diagonals $\mathcal{T}_{ac,bd}$ and $\mathcal{T}_{ad,bc}$.
Let $T(x, y)$ be any point on the line that we are looking for. Because of the mentioned proportion there is $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} x - \frac{ab+cd}{2} &= \lambda(a+b-c-d)^2, \\ x - \frac{ac+bd}{2} &= \lambda(a+c-b-d)^2, \\ x - \frac{ad+bc}{2} &= \lambda(a+d-b-c)^2 \end{aligned} \tag{10}$$

is valid. Subtracting the first two equations we get

$$ab + cd - ac - bd = 8\lambda(a-d)(b-c),$$

out of which we obtain

$$\lambda = \frac{1}{8}.$$

Applying $\lambda = \frac{1}{8}$ in (10) and adding them up the equation

$$6x = q + \frac{3s^2 - 8q}{4}$$

is obtained, that yields

$$x = \frac{s^2}{8} - \frac{q}{6}.$$

The equation of the line that we are looking for is of this form. □

Prior to researching the properties of the diagonal triangle let us introduce some notations. Let $M_{ab,cd}$ denote the midpoint of the line segment joining $\mathcal{A}\cap\mathcal{B}$ to $\mathcal{C}\cap\mathcal{D}$ and by analogy, let $M_{ab,bd}$ denote the midpoint of the one joining $\mathcal{A}\cap\mathcal{B}$ to $\mathcal{B}\cap\mathcal{D}$. The six points M_{bc}; M_{ad}; M_{ac}; M_{bd}; M_{ab}, and M_{cd} represent foci of the quadrilaterals $\mathcal{T}_{ac,bd}\mathcal{T}_{ab,cd}\mathcal{B}\mathcal{C}$; $\mathcal{T}_{ac,bd}\mathcal{T}_{ab,cd}\mathcal{A}\mathcal{D}$; $\mathcal{T}_{ad,bc}\mathcal{T}_{ab,cd}\mathcal{A}\mathcal{C}$; $\mathcal{T}_{ad,bc}\mathcal{T}_{ab,cd}\mathcal{B}\mathcal{D}$; $\mathcal{T}_{ad,bc}\mathcal{T}_{ac,bd}\mathcal{A}\mathcal{B}$, and $\mathcal{T}_{ad,bc}\mathcal{T}_{ac,bd}\mathcal{C}\mathcal{D}$.

Concerning these points the next theorem states

Theorem 8. *The circle passing through the points $\mathcal{B}\cap\mathcal{C}$, $M_{ab,bd}$, $M_{ac,cd}$ pass through its focus M_{bc} and the focus of the non-tangential quadrilateral $\mathcal{ABCD}$. The same statement holds for the circles passing through the other five triplets of corresponding points and the corresponding foci.*

For the Euclidean version of this theorem see [3].

Proof. Let us study for example the quadrilateral $\mathcal{T}_{ac,bd}\mathcal{T}_{ab,cd}\mathcal{BC}$. We have to find its focus M_{bc}. The circle passing through $\mathcal{B}\cap\mathcal{C}$, $M_{ab,bd}$ and $M_{ac,cd}$ has the equation

$$2bc(a+d)y = -2x^2 + [2bc + (a+d)(b+c)]x. \tag{11}$$

The point $\mathcal{B}\cap\mathcal{C}$ is given in (3) while $M_{ab,bd}, M_{ac,cd}$ have coordinates of the forms

$$\begin{aligned} M_{ab,bd} &= (\frac{b(a+d)}{2}, \frac{a+d+2b}{4}),\\ M_{ac,cd} &= (\frac{c(a+d)}{2}, \frac{a+d+2c}{4}). \end{aligned}$$

The equalities

$$\begin{aligned} bc(a+d)(b+c) &= -2b^2c^2 + [2bc + (a+d)(b+c)]bc,\\ bc(a+d)(a+d+2b) &= -b^2(a+d)^2 + b(a+d)[2bc + (a+d)(b+c)] \end{aligned}$$

verify that the coordinates of the points $\mathcal{B}\cap\mathcal{C}$ and $M_{ab,bd}$ satisfy equation (11). Clearly, the circle (11) is incident with the focus of the quadrilateral $O = (0,0)$, as well.

The vertices of the quadrilateral $\mathcal{T}_{ac,bd}\mathcal{T}_{ab,cd}\mathcal{BC}$ are

$$T_{ad,bc} = \left(\frac{ad(b+c) - bc(a+d)}{a+d-b-c}, \frac{ad-bc}{a+d-b-c}\right), \mathcal{A}\cap\mathcal{B}, \mathcal{A}\cap\mathcal{C}, \mathcal{B}\cap\mathcal{C}, \mathcal{B}\cap\mathcal{D},$$

and $\mathcal{C}\cap\mathcal{D}$.

Let us observe a circle circumscribed to the vertices of the triangle $\mathcal{BC}\mathcal{T}_{ac,bd}$. The vertices are $\mathcal{A}\cap\mathcal{C}$, $\mathcal{B}\cap\mathcal{C}$, and $\mathcal{B}\cap\mathcal{D}$ and their circumscribed circle is of the form

$$2bc(ac-bd)y = (b-c)x^2 + [c^2(a+b) - b^2(c+d)]x + b^2c^2(a-d). \tag{12}$$

Because of

$$bc(ac-bd)(a+c) = (b-c)a^2c^2 + [c^2(a+b) - b^2(c+d)]ac + b^2c^2(a-d),$$

the point $\mathcal{A}\cap\mathcal{C}$ is incident with the circle (12) and moreover, out of

$$bc(ac-bd)(b+c) = (b-c)b^2c^2 + [c^2(a+b) - b^2(c+d)]bc + b^2c^2(a-d)$$

follows that $\mathcal{B}\cap\mathcal{C}$ lies on (12) as well.

The circumscribed circle of the triangle $\mathcal{BC}\mathcal{T}_{ab,cd}$ is

$$2bc(ab-cd)y = (c-b)x^2 + [b^2(a+c) - c^2(b+d)]x + b^2c^2(a-d). \tag{13}$$

Therefore, the circles (12) and (13) intersect in two points, $\mathcal{B} \cap \mathcal{C}$ and the focus

$$M_{bc} = \left(\frac{bc(a+d)}{b+c}, \frac{2bc(b+c)+(a+d)(b^2+c^2)}{2(b+c)^2}\right)$$

of this quadrilateral. Out of

$$\begin{aligned} &2b^2c^2(ab-cd)(b+c)+bc(ab-cd)(a+d)(b^2+c^2)\\ &\quad = b^2c^2(a+d)^2(c-b)+bc(a+d)(b+c)[b^2(a+c)\\ &\quad\quad -c^2(b+d)]+b^2c^2(a-d)(b+c)^2 \end{aligned}$$

follows that M_{bc} lies on (13). The argument can be applied for (11) as well. □

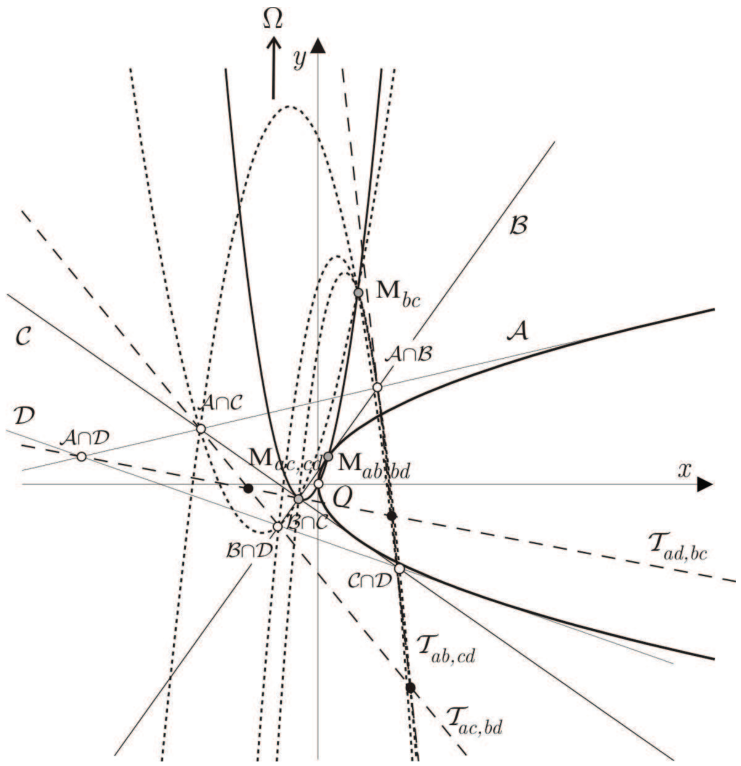

Figure 3: *Visualization of the proof of Theorem 8*

Visualization of Theorem 8 is given in Figure 3.

Theorem 9. *The circle that passes through $T_{ad,bc}$, $M_{ac,bd}$, and $M_{ab,cd}$ is incident with M_{bc} and M_{ad}, foci of the quadrilaterals $\mathcal{T}_{ab,cd}\mathcal{T}_{ac,bd}\mathcal{BC}$ and $\mathcal{T}_{ab,cd}\mathcal{T}_{ac,bd}\mathcal{AD}$. The equation of the circle passing through $T_{ad,bc}, M_{ac,bd}, M_{ab,cd}$ is of the form*

$$\begin{aligned} 4(ac-bd)(ab-cd)y = &-4(a+d-b-c)x^2+2(a+d)(b+c)(a+d-b-c)x\\ &+s(ac-bd)(ab-cd) \qquad (14)\\ &-(a+d-b-c)(ab+cd)(ac+bd). \end{aligned}$$

Proof. $T_{ad,bc}$ is given in (7). It is not difficult to check that its coordinates satisfy equation (14). In the same way, the equality

$$\begin{aligned} s(ac-bd)(ab-cd) = {} & (a+d)(b+c)(ac+bd)(a+d-b-c) \\ & -(ac+bd)^2(a-b-c+d) + s(ac-bd)(ab-cd) - \\ & -(a-b-c+d)(ac+bd)(ab+cd) \end{aligned}$$

proves that $M_{ac,bd}$ is incident with that circle as well.
Finally, by putting the coordinates of the point M_{bc} in (14) we get the equality

$$\begin{aligned} & \frac{2(ac-bd)(ab-cd)(2bc(b+c)+(b^2+c^2)(a+d))}{(b+c)^2} \\ & = 2bc(a+d)^2(a-b-c+d) - \frac{4b^2c^2(a+d)^2(a+d-b-c)}{(b+c)^2} \\ & \quad +s(ac-bd)(ab-cd)-(a-b-c+d)(ac+bd)(ab+cd), \end{aligned}$$

which proves the claim of the theorem. □

Theorem 10. *The circles passing through the triples of points $M_{ad}, M_{bc}, T_{ad,bc}$; M_{ac}, M_{bd}, $T_{ac,bd}$, and $M_{ab}, M_{cd}, T_{ab,cd}$ intersect in a point that the circle circumscribed to the diagonal triangle $T_{ab,cd}T_{ac,bd}T_{ad,bc}$ is incident with.*

Proof. The circle passing through $M_{ad}, M_{bc}, T_{ad,bc}$ is of the form

$$\begin{aligned} \mathcal{K}_{ad,bc}:\ 2(ab-cd)(ac-bd)y = {} & -2(a+d-b-c)x^2 + (a+d-b-c)(a+d)(b+c)x \\ & +a^2b^2(c-d) - c^2d^2(a-b) \\ & +a^2c^2(b-d) - b^2d^2(a-c). \end{aligned} \tag{15}$$

Indeed, inserting the coordinates of $M_{ad}, M_{bc}, T_{ad,bc}$ in (15), respectively, one gets three valid equalities. Because of the symmetry on a, b, c, and d the circles from the theorem are of the form

$$\begin{aligned} \mathcal{K}_{ac,bd}:\ 2(ab-cd)(ad-bc)y = {} & -2(a+c-b-d)x^2 \\ & +(a+c-b-d)(a+c)(b+d)x \\ & -a^2b^2(c-d) - c^2d^2(a-b) \\ & +a^2d^2(b-c) - b^2c^2(a-d), \\ \mathcal{K}_{ab,cd}:\ 2(ac-bd)(ad-bc)y = {} & -2(a+b-c-d)x^2 \\ & +(a+b-c-d)(a+b)(c+d)x \\ & -a^2d^2(b-c) - b^2c^2(a-d) \\ & -a^2c^2(b-d) - b^2d^2(a-c). \end{aligned}$$

These three circles intersect in the point

$$P = \left(\frac{r}{s}, \frac{sq-2r}{s^2}\right).$$

As a matter of fact

$$\begin{aligned}2(ab-cd)(ac-bd)(sq-2r) &= -2(a-b-c+d)r^2\\ &\quad +(a-b-c+d)(a+d)(b+c)rs\\ &\quad +s^2[a^2b^2(c-d)-c^2d^2(a-b)\\ &\quad +a^2c^2(b-d)-b^2d^2(a-c)]\end{aligned}$$

proves that the point P is incident with e.g. $\mathcal{K}_{ad,bc}$. By direct computation

$$\begin{aligned}&4(ab-cd)(ac-bd)(ad-bc)(sq-2r)\\ &\quad = r^2(a+b-c-d)(a+c-b-d)(a-b-c+d)\\ &\qquad +s^3(ab-cd)(ac-bd)(ad-bc)\\ &\qquad -ps^2(a+b-c-d)(a+c-b-d)(a-b-c+d)\end{aligned}$$

we see that the point P satisfies the equation of the circumscribed circle of the diagonal triangle (8). □

Corollary 2. *The point of intersection from Theorem 10 has the form*

$$P=(\frac{r}{s},\frac{sq-2r}{s^2}).$$

Theorem 11. *Circles passing through $O, M_{bc}, \mathcal{B}\cap\mathcal{C}$; $O, M_{ad}, \mathcal{A}\cap\mathcal{D}$, and $M_{ad}, M_{bc}, T_{ad,bc}$ have a common point of intersection; there are two more such sets of circles.*

Proof. The circle that passes through $O, M_{bc}, \mathcal{B}\cap\mathcal{C}$ is of the form

$$2bc(a+d)y=-2x^2+[(a+d)(b+c)+2bc]x.$$

It is easy to show that the point $\mathcal{B}\cap\mathcal{C}$ is incident with this circle, and

$$bc(a+d)^2(b^2+c^2)=-2b^2c^2(a+d)^2+bc(a+d)^2(b+c)^2$$

verifies that coordinates of M_{bc} satisfy the equation of the upper circle as well. Three circles from the theorem intersect in the point

$$R=(\frac{1}{2}(a+d)(b+c)+p\frac{a-b-c+d}{bc(a+d)-ad(b+c)},p\frac{(bc-ad)(a-b-c+d)}{[bc(a+d)-ad(b+c)]^2}).$$

As

$$\begin{aligned}&2bc(a+d)(\frac{p(a-b-c+d)(bc-ad)}{[bc(a+d)-ad(b+c)]^2}+\frac{(a+d)(b+c)(bc-ad)}{2[bc(a+d)-ad(b+c)]})\\ &\quad =(2bc+(a+d)(b+c))(\frac{1}{2}(b+c)(a+d)+\frac{p(a-b-c+d)}{bc(a+d)-ad(b+c)})\\ &\qquad -2(\frac{1}{2}(a+d)(b+c)+\frac{p(a-b-c+d)}{bc(a+d)-ad(b+c)})^2\end{aligned}$$

holds, R is incident with the circle passing through the points O, M_{bc}, and $\mathcal{B} \cap \mathcal{C}$. Because of

$$\begin{aligned} &2(ab-cd)(ac-bd)(\frac{p(a-b-c+d)(bc-ad)}{[bc(a+d)-ad(b+c)]^2}+\frac{(a+d)(b+c)(bc-ad)}{2[bc(a+d)-ad(b+c)]}) \\ &= a^2c^2(b-d)+a^2b^2(c-d)-b^2d^2(a-c)-c^2d^2(a-b) \\ &\quad +(a-b-c+d)(ab+ac+bd+cd)(\frac{1}{2}(a+d)(b+c)+\frac{p(a-b-c+d)}{bc(a+d)-ad(b+c)}) \\ &\quad -2(a-b-c+d)(\frac{1}{2}(a+d)(b+c)+\frac{p(a-b-c+d)}{bc(a+d)-ad(b+c)})^2, \end{aligned}$$

R also lies on the circle (15). □

3. Conclusions

In [8] we have studied the geometry of a non-tangential quadrilateral in the isotropic plane. Before that, the geometry of a quadrilateral has not been developed at all. It arised from the way the geometry of a triangle was studied in [1] and [4]. Hence, in [8] an elegant standardization of a non-tangential quadrilateral is given that helps to prove the properties of any non-tangential quadrilateral in the isotropic plane. Several properties related to the focus and the median of the non-tangential quadrilateral are represented.

This paper naturally follows the research given in [8]. The diagonal triangle is an important element of the non-tangential quadrilateral. Throughout the paper we prove several geometrical facts for the diagonal triangle valid in the isotropic plane. Being of great interest to us, we have studied special quadrilaterals formed by two diagonals and some two sides of the non-tangential quadrilateral. The results of [8] has helped us to obtain the forms of foci of these quadrilaterals in order to show some interesting properties related to them (Theorem 8, Theorem 9, and Theorem 10).

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