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Systems of relaxed cocoercive generalized variational inequalities via nonexpansive mappings*

Yan Hao[1,†], Xiaolong Qin[2] and Shin Min Kang[3]

[1] *School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan 316 004, P. R. China*
[2] *2School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450 011, P. R. China*
[3] *Department of Mathematics and the RINS, Gyeongsang National University, Jinju 660-701, Korea*

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Abstract. In this paper, we consider a system of generalized variational inequalities. We introduce an iterative algorithm for the system of generalized variational inequalities and study the algorithmic convergence analysis. Strong convergence theorems are established. The results presented in this paper mainly improve and extend the corresponding results announced by many others.

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1. Introduction

Variational inequalities and variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years. They have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, engineering sciences, etc. For details, we can refer to [1-29, 31] and the references therein. In the theory of variational inequalities, the development of an efficient and implementable iterative algorithm is interesting and important. Various kinds of iterative algorithms for solving variational inequalities have been developed by many authors. Among these methods, the projection methods which have been applied widely to problems arising especially from complementarity, convex quadratic programming, and variational problems are interesting and important. Recently, Chang et al. [2] and Huang and Noor [7] considered new systems of nonlinear variational inequalities and studied the approximate solvability of this system based on projection methods. In this paper, we consider, based on the projection method, the approximate solvability of a system of nonlinear relaxed cocoercive general variational inequalities

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†Corresponding author. *Email addresses:* `zjhaoyan@yahoo.cn` (Y. Hao), `ljjhqxl@yahoo.com.cn` (X. Qin), `smkang@gnu.ac.kr` (S. M. Kang)

within the framework of Hilbert spaces. The results obtained in this paper generalize the results of Chang et al. [2], Huang and Noor [7] and some others.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let C be a nonempty closed and convex subset of H. Given nonlinear operators $T : C \to H$ and $g : C \to C$, we consider the problem of finding $u \in C$ such that

$$\langle Tu, v - g(u)\rangle \geq 0, \quad \forall v \in C. \tag{1}$$

The generalized variational inequality (1) was introduced by Noor in 1988, see [11, 9] for more details.

(I) If $g = I$, the identity operator, then the generalized variational inequality (1) is equivalent to finding $u \in C$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in C, \tag{2}$$

which is known as the classical variational inequality, originally introduced and studied by Stampacchia [26].

(II) If $C^* = \{u \in H : \langle u, v\rangle \geq 0, \forall v \in C\}$ is a polar cone of the convex cone C in H, then the generalized variational inequality (1) is equivalent to finding $u \in C$ such that

$$Tu \in C^* \text{ and } \langle Tu, g(u)\rangle = 0,$$

which is known as the general nonlinear complementarity problem, which includes many previously known complementarity problems as special cases.

Next, we consider the following definition:

Definition 1 (see [16, 15]). *Let C be a nonempty convex subset of a Hilbert space H. C is said to be g-convex if and only if there exists a mapping $g : C \to C$ such that*

$$g(x) + \lambda(y - g(x)) \in C, \quad \forall x, y \in C,$$

where $\lambda \in (0, 1)$. Note that every convex set is g-convex, but the converse is not true, see [5, 30] for more details.

Definition 2 (see [16, 15]). *The function $F : C \to H$ is said to be g-convex, if there exists a function $g : C \to C$ such that*

$$F(g(x) + \lambda(y - g(x))) \leq (1 - \lambda)F(g(x)) + \lambda F(y)$$

for all $x, y \in C$ and $\lambda \in (0, 1)$.

Next, we show that the minimum of a differentiable g-convex function on a g-convex subset C can be characterized by the generalized variational inequality (1).

Lemma 1. *Let C be a nonempty g-convex subset of a real Hilbert space H and $F : C \to H$ a differentiable g-convex function. Then $g(u) \in C$ is the minimum of the function F on C if and only if $g(u) \in C$ satisfies:*

$$\langle F'\big(g(u)\big), v - g(u)\rangle \geq 0, \quad \forall v \in C, \tag{3}$$

where $F'(\cdot)$ is the differential of F at $g(u) \in C$.

Proof. Let $g(u) \in C$ be a minimum of the differentiable g-convex function F. Since the subset C is g-convex, for any $v \in C$ we have that $g(u) + \lambda(v - g(u)) \in C$, where $\lambda \in (0, 1)$. It follows that

$$F(g(u) + \lambda(v - g(u))) - F(g(u)) \geq 0.$$

Dividing the above inequality by λ and letting $\lambda \to 0$ we have that (3) holds.

Conversely, let $g(u)$ satisfy the inequality (3). Since F is a g-convex function, for all $u, v \in C$ and $\lambda \in (0, 1)$, we have

$$F(g(u) + \lambda(v - g(u))) \leq (1 - \lambda)F(g(u)) + \lambda F(v),$$

which implies that

$$F(v) - F(g(u)) \geq \frac{F(g(u) + \lambda(v - g(u))) - F(g(u))}{\lambda}$$

Letting $\lambda \to 0$, from (3) we obtain that

$$F(v) - F(g(u)) \geq \langle F'\big(g(u)\big), v - g(u)\rangle \geq 0,$$

from which it follows that

$$F(g(u)) \leq F(v), \quad \forall v \in C.$$

This shows that $g(u)$ is the minimum of F on C. This completes the proof. □

Lemma 1 implies that g-convex programming problems can be studied via the generalized variational inequality (1) with $Tu = F'(g(u))$

Recall the following definitions:

(1) T is said to be monotone if

$$\langle Tu - Tv, u - v\rangle \geq 0, \quad \forall u, v \in C.$$

(2) T is said to be r-strongly monotone if there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|^2, \quad \forall x, y \in C.$$

This implies that

$$\|Tx - Ty\| \geq r\|x - y\|, \quad \forall x, y \in C.$$

(3) T is said to be γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \gamma\|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Clearly, every γ-cocoercive mapping T is $\frac{1}{\gamma}$-Lipschitz continuous.

(4) T is said to be relaxed γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma)\|Tx - Ty\|^2, \quad \forall x, y \in C.$$

(5) T is said to be relaxed (γ, r)-cocoercive if there exist two constants $\gamma, r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma)\|Tx - Ty\|^2 + r\|x - y\|^2, \quad \forall x, y \in C.$$

(6) Recall that a mapping $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In this paper, we use $F(S)$ to denote the set of fixed points of S.

Next, we now recall the following well-known results.

Lemma 2. *For given $z \in H$ and $u \in C$, we see that the following inequality holds*

$$\langle z - u, u - v\rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = P_C z$, where P_C is the projection from H onto C.

Lemma 3. *$u \in C$ satisfies the generalized variational inequality* (1)*, if and only if u satisfies the relation:*

$$g(u) = P_C[g(u) - \rho Tu], \tag{4}$$

where $\rho > 0$ is a constant and P_C is the metric projection of H onto C.

Proof. Let $u \in C$ be a solution of (1). Then we have the following:

$$\langle g(u) - \rho Tu - g(u), g(u) - v\rangle \geq 0, \quad \forall v \in C.$$

From Lemma 1, we see that (4) holds. This completes the proof.

Next, we assume that $u \in C$ satisfies (4). From Lemma 2, we have

$$\langle Tu, v - g(u)\rangle \geq 0, \quad \forall v \in C.$$

This completes the proof. □

Lemma 3 implies that (1) and (4) are equivalent. This alternative formulation is very important from the numerical analysis point of view. Let $T : C \to H$ be a nonlinear mapping.

Let $T_1, T_2 : C \times C \to H$ and $g_1, g_2 : C \to C$ be nonlinear mappings. Consider a system of nonlinear variational inequality (SNVI) problem as follows:

Find $(x^*, y^*) \in C \times C$ such that

$$\langle sT_1(y^*, x^*) + g_1(x^*) - g_1(y^*), x - g_1(x^*)\rangle \geq 0, \quad \forall x \in C,\ s > 0, \tag{5}$$

$$\langle tT_2(x^*, y^*) + g_2(y^*) - g_2(x^*), x - g_2(y^*)\rangle \geq 0, \quad \forall x \in C,\ t > 0. \tag{6}$$

In this paper, we denote the solution set of the SNVI problem (5)-(6) by Ω_1.

If $T_1 = T_2$ are univariate mappings in the system of general variational inequalities (5)-(6), then we see that the system of general variational inequalities is reduced to the general variational inequality (1) by adding up the requirement $x^* = y^*$.

From Lemma 3, one can easily see that the SNVI problems (5) and (6) are equivalent to the following projection formulas:

$$g_1(x^*) = P_C[g_1(y^*) - sT_1(y^*, x^*)], \quad s > 0, \tag{7}$$
$$g_2(y^*) = P_C[g_2(x^*) - tT_2(x^*, y^*)], \quad t > 0, \tag{8}$$

respectively, where P_C is the projection of H onto C.

Next, we consider some special cases of the SNVI problems (5) and (6) as follows:

(I) If $g_1 = g_2 = I$, the identity operator, then the SNVI problems (5) and (6) reduce to the following SNVI problem:

Find $(x^*, y^*) \in C \times C$ such that

$$\langle sT_1(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\ s > 0, \tag{9}$$
$$\langle tT_2(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C,\ t > 0, \tag{10}$$

which was considered by Huang and Noor [8]. Next, we shall denote the set of solutions of the SNVI problem (9)-(10) by Ω_2.

(II) If $g_1 = g_2 = I$, the identity operator and $T_1 = T_2 = T$, then the SNVI problems (5) and (6) reduce to the following SNVI problem:

Find $(x^*, y^*) \in C \times C$ such that

$$\langle sT(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\ s > 0, \tag{11}$$
$$\langle tT(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C,\ t > 0, \tag{12}$$

which was considered by Chang et al. [2]. Next, we shall denote the set of solutions of the SNVI problem (11)-(12) by Ω_3.

(III) If C is a closed convex cone of H, then the SNVI problems (5) and (6) are equivalent to the following system of nonlinear complementarity (SNC) problems:

Find $(x^*, y^*) \in C \times C$ such that

$$T_1(y^*, x^*) \in C^* \text{ and } \quad T_2(x^*, y^*) \in C^*,$$
$$\langle sT_1(y^*, x^*) + g_1(x^*) - g_1(y^*), g_1(x^*) \rangle = 0, \quad s > 0, \tag{13}$$
$$\langle tT_2(x^*, y^*) + g_2(y^*) - g_2(x^*), g_2(x^*) \rangle = 0, \quad t > 0, \tag{14}$$

where C^* is the polar cone to C defined by

$$C^* = \{f \in H : \langle f, x \rangle \geq 0, \quad \forall x \in C\}.$$

(IV) If $g_1 = g_2 = g$ and $T_1 = T_2 = T$ are univariate mappings, then the SVNI problems (5) and (6) reduce to the following SNVI problems:

Find $(x^*, y^*) \in H \times H$ such that

$$\langle sT(y^*) + g(x^*) - g(y^*), x - g(x^*) \rangle \geq 0, \quad \forall x \in C,\ s > 0, \tag{15}$$
$$\langle tT(x^*) + g(y^*) - g(x^*), x - g(x^*) \rangle \geq 0, \quad \forall x \in C,\ t > 0. \tag{16}$$

Next, we use Ω_4 to denote the set of solutions of the SNVI problem (15)-(16).

One can easily get that the SNVI problems (15) and (16) are equivalent to the following projection formulas:

$$g(x^*) = P_C[g(y^*) - sT(y^*)], \quad s > 0, \tag{17}$$

$$g(y^*) = P_C[g(x^*) - tT(x^*)], \quad t > 0. \tag{18}$$

By rewriting (17) and (18), we have

$$\begin{cases} x^* = x^* - g(x^*) + P_C[g(y^*) - sT(y^*, x^*)], \\ y^* = y^* - g(y^*) + P_C[g(x^*) - tT(x^*, y^*)]. \end{cases}$$

2. Algorithms

Let $S_1, S_2 : C \to C$ be nonexpansive mappings. Assume that $\Omega_1 \neq \emptyset$, $F(S_1) \neq \emptyset$ and $F(S_2) \neq \emptyset$, respectively. Let $x^* \in F(S_1)$, $y^* \in F(S_2)$ and $(x^*, y^*) \in \Omega_1$. From (7) and (8), we arrive at

$$\begin{cases} x^* = S_1\{x^* - g_1(x^*) + P_C[g_1(y^*) - sT_1(y^*, x^*)]\}, \quad s > 0, \\ y^* = S_2\{y^* - g_2(y^*) + P_C[g_2(x^*) - tT_2(x^*, y^*)]\}, \quad t > 0. \end{cases}$$

Algorithm 1. *For any $(x_0, y_0) \in C \times C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative process:*

$$\begin{cases} y_n = S_2\{y_n - g_2(y_n) + P_C[g_2(x_n) - tT_2(x_n, y_n)]\}, \quad n \geq 1, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S_1\{x_n - g_1(x_n) + P_C[g_1(y_n) - sT_1(y_n, x_n)]\}, \quad n \geq 0, \end{cases} \tag{19}$$

where $s, t > 0$ are two constants.

(I) If $g_1 = g_2 = I$, the identity mapping, then Algorithm 1 reduces to the following:

Algorithm 2. *For any $(x_0, y_0) \in C \times C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative process:*

$$\begin{cases} y_n = S_2 P_C[x_n - tT_2(x_n, y_n)], \quad n \geq 1, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S_1 P_C[y_n - sT_1(y_n, x_n)], \quad n \geq 0, \end{cases} \tag{20}$$

where $s, t > 0$ are two constants.

(II) If $T_1 = T_2 = T$, $S_1 = S_2 = S$ and $g_1 = g_2 = I$, the identity mapping, in Algorithm 2.1, then we have the following:

Algorithm 3. *For any $(x_0, y_0) \in C \times C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative process:*

$$\begin{cases} y_n = SP_C[x_n - tT(x_n, y_n)], \quad n \geq 1, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n SP_C[y_n - sT(y_n, x_n)], \quad n \geq 0, \end{cases} \tag{21}$$

where $s, t > 0$ are two constants.

(III) If $g_1 = g_2 = g$, $S_1 = S_2 = I$, the identity operator and $T_1 = T_2 = T$ are univariate mappings, in Algorithm 1, then we have the following:

Algorithm 4. *For any $x_0 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative process:*

$$\begin{cases} g(y_n) = P_C[g(x_n) - tTx_n], \quad n \geq 1, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\{x_n - g(x_n) + P_C[g(y_n) - sTy_n]\}, \quad n \geq 0, \end{cases} \tag{22}$$

where $s, t > 0$ are two constants.

In order to prove our main results, we need the following lemmas and definitions.

Definition 3. *A mapping $T : C \times C \to H$ is said to be relaxed (γ, r)-cocoercive in the first variable if there exist constants $\gamma, r > 0$ such that, for all $x, x' \in C$,*

$$\langle T(x,y) - T(x',y'), x - x'\rangle \geq (-\gamma)\|T(x,y) - T(x',y')\|^2 + r\|x - x'\|^2, \quad \forall y, y' \in C.$$

Definition 4. *A mapping $T : C \times C \to H$ is said to be μ-Lipschitz continuous in the first variable if there exists a constant $\mu > 0$ such that, for all $x, x' \in C$,*

$$\|T(x,y) - T(x',y')\| \leq \mu\|x - x'\|, \quad \forall y, y' \in C.$$

Definition 5 (see [29])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\lambda_n)a_n + b_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = \circ(\lambda_n)$, then $\lim_{n\to\infty} a_n = 0$.

3. Convergence analysis on algorithms

Theorem 1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C \times C \to H$ be a relaxed (γ_i, r_i)-cocoercive and μ_i-Lipschitz continuous mapping in the first variable and $g_i : C \to C$ a relaxed (λ_i, δ_i)-cocoercive and ν_i-Lipschitz continuous mapping for each $i = 1, 2$. Let $S_1, S_2 : C \to C$ be nonexpansive mappings. Assume that $\Omega_1 \neq \emptyset$, $F(S_1) \neq \emptyset$ and $F(S_2) \neq \emptyset$, respectively. Let $x^* \in F(S_1)$, $y^* \in F(S_2)$ and $(x^*, y^*) \in \Omega_1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm 1. If the following conditions are satisfied:*

(i) $\{\alpha_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\theta_4 < 1$ and $(\theta_1 + \theta_3)(\theta_2 + \theta_4) < (1-\theta_3)(1-\theta_4)$, where

$$\theta_1 = \sqrt{1 + s^2\mu_1^2 - 2sr_1 + 2s\gamma_1\mu_1^2}, \quad \theta_2 = \sqrt{1 + t^2\mu_2^2 - 2tr_2 + 2t\gamma_2\mu_2^2}$$

and

$$\theta_3 = \sqrt{1 + \nu_1^2 - 2\delta_1 + 2\lambda_1\nu_1^2}, \quad \theta_4 = \sqrt{1 + \nu_2^2 - 2\delta_2 + 2\lambda_2\nu_2^2},$$

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.

Proof. From the assumption, for all $s, t > 0$, we have

$$\begin{cases} y^* = S_2\{y^* - g_2(y^*) + P_C[g_2(x^*) - tT_2(x^*, y^*)]\} \\ x^* = (1-\alpha_n)x^* + \alpha_n S_1\{x^* - g_1(x^*) + P_C[g_1(y^*) - sT_1(y^*, x^*)]\}, \end{cases}$$

From (19), we obtain that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n S_1\{x_n - g_1(x_n) + P_C[g_1(y_n) - sT_1(y_n, x_n)]\} - x^*\| \\ &= \|(1-\alpha_n)x_n + \alpha_n S_1\{x_n - g_1(x_n) + P_C[g_1(y_n) - sT_1(y_n, x_n)]\} \\ &\quad -(1-\alpha_n)x^* - \alpha_n S_1\{x^* - g_1(x^*) + P_C[g_1(y^*) - sT_1(y^*, x^*)]\}\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - g_1(x_n) + P_C[g_1(y_n) - sT_1(y_n, x_n)] \\ &\quad -\{x^* - g_1(x^*) + P_C[g_1(y^*) - sT_1(y^*, x^*)]\}\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - [g_1(x_n) - g_1(x^*)]\| \\ &\quad +\alpha_n\|y_n - y^* - [g_1(y_n) - g_1(y^*)]\| \\ &\quad +\alpha_n\|[y_n - y^* - s[T_1(y_n, x_n) - T_1(y^*, x^*)]\|. \end{aligned} \tag{23}$$

By the assumption that T_1 is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitz continuous in the first variable, we obtain that

$$\begin{aligned} &\|y_n - y^* - s[T_1(y_n, x_n) - T_1(y^*, x^*)]\|^2 \\ &\quad = \|y_n - y^*\|^2 - 2s\langle y_n - y^*, T_1(y_n, x_n) - T_1(y^*, x^*)\rangle \\ &\qquad +s^2\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\ &\quad \le \|y_n - y^*\|^2 - 2s[-\gamma_1\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 + r_1\|y_n - y^*\|^2] \\ &\qquad +s^2\mu_1^2\|y_n - y^*\|^2 \\ &\quad \le \|y_n - y^*\|^2 + 2s\gamma_1\mu_1^2\|y_n - y^*\|^2 - 2sr_1\|y_n - y^*\|^2 + s^2\mu_1^2\|y_n - y^*\|^2 \\ &\quad = \theta_1^2\|y_n - y^*\|^2, \end{aligned} \tag{24}$$

where $\theta_1 = \sqrt{1 + s^2\mu_1^2 - 2sr_1 + 2s\gamma_1\mu_1^2}$. By the assumption that g_1 is relaxed (λ_1, δ_1)-cocoercive and ν_1-Lipschitz continuous, we arrive at

$$\begin{aligned} &\|y_n - y^* - [g_1(y_n) - g_1(y^*)]\|^2 \\ &\quad = \|y_n - y^*\|^2 - 2\langle y_n - y^*, g_1(y_n) - g_1(y^*)\rangle + \|g_1(y_n) - g_1(y^*)\|^2 \\ &\quad \le \|y_n - y^*\|^2 - 2(-\lambda_1\|g_1(y_n) - g_1(y^*)\|^2 + \delta_1\|y_n - y^*\|^2) + \nu_1^2\|y_n - y^*\|^2 \\ &\quad \le \|y_n - y^*\|^2 + 2\lambda_1\nu_1^2\|y_n - y^*\|^2 - 2\delta_1\|y_n - y^*\|^2 + \nu_1^2\|y_n - y^*\|^2 \\ &\quad = \theta_3^2\|y_n - y^*\|^2, \end{aligned} \tag{25}$$

where $\theta_3 = \sqrt{1 + \nu_1^2 - 2\delta_1 + 2\lambda_1\nu_1^2}$. In a similar way, one can show that

$$\|x_n - x^* - [g_1(x_n) - g_1(x^*)]\| \le \theta_3\|x_n - x^*\|. \tag{26}$$

Substituting (24), (25) and (26) into (24) yields

$$\|x_{n+1} - x^*\| \le [1 - \alpha_n(1 - \theta_3)]\|x_n - x^*\| + \alpha_n(\theta_3 + \theta_1)\|y_n - y^*\|. \tag{27}$$

Next, we estimate

$$\begin{aligned}\|y_n - y^*\| &= \|S_2\{y_n - g_2(y_n) + P_C[g_2(x_n) - tT_2(x_n, y_n)]\}\\ &\quad - S_2\{y^* - g_2(y^*) + P_C[g_2(x^*) - tT_2(x^*, y^*)]\}\|\\ &\le \|y_n - y^* - [g_2(y_n) - g_2(y^*)]\| + \|x_n - x^* - [g_2(x_n) - g_2(x^*)]\|\\ &\quad + \|x_n - x^* - t[T_2(x_n, y_n) - T_2(x^*, y^*)]\|. \qquad (28)\end{aligned}$$

By the assumption that T_2 is relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitz continuous in the first variable, we see that

$$\begin{aligned}&\|x_n - x^* - t[T_2(x_n, y_n) - T_2(x^*, y^*)]\|^2\\ &\quad = \|x_n - x^*\|^2 - 2t\langle x_n - x^*, T_2(x_n, y_n) - T_2(x^*, y^*)\rangle\\ &\quad\quad + t^2\|T_2(x_n, y_n) - T_2(x^*, y^*)\|^2\\ &\quad \le \|x_n - x^*\|^2 - 2t[-\gamma_2\|T(x_n, y_n) - T(x^*, y^*)\|^2 + r_2\|x_n - z^*\|^2]\\ &\quad\quad + t^2\mu_2^2\|x_n - x^*\|^2\\ &\quad \le \|x_n - x^*\|^2 + 2t\gamma_2\mu_2^2\|x_n - x^*\|^2 - 2tr_2\|x_n - x^*\|^2 + t^2\mu_2^2\|x_n - x^*\|^2\\ &\quad = \theta_2^2\|x_n - x^*\|^2, \qquad (29)\end{aligned}$$

where $\theta_2 = \sqrt{1 + t^2\mu_2^2 - 2tr_2 + 2t\gamma_2\mu_2^2}$. From proof (25), we can obtain that

$$\|x_n - x^* - [g_2(x_n) - g_2(x^*)]\| \le \theta_4\|x_n - x^*\| \qquad (30)$$

and

$$\|y_n - y^* - [g_2(y_n) - g_2(y^*)]\| \le \theta_4\|y_n - y^*\|, \qquad (31)$$

where $\theta_4 = \sqrt{1 + \nu_2^2 - 2\delta_2 + 2\lambda_2\nu_2^2}$. Substituting (29), (30) and ((31) into (28), we arrive at

$$\|y_n - y^*\| \le \theta_4\|y_n - y^*\| + (\theta_4 + \theta_2)\|x_n - x^*\|. \qquad (32)$$

Since $\theta_4 < 1$, we see that

$$\|y_n - y^*\| \le \frac{\theta_4 + \theta_2}{1 - \theta_4}\|x_n - x^*\|. \qquad (33)$$

Substituting (33) into (27) yields

$$\begin{aligned}\|x_{n+1} - x^*\| &\le [1 - \alpha_n(1 - \theta_3)]\|x_n - x^*\| + \alpha_n(\theta_3 + \theta_1)\frac{\theta_4 + \theta_2}{1 - \theta_4}\|x_n - x^*\|\\ &= [1 - \alpha_n(1 - \theta_3 - \frac{(\theta_1 + \theta_3)(\theta_2 + \theta_4)}{1 - \theta_4})]\|x_n - x^*\|. \qquad (34)\end{aligned}$$

Noticing the condition (ii) and applying Lemma 5 to (34), we can get the desired conclusion easily. This completes the proof. □

As some applications of Theorem 1, we can obtain the following results immediately.

Corollary 1. *Let C be a closed convex subset of a real Hilbert space H. Let $T_i : H \times H \to H$ be a relaxed (γ_i, r_i)-cocoercive and μ_i-Lipschitz continuous mapping in the first variable for each $i = 1, 2$. Let $S_1, S_2 : C \to C$ be nonexpansive mappings. Assume that $\Omega_1 \neq \emptyset$, $F(S_1) \neq \emptyset$ and $F(S_2) \neq \emptyset$, respectively. Let $x^* \in F(S_1)$, $y^* \in F(S_2)$ and $(x^*, y^*) \in \Omega_1$. Let $\{x_n\}$, $\{y_n\}$ be two sequences generated by Algorithm 2. If the following conditions are satisfied:*

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $(1 + s^2\mu_1^2 - 2sr_1 + 2s\gamma_1\mu_1^2)(1 + t^2\mu_2^2 - 2tr_2 + 2t\gamma_2\mu_2^2) < 1$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.

Remark 1. *Corollary 1 mainly improves Theorem 3.3 of Chang et al. [2]. To be more precise, Corollary 1 is reduced to the above result when $S = I$, the identity mapping and $T_1 = T_2$; see [2] for more details.*

Corollary 2. *Let C be a closed convex subset of a real Hilbert space H. Let $T : H \times H \to H$ be a relaxed (γ, r)-cocoercive and μ-Lipschitz continuous mapping in the first variable. Let S be a nonexpansive mapping on C with a fixed point. Assume that $\Omega_3 \neq \emptyset$. Let $(x^*, y^*) \in \Omega_3$ and $x^*, y^* \in F(S)$. Let $\{x_n\}$, $\{y_n\}$ be two sequences generated by Algorithm 3. If the following conditions are satisfied:*

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $(1 + s^2\mu^2 - 2sr + 2s\gamma\mu^2)(1 + t^2\mu^2 - 2tr + 2t\gamma\mu^2) < 1$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.

Corollary 3. *Let C be a closed convex subset of a real Hilbert space H. Let $T : H \times H \to H$ be a relaxed (γ, r)-cocoercive and μ-Lipschitz continuous mapping and $g : H \to H$ a relaxed (λ, δ)-cocoercivei and ν-Lipschitz continuous mapping. Assume that $\Omega_4 \neq \emptyset$. Let $(x^*, y^*) \in \Omega_4$. Let $\{x_n\}$, $\{y_n\}$ be the sequences generated by Algorithm 4. Assume that the following conditions are satisfied:*

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\theta_3 < 1$ *and* $(\theta_1 + \theta_3)(\theta_2 + \theta_3) \leq (1 - \theta_3)^2$, *where*

$$\theta_1 = \sqrt{1 + s^2\mu^2 - 2sr + 2s\gamma\mu^2}, \quad \theta_2 = \sqrt{1 + t^2\mu^2 - 2tr + 2t\gamma\mu^2}$$

and

$$\theta_3 = \sqrt{1 + \nu^2 - 2\delta + 2\lambda\nu^2},$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.

Remark 2. *In this paper, we show that the problem of the generalized variational inequality is equivalent to a fixed point problem. This alternative formulation is important from the numerical analysis point of view. Iterative algorithms are suggested and analyzed. It is of interest to use the technique in this paper to develop other new iterative algorithms for solving a generalized system of nonlinear variational inequalities via nonexpansive mappings or other important nonlinear mappings.*

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