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Equi-ideal convergence of positive linear operators for analytic p-ideals

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Abstract. In this paper, using equi-ideal convergence, we introduce a non-trivial generalization of the classical and the statistical cases of the Korovkin approximation theorem. We also compute the rates of equi-ideal convergence of sequences of positive linear operators. Furthermore, we obtain a Voronovskaya-type theorem in the equi-ideal sense for a sequence of positive linear operators constructed by means of the Meyer-König and Zeller polynomials.

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1. Introduction

A generalization of statistical convergence is based on the structure of the ideal $\mathcal{I}$ of subsets of $\mathbb{N}$, the set of natural numbers. A non-void class $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called the ideal if $\mathcal{I}$ is additive (i.e., $A,B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$) and hereditary (i.e., $A \in \mathcal{I}$, $B \subset A \Rightarrow B \in \mathcal{I}$). Throughout in this paper we consider ideals which are different from $\mathcal{P}(\mathbb{N})$ and contain all finite sets. Equip $\mathcal{P}(\mathbb{N})$ with the Cantor space topology, identifying subsets of $\mathbb{N}$ with their characteristic functions. The ideal which consists of all finite sets is denoted by Fin. An ideal $\mathcal{I}$ is a P-ideal if for every sequence $(A_n)_{n\in\mathbb{N}}$ of sets from $\mathcal{I}$ there is an $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n. Also, an ideal $\mathcal{I}$ is analytic if it is a continuous image of a G_δ subset of the Cantor-space.

A map $\phi : \mathcal{P}(\mathbb{N}) \to [0,\infty]$ is a submeasure on $\mathbb{N}$ if for all $A, B \subset \mathbb{N}$,

$$\phi(\emptyset) = 0,$$
$$\phi(A) \le \phi(A \cup B) \le \phi(A) + \phi(B).$$

It is lower semicontinuous if for all $A \subset \mathbb{N}$, we have

$$\phi(A) = \lim_{n\to\infty} \phi(A \cap n).$$

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For any lower semicontinuous submeasure on $\mathbb{N}$, let $\|A\|_{\phi} : \mathcal{P}(\mathbb{N}) \to [0,\infty]$ be a submeasure defined by

$$\|A\|_{\phi} = \limsup_{n\to\infty} \phi\left(A \setminus n\right) = \lim_{n\to\infty} \phi\left(A \setminus n\right),$$

where the second equality follows by the monotonicity of ϕ. Let

$$Exh\left(\phi\right) = \left\{A \subseteq \mathbb{N} : \|A\|_{\phi} = 0\right\},$$
$$Fin(\phi) = \left\{A \subseteq \mathbb{N} : \phi\left(A\right) < \infty\right\}.$$

It is clear that $Exh\left(\phi\right)$ and $Fin(\phi)$ are ideals (not necessarily proper) for an arbitrary submeasure ϕ (for detail, see [14], [15]). All analytic P-ideals are characterized by Solecki [15] as follows:

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. $\mathcal{I}$ is an analytic P-ideal iff $\mathcal{I} = Exh\left(\phi\right)$ for some lower semicontinuous submeasure ϕ on $\mathbb{N}$.

Let us introduce the following examples of analytic P-ideals [16], (see [9] for more examples).

• A nontrivial analytic P-ideal is the ideal of sets of statistical density zero, i.e.

$$\mathcal{I}_d = \left\{A \subset \mathbb{N} : \limsup_{j\to\infty} \; d_j\left(A\right) = 0\right\},$$

where $d_j\left(A\right) = \frac{|A\cap j|}{j}$ is the jth partial density of A, where the symbol $|B|$ denotes the cardinality of the set B. If we denote $\phi_d(A) = \sup\left\{\frac{|A\cap j|}{j} : j \in \mathbb{N}\right\}$, then $\mathcal{I}_d = Exh\left(\phi_d\right)$.

• Let

$$\mathcal{I}_{\frac{1}{n}} = \left\{A \subseteq \mathbb{N} : \sum_{n\in A} \frac{1}{n+1} < \infty\right\}.$$

If ϕ is a submeasure defined by $\phi(A) = \sum\limits_{n\in A} \frac{1}{n+1}$, then $\mathcal{I}_{\frac{1}{n}} = Fin(\phi)$.

Recently various kinds of ideal convergence (equi-ideal convergence), which is an extension of equi-statistical convergence to the class of all analytic P-ideals for sequences of functions, have been introduced by Mrożek [14].

An analytic P-ideal on $\mathbb{N}$ need not be determined by a unique lower semicontinuous submeasure ϕ on $\mathbb{N}$. Mrożek proved that equi-ideal convergence does not depend on the choice of ϕ ([14], Prop. 2.1), and he observed that a similar property holds for pointwise and uniform ideal convergence. This fact will be used in the proof of Theorem 2.1 where a fixed function ϕ associated with an ideal $\mathcal{I}$ is considered. We first recall these convergence methods.

Let f and f_n belong to $C(X)$, which is the space of all continuous real valued function on a compact subset X of the real numbers. Throughout the paper, we use the following notations.

$$\Psi(x,\varepsilon) := \left\{n \in \mathbb{N} : \; |f_n(x) - f(x)| \geq \varepsilon\right\}, (x \in X) \tag{1}$$
$$\Phi(\varepsilon) := \left\{n \in \mathbb{N} : \; ||f_n - f||_{C(X)} \geq \varepsilon\right\},$$

where $\varepsilon > 0$ and $||f||_{C(X)}$ denotes the usual supremum norm of f in $C(X)$.

Definition 1 (see [14]). *Let $\mathcal{I}$ be an analytic P-ideal on $\mathbb{N}$ with $\mathcal{I} = Exh\,(\phi)$ for a lower semicontinuouos submeasure ϕ on $\mathbb{N}$. (f_n) is said to be pointwise ideal convergent to f on X if for every $\varepsilon > 0$ and for each $x \in X$, $\lim_{k\to\infty} \phi\,(\Psi(x,\varepsilon) \setminus k) = 0$. In this case we write $f_n \to_{\mathcal{I}} f(ideal)$ on X.*

Definition 2 (see [14]). *Let $\mathcal{I}$ be an analytic P-ideal on $\mathbb{N}$ with $\mathcal{I} = Exh\,(\phi)$ for a lower semicontinuouos submeasure ϕ on $\mathbb{N}$. (f_n) is said to be equi-ideal convergent to f on X if for every $\varepsilon > 0$,*

$$\lim_{k\to\infty} \phi\,(\Psi(x,\varepsilon) \setminus k) = 0$$

uniformly with respect to $x \in X$. In this case we write $f_n \to_{\mathcal{I}} f$ $(equi - ideal)$ on X.

Definition 3 (see [14]). *Let $\mathcal{I}$ be an analytic P-ideal on $\mathbb{N}$ with $\mathcal{I} = Exh\,(\phi)$ for a lower semicontinuouos submeasure ϕ on $\mathbb{N}$. (f_n) is said to be uniform ideal convergent to f on X if for every $\varepsilon > 0$, $\lim_{k\to\infty} \phi\,(\Phi(\varepsilon) \setminus k) = 0$. In this case we write $f_n \rightrightarrows_{\mathcal{I}} f$ $(ideal)$ on X.*

Using the definitions, the next result follows immediately.

Lemma 1. *Let $\mathcal{I}$ be an analytic P-ideal on $\mathbb{N}$ with $\mathcal{I} = Exh\,(\phi)$ for a lower semicontinuouos submeasure ϕ on $\mathbb{N}$. $f_n \rightrightarrows f$ on X implies $f_n \rightrightarrows_{\mathcal{I}} f$ $(ideal)$ on X, which also implies $f_n \to_{\mathcal{I}} f$ $(equi - ideal)$ on X. Furthermore, $f_n \to_{\mathcal{I}} f$ $(equi - ideal)$ on X implies $f_n \to_{\mathcal{I}} f$ $(ideal)$ on X, and $f_n \to f$ on X (in the ordinary sense) implies $f_n \to_{\mathcal{I}} f$ $(ideal)$ on X.*

Definition 4 (see [2]). *(f_n) is said to be equi-statistically convergent to f on X if $\forall \varepsilon > 0$, $\lim_{n\to\infty} \frac{|\Psi(x,\varepsilon)|}{n} = 0$ uniformly with respect to $x \in X$. In this case we write $f_n \to f\,(equi - stat)$ on X.*

Definition 5 (see [11]). *(f_n) is said to be statistically uniform convergent to f on X if $\forall \varepsilon > 0$, $\lim_{n\to\infty} \frac{|\Phi(\varepsilon)|}{n} = 0$. In this case we write $f_n \rightrightarrows f$ $(stat)$ on X.*

However, one can construct examples which guarantee that the converses of Lemma 1 are not always true. Such an example was given Balcerzak et al. [2] as follows.

Example 1. *Let $X = [0,1]$ and h is a function by $h\,(x) = 0$ for $x \in [0,1]$. For each $n \in \mathbb{N}$, define $h_n \in C\,[0,1]$ by*

$$h_n\,(x) = \begin{cases} 2^{n+1}\left(x - \frac{1}{2^n}\right) & , \ \text{if } x \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} - \frac{1}{2^{n+1}}\right], \\ -2^{n+1}\left(x - \frac{1}{2^{n-1}}\right) & , \ \text{if } x \in \left[\frac{1}{2^{n-1}} - \frac{1}{2^{n+1}}, \frac{1}{2^{n-1}}\right], \\ 0 & , \ \text{otherwise.} \end{cases}$$

Then it is easy to show that h_n is equi-ideal (equi-statistical) convergent to h on X with respect to the ideal $\mathcal{I}_d$. But (h_n) is not uniform ideal (statistical uniform) convergent and uniform convergent to the function $h = 0$ on X.

The classical Korovkin theory is mainly connected with the approximation of continuous functions by means of positive linear operators (see, for instance [1, 12]). In recent years, with the help of the concept of statistical convergence [10], various statistical approximation results have been proved (see [5, 6, 7, 8, 11]).

2. A Korovkin-type approximation theorem

In this section, using a similar technique in the proof of Theorem 2.1 in [11], we give a Korovkin-type theorem for sequences of positive linear operators defined on $C(X)$ using the concept of equi-ideal convergence.

Theorem 1. *Let X be a compact subset of the real numbers, and let $\{L_n\}$ be a sequence of positive linear operators acting from $C(X)$ into itself. Assume that $\mathcal{I}$ is an analytic P-ideal on $\mathbb{N}$ with $\mathcal{I} = Exh(\phi)$ for a lower semicontinuouos submeasure ϕ on $\mathbb{N}$. Then for all $f \in C(X)$,*

$$L_n(f) \to_{\mathcal{I}} f \ (equi - ideal) \ on \ X \ , \tag{2}$$

if and only if

$$L_n(e_i) \to_{\mathcal{I}} e_i(equi - ideal) \ on \ X \ with \ e_i(x) = x^i, \ i = 0, 1, 2. \tag{3}$$

Proof. Since each $e_i \in C(X)$, i = 0,1,2, the implication (2)⇒(3) is obvious. Assume now that (3) holds. Since f is bounded on X, we can write

$$|f(x)| \le M,$$

where $M = \|f\|_{C(X)}$. Also, since f is continuous on X, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ for all $x \in X$ satisfying $|t - x| < \delta$. Hence, we get

$$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2}(t - x)^2 . \tag{4}$$

Since L_n is linear and positive, we obtain

$$\begin{aligned} |L_n(f; x) - f(x)| &\le L_n(|f(t) - f(x)|; x) + M |L_n(e_0; x) - e_0(x)| \\ &\le \left| L_n\left(\varepsilon + \frac{2M}{\delta^2}(t - x)^2; x\right)\right| + M |L_n(e_0; x) - e_0(x)| \\ &\le \varepsilon + \left(\varepsilon + M + \frac{2Mx^2}{\delta^2}\right) |L_n(e_0; x) - e_0(x)| \\ &\quad + \frac{4Mx}{\delta^2} |L_n(e_1; x) - e_1(x)| + \frac{2M}{\delta^2} |L_n(e_2; x) - e_2(x)|, \end{aligned}$$

which implies that

$$|L_n(f; x) - f(x)| \le \varepsilon + N \sum_{i=0}^{2} |L_n(e_i; x) - e_i(x)|, \tag{5}$$

where $N := \varepsilon + M + \frac{2M}{\delta^2}\left(||e_2||_{C(X)} + 2\,||e_1||_{C(X)} + 1\right)$. Now, for a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. Then define

$$\Psi(x, r) = \{n \in \mathbb{N} : \ |L_n(f; x) - f(x)| \ge r\}$$

and

$$\Psi_i(x, \frac{r-\varepsilon}{3N}) := \left\{ n \in \mathbb{N} : \ |L_n(e_i; x) - e_i(x)| \geq \frac{r-\varepsilon}{3N} \right\} (i = 0, 1, 2).$$

It is easy to see that $\Psi(x,r) \subset \bigcup_{i=0}^{2} \Psi_i(x, \frac{r-\varepsilon}{3N})$. Thus, from the monotonicity of ϕ, it follows from (5) that

$$\begin{aligned} \phi\left(\Psi(x,r) \setminus k\right) &\leq \phi\left(\left[\bigcup_{i=0}^{2} \Psi_i(x, \frac{r-\varepsilon}{3N})\right] \setminus k\right) \\ &\leq \sum_{i=0}^{2} \phi\left(\Psi_i(x, \frac{r-\varepsilon}{3N}) \setminus k\right). \end{aligned} \tag{6}$$

Then using the hypothesis (3) and considering Definition 2, the right-hand side of (6) tends to zero as $k \to \infty$. The proof is completed. □

3. Remarks

1. If we take $\mathcal{I}_d = Exh\,(\phi)$ where $\phi\,(A) = \sup_{j\in\mathbb{N}} d_j\,(A)$ and $Fin = Exh\,(\phi)$ where

$$\phi\,(A) = \begin{cases} |A|\ , & \text{if } A \text{ is finite,} \\ \infty\ , & \text{if } A \text{ is infinite,} \end{cases}$$

then equi-ideal convergence is reduced to equi-statistical convergence and uniform convergence from Propositions 2.2 and 2.3 in [14]. Hence, we immediately get the equi-statistical Korovkin-type approximation theorem which was introduced by Karakuş, Demirci and Duman [11] and the classical Korovkin-type approximation theorem which was introduced by Korovkin [12].
2. Now we present a example such that our new approximation result works but its classical case and statistical case do not work. Let $X = [0,1]$. To see this, first consider the following Meyer-König and Zeller polynomials introduced by W. Meyer-König and K. Zeller [13]:

$$M_n(f;x) = \sum_{k=0}^{\infty} p_{nk}(x) f\left(\frac{k}{n+k}\right),\ f \in C\,[0,1]\,,$$

where $p_{nk}(x) = \binom{n+k}{k} x^k\,(1-x)^{n+1}$. It is known that

$$\begin{aligned} M_n(e_0;x) &= e_0(x), \\ M_n(e_1;x) &= e_1(x), \\ M_n(e_2;x) &= e_2(x) + \eta_n\,(x) \leq e_2(x) + \frac{x(1-x)}{n+1}, \end{aligned}$$

where $\eta_n\,(x) = x(1-x)^{n+1} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \frac{x^k}{n+k+1}$. Let $\mathcal{I}_d = Exh\,(\phi)$ where $\phi\,(A) = \sup_{j\in\mathbb{N}} d_j\,(A)$. Using these polynomials, we introduce the following positive linear operators on $C\,[0,1]$:

$$D_n(f;x) = (1+h_n(x))M_n(f;x),\ x \in [0,1] \ \text{ and } f \in C\,[0,1]\,, \qquad (7)$$

where $h_n(x)$ is defined as in Example 1. Then observe the Korovkin result that

$$\begin{aligned} D_n(e_0;x) &= (1+h_n(x))e_0(x), \\ D_n(e_1;x) &= (1+h_n(x))e_1(x), \\ D_n(e_2;x) &\leq (1+h_n(x))\left[e_2(x) + \frac{x(1-x)}{n+1}\right]. \end{aligned}$$

Since $h_n \rightarrow_{\mathcal{I}_d} h = 0$ $(equi-ideal)$ on $[0,1]$, we conclude that

$$D_n(e_i) \rightarrow_{\mathcal{I}_d} e_i(equi-ideal) \text{ on } [0,1] \text{ for each } i = 0,1,2.$$

So, by Theorem 1, we immediately see that

$$D_n(f) \rightarrow_{\mathcal{I}_d} f(equi-ideal) \text{ on } [0,1] \text{ for all } i = 0,1,2.$$

However, since (h_n) is not uniform ideal (uniform statistical) convergent to the function $h = 0$ on the interval $[0,1]$, we can say that Theorem 1 of [8] does not work for our operators defined by (7). Furthermore, since (h_n) is not uniformly convergent (in the ordinary sense) to the function $h = 0$ on $[0,1]$, the classical Korovkin-type approximation theorem does not work either. Therefore, this application clearly shows that our Theorem 1 is a non-trivial generalization of the classical and the statistical cases of the Korovkin results.

4. Rate of convergence

In this section, we compute the rates of equi-ideal convergence of a sequence of positive linear operators defined on $C\,(X)$ by means of the modulus of continuity.

Now we give the following definition.

Definition 6. *Let $\mathcal{I}$ be an analytic P-ideal on $\mathbb{N}$ with $\mathcal{I} = Exh\,(\phi)$ for a lower semicontinuouos submeasure ϕ on $\mathbb{N}$. The sequence (f_n) is equi-ideal convergent to f with degree $0 < \beta < 1$ if for each $\varepsilon > 0$,*

$$\lim_{k\to\infty} \frac{\phi\left(\Psi(x,\varepsilon) \setminus k\right)}{k^{1-\beta}} = 0$$

uniformly with respect to x. In this case we write $f_k - f = o\left(k^{-\beta}\right)(equi-ideal)$ on X.

The fact that the notion introduced in this definition does not depend on ϕ can be easily shown by Proposition 2.1 given in [14].

Now we remind of the concept of the modulus of continuity. For $f \in C\,(X)$, the modulus of continuity of f, denoted by $\omega\,(f;\delta)$, is defined to be

$$\omega\,(f;\delta) = \sup_{|y-x|<\delta,\ x,y\in X} \left|f\,(y) - f\,(x)\right|.$$

It is also well known that for any $\delta > 0$ and each $x, y \in X$

$$|f(y) - f(x)| \le \omega(f;\delta)\left(\frac{|y-x|}{\delta} + 1\right).$$

We will need the following lemma.

Lemma 2. *Let (f_n) and (g_n) be function sequences belonging to $C(X)$. Assume that $f_k \to f = o\left(k^{-\beta_0}\right)$ (equi − ideal) on X and $g_k - g = o(k^{-\beta_1})$ (equi − ideal) on X. Let $\beta = \min\{\beta_0, \beta_1\}$. Then the following statements hold:*

(i) $(f_k + g_k) - (f + g) = o(k^{-\beta})$ *(equi − ideal) on X,*

(ii) $(f_k - f) - (g_k - g) = o(k^{-\beta})$ *(equi − ideal) on X,*

(iii) $\lambda(f_k - f) = o(k^{-\beta_0})$ *(equi − ideal) on X, for any real number λ,*

(iv) $\sqrt{|f_k - f|} = o(k^{-\beta_0})$ *(equi − ideal) on X.*

Proof. (i) Assume that $f_k - f = o(k^{-\beta_0})$ $(equi - ideal)$ on X and that $g_k - g = o(k^{-\beta_1})$ $(equi - ideal)$ on X. Also, for $\varepsilon > 0$ and $x \in X$ define

$$\begin{aligned}\Psi(x,\varepsilon) &: = \{n : |(f_n + g_n)(x) - (f+g)(x)| \ge \varepsilon\} \\ \Psi_0\left(x, \frac{\varepsilon}{2}\right) &: = \left\{n : |f_n(x) - f(x)| \ge \frac{\varepsilon}{2}\right\}, \\ \Psi_1\left(x, \frac{\varepsilon}{2}\right) &: = \left\{n : |g_n(x) - g(x)| \ge \frac{\varepsilon}{2}\right\}.\end{aligned}$$

Then, observe that

$$\Psi(x,\varepsilon) \subset \Psi_0\left(x, \frac{\varepsilon}{2}\right) \cup \Psi_1\left(x, \frac{\varepsilon}{2}\right),$$

which gives

$$\frac{\phi(\Psi(x,\varepsilon) \setminus k)}{k^{1-\beta}} \le \frac{\phi\left(\Psi_0\left(x, \frac{\varepsilon}{2}\right) \setminus k\right)}{k^{1-\beta_0}} + \frac{\phi\left(\Psi_1\left(x, \frac{\varepsilon}{2}\right) \setminus k\right)}{k^{1-\beta_1}}, \tag{8}$$

where $\beta = \min\{\beta_0, \beta_1\}$. Now by taking limit as $k \to \infty$ in (8) and using the hypotheses, we conclude that

$$\lim_{k\to\infty} \frac{\phi(\Psi(x,\varepsilon) \setminus k)}{k^{1-\beta}} = 0, \text{ for all } x \in X,$$

which completes the proof of (i). Since the proofs of (ii), (iii) and (iv) are similar, they are omitted. ☐

Then we have the following result.

Theorem 2. *Let X be a compact subset of the real numbers, and let $\{L_n\}$ be a sequence of positive linear operators acting from $C(X)$ into itself. Assume that $\mathcal{I}$ is an analytic P-ideal on $\mathbb{N}$ with $\mathcal{I} = Exh(\phi)$ for a lower semicontinuouos submeasure ϕ on $\mathbb{N}$. Assume that the following conditions hold:*

(i) $L_k(e_0) - e_0 = o(k^{-\beta_0})$ *(equi − ideal) on* X,

(ii) $\omega(f, \alpha_k) = o(k^{-\beta_1})$ *(equi − ideal) on* X, *where* $\alpha_k(x) = \sqrt{L_k(\varphi_x; x)}$ *with* $\varphi_x(y) = (y-x)^2$.

Then we have, for all $f \in C(X)$,

$$L_k(f) - f = o(k^{-\beta})(equi - ideal) \ on \ X,$$

where $\beta = \min\{\beta_0, \beta_1\}$.

Proof. Let $f \in C(X)$ and $x \in X$. It is known that ([1],[3]),

$$|L_n(f;x) - f(x)| \le M\,|L_n(e_0;x) - e_0(x)| + \left\{L_n(e_0;x) + \sqrt{L_n(e_0;x)}\right\} w(f, \alpha_n),$$

where $M := \|f\|_{C(X)}$. Then, we get

$$\begin{aligned}|L_n(f;x) - f(x)| \le\ & M\,|L_n(e_0;x) - e_0(x)| + 2w(f,\alpha_n) + |L_n(e_0;x) - e_0(x)|\, w(f,\alpha_n) \\ & + \sqrt{|L_n(e_0;x) - e_0(x)|}\, w(f,\alpha_n).\end{aligned}$$

Using the hypotheses (i), (ii), Lemma 2 and the monotonicity of ϕ in the above inequality, the proof is completed at once. □

5. A Voronovskaya-type theorem

In this section, we obtain a Voronovskaya-type theorem equi-ideal case for the positive linear operators $\{D_n\}$ given by (7) with respect to the ideal $\mathcal{I}_d$.

Theorem 3. *For every* $f \in C[0,1]$ *such that* $f', f'' \in C[0,1]$, *we have*

$$n\{D_n(f) - f\} = \frac{x(1-x)^2}{2} f''(x) \ \ (equi - ideal) \ \ on \ \ [0,1].$$

Proof. Let $x \in [0,1]$ and $f, f', f'' \in C[0,1]$. Define the function ξ_x by

$$\xi_x(t) = \begin{cases} \frac{f(t) - f(x) - f'(x)(t-x) - \frac{1}{2}f''(x)(t-x)^2}{(t-x)^2} & , t \neq x, \\ 0 & , t = x. \end{cases}$$

Then by assumption we get $\xi_x(t) = 0$ and $\xi_x \in C[0,1]$. By the Taylor formula for $f \in C[0,1]$, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \xi_x(t)(t-x)^2.$$

From the linearity D_n, we obtain

$$\begin{aligned} D_n(f;x) = &\ f(x) D_n(1;x) + f'(x) D_n(t-x;x) + \frac{1}{2}f''(x) D_n\left((t-x)^2; x\right) \\ & + D_n\left(\xi_x(t)(t-x)^2; x\right). \end{aligned}$$

Since $M_n\left((t-x)^2;x\right)=\frac{x(1-x)^2}{n}+O\left(\frac{1}{n^2}\right)$ (see, [4],[13]), we obtain

$$\begin{aligned}D_n(f;x)-f(x) &= f(x)\,h_n(x)+\frac{1}{2}f''(x)\,\frac{x(1-x)^2}{n}+\frac{1}{2}f''(x)\,O\left(\frac{1}{n^2}\right)\\ &\quad+\frac{1}{2}f''(x)\,h_n(x)\left\{\frac{x(1-x)^2}{n}+O\left(\frac{1}{n^2}\right)\right\}\\ &\quad+D_n\left(\xi_x(t)(t-x)^2;x\right). \qquad (9)\end{aligned}$$

Applying the Cauchy-Schwarz inequality for the last term on the right-hand side of (9), we get

$$\left|D_n\left(\xi_x(t)(t-x)^2;x\right)\right|\le\left(D_n\left(\xi_x^2(t);x\right)\right)^{1/2}.\left(D_n\left((t-x)^4;x\right)\right)^{1/2}:=g_n(x).$$

Let $\varphi_x(t)=\xi_x^2(t)$. In this case, we will show that $\varphi_x(x)=0$ and $\varphi_x\in C[0,1]$. From Theorem 1,

$$D_n(\varphi_x(t);x)=D_n\left(\xi_x^2(t);x\right)\to\varphi_x(x)=0\ \ (equi-ideal)\ \text{ on }\ [0,1]. \qquad (10)$$

Since for every $f\in C[0,1]$, $\|D_n(f)\|_{C[0,1]}\le 2\|f\|_{C[0,1]}$ and from (10), it follows that

$$g_n(x)=o\left(\frac{1}{n}\right)\to 0\ \ (equi-ideal)\ \text{ on }\ [0,1]. \qquad (11)$$

Considering (9), (11) and also $h_n\to h=0$ $(equi-ideal)$ on $[0,1]$, we have

$$n\{D_n(f;x)-f(x)\}=\frac{x(1-x)^2}{2}f''(x)\ \ (equi-ideal)\ \text{ on }\ [0,1].$$

Thus the proof is completed. □

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References

[1] F. Altomare, M. Campiti, *Korovkin type approximation theory and its application*, Walter de Gryter Publ., Berlin, 1994.

[2] M. Balcerzak, K. Dems, A. Komisarski, *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl. **328**(2007), 715–729.

[3] R. A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators*, Lecture Notes in Mathematics 293, Spinger-Verlag, New York, 1972.

[4] R. A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators for Functions of Bounded Variation*, J. Approx. Theory **56**(1989), 245–255.

[5] O. Duman, M. K. Khan, C. Orhan, *A-Statistical convergence of approximating operators*, Math. Inequal. Appl. **4**(2003), 689–699.
[6] E. Erkuş, O. Duman, H. M. Srivastava, *Statistical approximation of certain positive linear operators constructed by means of the Chan-Chyan-Srivastava polynomials*, Appl. Math. Comput. **182**(2006), 213–222.
[7] E. Erkuş, O. Duman, *A Korovkin type approximation theorem in statistical sense*, Studia. Sci. Math. Hungar. **43**(2006), 285–244.
[8] A. D. Gadjiev, C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32**(2002), 129–138.
[9] I. Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. **148**(2000), p.xvi+177.
[10] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2**(1951), 241–244.
[11] S. Karakuş, K. Demirci, O. Duman, *Equi-statistical convergence of positive linear operators*, J. Math. Anal. Appl. **339**(2008), 1065-1072.
[12] P. P. Korovkin, *Linear operators and approximation theory*, Hindustan Publ. Co., Delhi, 1960.
[13] W. Meyer-König, K. Zeller, *Bernsteiniche Potenzreihen*, Studia Math. **19**(1960), 89–94.
[14] N. Mrożek, *Ideal version of Egorov's theorem for analytic P-ideals*, J. Math. Anal. Appl. **349**(2009), 452–458.
[15] S. Solecki, *Analytic ideals and their applications*, Ann. Pure Appl. Logic **99**(1999), 51–72.
[16] S. Solecki, *Local inverses of Borel homomorphisms and analytic P-ideals*, Abst. Appl. Anal. (2005), 207–219.