MATHEMATICAL COMMUNICATIONS
Math. Commun. **16**(2011), 191–203.

Jacobi's triple product identity and theta function identities

WENCHANG CHU[1,*] AND QINGLUN YAN[2,‡]

[1] *Institute of Combinatorial Mathematics, Hangzhou Normal University, Hangzhou–310 036, P.R. China*
[2] *College of Mathematics and Physics, Nanjing University of Posts and Telecommunications, Nanjing–210 046, P.R. China*

Received March 5, 2010; accepted September 4, 2010

Abstract. As a unified approach, Jacobi's triple product identity will be utilized to derive theta function formulae due to Baruah–Berndt (2007), identities of Rogers–Ramanujan functions and modular equations due to Ramanujan.

AMS subject classifications: Primary 05A30; Secondary 11P83

Key words: Jacobi's triple product identity, theta functions, modular equations

For two indeterminate q and z, the q-shifted factorial reads

$$(z;q)_0 = 1 \quad \text{and} \quad (z;q)_m = \prod_{n=0}^{m-1}(1-zq^n) \quad \text{for} \quad m = 1, 2, \cdots .$$

When $|q| < 1$, the following products of infinite order are well defined:

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n) \quad \text{and} \quad \langle z;q\rangle_\infty = (z;q)_\infty (q/z;q)_\infty.$$

For brevity, their multi–parameter forms are abbreviated as

$$[\alpha, \beta, \cdots, \gamma; q]_\infty = (\alpha;q)_\infty(\beta;q)_\infty \cdots (\gamma;q)_\infty,$$
$$\langle \alpha, \beta, \cdots, \gamma; q\rangle_\infty = \langle \alpha;q\rangle_\infty \langle \beta;q\rangle_\infty \cdots \langle \gamma;q\rangle_\infty .$$

There are several important theta function identities in mathematical literature. Perhaps the simplest and the most significant one is Jacobi's triple product identity [13] (see also [12] §1.6)

$$[q, x, q/x; q]_\infty = \sum_{n=-\infty}^{+\infty} (-1)^n q^{\binom{n}{2}} x^n,$$

which has several important applications in number theory, combinatorics and physics (see e.g. Andrews [1]).

*Corresponding author. *Email addresses:* chu.wenchang@unisalento.it (W. Chu), yanqinglun@yahoo.com.cn (Q. L. Yan)
‡Partially supported by Foundation of NJUPT under Grant No. NY208068 and the Natural Science Foundation of China under Grant No. 11026073 and 61070234.

The purpose of this paper is to show that Jacobi's triple product identity can be utilized in a unified manner to prove theta function formulae due to

Baruah–Berndt [3, 4], identities of Rogers–Ramanujan functions (cf. [8, 16, 17]) and the modular equations due to Ramanujan (cf. Andrews-Berndt [2], Theorem 1.6.1). Several new identities are also derived even though it is not our primary concern.

1. Theta function identities

By using elementary manipulations of theta functions, Berndt [5], Entry 29, p. 45 proved the following pair of identities.

Lemma 1. *For two variables x and y, there hold the theta function identities*

$$\langle -x,-y;q\rangle_\infty + \langle x,y;q\rangle_\infty = 2(-q;q)^2_\infty \langle -xy,-qx/y;q^2\rangle_\infty, \tag{1a}$$

$$\langle -x,-y;q\rangle_\infty - \langle x,y;q\rangle_\infty = 2x(-q;q)^2_\infty \langle -qxy,-y/x;q^2\rangle_\infty. \tag{1b}$$

Because these two formulae are very important in deriving theta function identities (cf. e.g. Berndt [5], Chapter 16, and Chu [11]), we present an elementary proof through Jacobi's triple product identity.

Proof. By means of Jacobi's triple product identity, we have the following equation

$$(q;q)^2_\infty\Big\{\langle -x,-y;q\rangle_\infty \pm \langle x,y;q\rangle_\infty\Big\} = \sum_{i,j=-\infty}^{+\infty} q^{\binom{i}{2}+\binom{j}{2}}\bigg\{\Big(\frac{y}{x}\Big)^i \pm (-1)^{i+j}\Big(\frac{y}{x}\Big)^j\bigg\}x^{i+j}.$$

Performing the replacements on summation indices

$$\left.\begin{matrix} i+j=m \\ i-j=n \end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix} i=\frac{m+n}{2} \\ j=\frac{m-n}{2} \end{matrix}\right.$$

and then simplifying the result, we can reformulate the last equation as follows

$$(q;q)^2_\infty\Big\{\langle -x,-y;q\rangle_\infty \pm \langle x,y;q\rangle_\infty\Big\} \tag{2a}$$

$$= \sum_{m\equiv_2 n} q^{\frac{m^2+n^2-2m}{4}}\bigg\{\Big(\frac{y}{x}\Big)^{\frac{m+n}{2}} \pm (-1)^m\Big(\frac{y}{x}\Big)^{\frac{m-n}{2}}\bigg\}x^m, \tag{2b}$$

where $m\equiv_2 n$ stands for the summation indices m and n with the same parity.

When both m and n are even, we can evaluate the double sum displayed in (2b) by making the replacements $m\to 2m$ and $n\to 2n$ as follows:

$$\begin{aligned}\sum_{m,n=-\infty}^{+\infty} q^{m^2+n^2-m}(xy)^m\Big\{(y/x)^n \pm (x/y)^n\Big\} &= (1\pm 1)\sum_{m=-\infty}^{+\infty} q^{2\binom{m}{2}}(xy)^m \\ &\quad\times \sum_{n=-\infty}^{+\infty} q^{2\binom{n}{2}+n}(x/y)^n \\ &= (1\pm 1)(q^2;q^2)^2_\infty \langle -xy,-qx/y;q^2\rangle_\infty,\end{aligned}$$

which leads to the first identity stated in the lemma.

When both m and n are odd, we can similarly evaluate the double sum displayed in (2b) by making the replacements $m \to 2m+1$ and $n \to 2n-1$ as follows:

$$\begin{aligned} x\sum_{m,n=-\infty}^{+\infty} q^{m^2+n^2-n}(xy)^m\Big\{(y/x)^n \mp (x/y)^{n-1}\Big\} &= (1\mp 1)\,x\sum_{m=-\infty}^{+\infty} q^{2\binom{m}{2}+m}(xy)^m \\ &\quad\times \sum_{n=-\infty}^{+\infty} q^{2\binom{n}{2}}(y/x)^n \\ &= (1\mp 1)x(q^2;q^2)_\infty^2 \left\langle -qxy, -y/x; q^2\right\rangle_\infty . \end{aligned}$$

This leads us to the second identity displayed in the lemma. □

By means of Lemma 1, we shall derive several theta function identities including some due to Baruah–Berndt [3, 4]. Throughout this section, Euler's famous identity $(-q;q)_\infty = 1/(q;q^2)_\infty$ will be frequently invoked without explanation. In addition, we shall fix two symbols $\omega = \exp(2\pi i/3)$ and $\varpi = \exp(2\pi i/5)$, for the cubic and fifth roots of unity, respectively. Because all the proofs are routine matter, which consist of reformulating the expressions in terms of the differences displayed in Lemma 1 and then factorizing the results, we shall only sketch the proofs without going into much detail.

Theorem 1 (Baruah–Berndt [3] Eqs 3.4 and 3.6)**.**

$$\begin{aligned} \frac{(-q;q^2)_\infty}{(q^3;q^6)_\infty} + \frac{(q;q^2)_\infty}{(-q^3;q^6)_\infty} &= 2(-q^4;q^4)_\infty^2(-q^6;q^6)_\infty(-q^6;q^{12})_\infty^2, \\ \frac{(-q;q^2)_\infty}{(q^3;q^6)_\infty} - \frac{(q;q^2)_\infty}{(-q^3;q^6)_\infty} &= 2q(-q^2;q^4)_\infty^2(-q^6;q^6)_\infty(-q^{12};q^{12})_\infty^2, \\ \frac{(-q;q^2)_\infty^2}{(q^3;q^6)_\infty^2} - \frac{(q;q^2)_\infty^2}{(-q^3;q^6)_\infty^2} &= 4q(-q^2;q^2)_\infty^2(-q^6;q^6)_\infty^4. \end{aligned}$$

Proof. Applying the relation $(q^3;q^6)_\infty = (q;q^2)_\infty \left\langle q\omega;q^2\right\rangle_\infty$, we have

$$\frac{(-q;q^2)_\infty}{(q^3;q^6)_\infty} \pm \frac{(q;q^2)_\infty}{(-q^3;q^6)_\infty} = \frac{\left\langle -q,-q\omega;q^2\right\rangle_\infty \pm \left\langle q,q\omega;q^2\right\rangle_\infty}{(q^6;q^{12})_\infty},$$

which leads to the first two identities stated in the theorem. The third one follows from the product of the first two. □

Theorem 2 (Baruah–Berndt [3] Eqs 3.9 and 3.11 and [4] Eq 6.19)**.**

$$\begin{aligned} \frac{(-q;q^2)_\infty^2}{(-q^3;q^6)_\infty^2} + \frac{(q;q^2)_\infty^2}{(q^3;q^6)_\infty^2} &= 2(-q^2;q^4)_\infty(-q^6;q^6)_\infty^2(-q^6;q^{12})_\infty, \\ \frac{(-q;q^2)_\infty^2}{(-q^3;q^6)_\infty^2} - \frac{(q;q^2)_\infty^2}{(q^3;q^6)_\infty^2} &= 4q(-q^4;q^4)_\infty(-q^6;q^6)_\infty^2(-q^{12};q^{12})_\infty, \\ \frac{(-q;q^2)_\infty^4}{(-q^3;q^6)_\infty^4} - \frac{(q;q^2)_\infty^4}{(q^3;q^6)_\infty^4} &= 8q(-q^2;q^2)_\infty(-q^6;q^6)_\infty^5. \end{aligned}$$

Proof. Similarly, there holds the relation

$$\frac{(-q;q^2)_\infty^2}{(-q^3;q^6)_\infty^2} \pm \frac{(q;q^2)_\infty^2}{(q^3;q^6)_\infty^2} = \frac{(q^2;q^4)_\infty^2}{(q^6;q^{12})_\infty^2}\Big\{ \left\langle q\omega, q\omega; q^2\right\rangle_\infty \pm \left\langle -q\omega, -q\omega; q^2\right\rangle_\infty \Big\},$$

which leads to the first two identities stated in the theorem. Multiplying the first two gives the third one. □

Theorem 3 (Baruah–Berndt [3] Eqs 4.3 and 4.5 and [4] Eq 7.19)**.**

$$\frac{(-q;q^2)_\infty}{(-q^5;q^{10})_\infty} + \frac{(q;q^2)_\infty}{(q^5;q^{10})_\infty} = 2(-q^4;q^4)_\infty(-q^{10};q^{10})_\infty(-q^{10};q^{20})_\infty,$$
$$\frac{(-q;q^2)_\infty}{(-q^5;q^{10})_\infty} - \frac{(q;q^2)_\infty}{(q^5;q^{10})_\infty} = 2q(-q^2;q^4)_\infty(-q^{10};q^{10})_\infty(-q^{20};q^{20})_\infty,$$
$$\frac{(-q;q^2)_\infty^2}{(-q^5;q^{10})_\infty^2} - \frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2} = 4q(-q^2;q^2)_\infty(-q^{10};q^{10})_\infty^3.$$

Proof. By means of the relation $(q^5;q^{10})_\infty = (q;q^2)_\infty \left\langle q\varpi, q\varpi^2; q^2\right\rangle_\infty$, we get

$$\frac{(-q;q^2)_\infty}{(-q^5;q^{10})_\infty} \pm \frac{(q;q^2)_\infty}{(q^5;q^{10})_\infty} = \frac{(q^2;q^4)_\infty}{(q^{10};q^{20})_\infty}\Big\{ \left\langle q\varpi, q\varpi^2; q^2\right\rangle_\infty \pm \left\langle -q\varpi, -q\varpi^2; q^2\right\rangle_\infty \Big\},$$

which leads to the first two identities stated in the theorem. The third one follows from the product of the first two. □

Theorem 4 (Theta function identities)**.**

$$\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} + \frac{(q;q)_\infty}{(q^3;q^3)_\infty} = 2\frac{(-q^6;q^6)_\infty}{(q^3;q^6)_\infty}\left\langle -q^5;q^{12}\right\rangle_\infty,$$
$$\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} - \frac{(q;q)_\infty}{(q^3;q^3)_\infty} = 2q\frac{(-q^6;q^6)_\infty}{(q^3;q^6)_\infty}\left\langle -q;q^{12}\right\rangle_\infty,$$
$$\frac{(-q;q)_\infty^2}{(-q^3;q^3)_\infty^2} + \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty^2} = 2\frac{(-q^3;q^6)_\infty^2}{(q^3;q^6)_\infty^2}\left\langle -q^2;q^6\right\rangle_\infty,$$
$$\frac{(-q;q)_\infty^2}{(-q^3;q^3)_\infty^2} - \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty^2} = 4q\frac{(-q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}\left\langle -q;q^6\right\rangle_\infty,$$
$$\frac{(-q;q)_\infty^4}{(-q^3;q^3)_\infty^4} - \frac{(q;q)_\infty^4}{(q^3;q^3)_\infty^4} = 8q\frac{(-q;q)_\infty}{(q^3;q^6)_\infty^5}.$$

Proof. Applying the relation $(\pm q;q)_\infty = (\pm q^3;q^3)_\infty \left\langle \pm q;q^3\right\rangle_\infty$, we can reformulate the first four differences as

$$\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} \pm \frac{(q;q)_\infty}{(q^3;q^3)_\infty} = \left\langle -q,-q^2;q^6\right\rangle \pm \left\langle q,q^2;q^6\right\rangle,$$
$$\frac{(-q;q)_\infty^2}{(-q^3;q^3)_\infty^2} \pm \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty^2} = \left\langle -q,-q;q^3\right\rangle \pm \left\langle q,q;q^3\right\rangle;$$

which lead to the first four identities displayed in the theorem. The last one follows from the product of the third and the fourth. □

Recall Ramanujan's ϕ-function (cf. Berndt [8], Entry 22, p. 36)

$$\phi(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = [q^2, -q, -q; q^2]_\infty.$$

It is trivial to see that $\phi(-q) = (q;q)_\infty/(-q;q)_\infty$. Then we can show the following identities.

Theorem 5 (Ramanujan's ϕ-function identities)**.**

$$\frac{\phi(-q^3)}{\phi(-q)} + \frac{\phi^2(-q^6)}{\phi^2(-q^2)} = 2\frac{(q^3;q^3)_\infty(q^{12};q^{12})_\infty \langle q^3, q^5; q^{12}\rangle_\infty}{(q;q)_\infty(q^2;q^2)_\infty(q^2;q^4)_\infty},$$
$$\frac{\phi(-q^3)}{\phi(-q)} - \frac{\phi^2(-q^6)}{\phi^2(-q^2)} = 2q\frac{(q^3;q^3)_\infty(q^{12};q^{12})_\infty \langle q, q^3; q^{12}\rangle_\infty}{(q;q)_\infty(q^2;q^2)_\infty(q^2;q^4)_\infty}.$$

Proof. Reformulate the differences

$$\begin{aligned}\frac{\phi(-q^3)}{\phi(-q)} \pm \frac{\phi^2(-q^6)}{\phi^2(-q^2)} &= \frac{(-q;q)_\infty}{(q;q)_\infty}\frac{(q^3;q^3)_\infty}{(-q^3;q^3)_\infty} \pm \frac{(-q^2;q^2)^2_\infty}{(q^2;q^2)^2_\infty}\frac{(q^6;q^6)^2_\infty}{(-q^6;q^6)^2_\infty}\\ &= \frac{\langle -q;q^3\rangle_\infty}{\langle q;q^3\rangle_\infty} \pm \frac{\langle -q^2;q^6\rangle^2_\infty}{\langle q^2;q^6\rangle^2_\infty}\\ &= \frac{\langle -q,q^2;q^6\rangle_\infty \pm \langle q,-q^2;q^6\rangle_\infty}{\langle q;q^3\rangle_\infty\langle q^2;q^6\rangle_\infty\langle q^2;q^{12}\rangle_\infty}.\end{aligned}$$

Then the equations in the theorem follow from the factorizations. □

Theorem 6 (Ramanujan's ϕ-function identities)**.**

$$\frac{\phi(-q^3)}{\phi(-q)} + \frac{\phi(-q^9)}{\phi(-q^3)} = 2\frac{(q^4;q^4)_\infty(q^{18};q^{18})_\infty}{(q;q)_\infty(q^3;q^3)_\infty(-q^{18};q^{18})_\infty},$$
$$\frac{\phi(-q^3)}{\phi(-q)} - \frac{\phi(-q^9)}{\phi(-q^3)} = 2q\frac{(-q;q)^2_\infty(q^{36};q^{36})_\infty}{(-q^2;q^2)_\infty(q^3;q^3)_\infty(-q^9;q^9)_\infty}.$$

Proof. Rewriting the differences as follows

$$\begin{aligned}\frac{\phi(-q^3)}{\phi(-q)} \pm \frac{\phi(-q^9)}{\phi(-q^3)} &= \frac{(-q;q)_\infty}{(q;q)_\infty}\frac{(q^3;q^3)_\infty}{(-q^3;q^3)_\infty} \pm \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty}\\ &= \frac{\langle -q;q^3\rangle_\infty}{\langle q;q^3\rangle_\infty} \pm \frac{\langle -q^3;q^9\rangle_\infty}{\langle q^3;q^9\rangle_\infty}\\ &= \frac{\langle q\omega, q\omega^2;q^3\rangle_\infty \pm \langle -q\omega,-q\omega^2;q^3\rangle_\infty}{(q;q^2)_\infty(-q^3;q^3)_\infty\langle q^3;q^9\rangle_\infty}\end{aligned}$$

leads us to the two identities displayed in the theorem. □

2. Rogers–Ramanujan functions

The Rogers–Ramanujan functions are connected with the following celebrated Rogers–Ramanujan identities [12], p. 44:

$$G(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{\langle q;q^5\rangle_\infty}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty},$$
$$H(q)=\sum_{n=0}^{\infty}\frac{q^{n+n^2}}{(q;q)_n}=\frac{1}{\langle q^2;q^5\rangle_\infty}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

Ramanujan recorded forty intriguing identities about them (see e.g. Birch [9]). By means of Jacobi's triple product identity, this section will exemplify a few identities related to G and H functions.

By considering the sum or difference of (1a) and (1b), it is almost trivial to derive the following factorization equation.

Lemma 2 (Dual lemma of Lemma 1)**.** *For two variables x and y, there holds the following theta function identity:*

$$\langle -xy,-qx/y;q^2\rangle_\infty+x\,\langle -qxy,-y/x;q^2\rangle_\infty = \frac{\langle -x,-y;q\rangle_\infty}{(-q;q)^2_\infty}.$$

In particular, performing the replacements $q\to q^2$ and $y\to qx$, from this lemma we recover the following useful equation

$$\langle -qx^2;q^4\rangle_\infty+x\,\langle -q/x^2;q^4\rangle_\infty=\frac{(q;q)_\infty}{(q^4;q^4)_\infty}\,\langle -x;q\rangle_\infty\,, \tag{3}$$

whose equivalent form can be found in [10], p. 61. Under the same parameter setting, Lemma 1 reduces further to two other theta function equations [5] Entry 30 (ii)-(iii); P 46 as follows:

$$\langle -x;q\rangle_\infty+\langle x;q\rangle_\infty=2\frac{(q^4;q^4)_\infty}{(q;q)_\infty}\,\langle -qx^2;q^4\rangle_\infty\,, \tag{4a}$$
$$\langle -x;q\rangle_\infty-\langle x;q\rangle_\infty=2x\frac{(q^4;q^4)_\infty}{(q;q)_\infty}\,\langle -q/x^2;q^4\rangle_\infty\,. \tag{4b}$$

The three equations just displayed can also be verified from Jacobi's triple product identity by splitting the infinite series according to the parity of summation index.

Theorem 7 (Watson [17]: see also Berndt *et al.* [8], Eqs 4.23 and 4.24)**.**

$$G(q^{16})+qH(-q^4)=\frac{(q^2;q^2)_\infty}{(q^8;q^8)_\infty}G(q),$$
$$G(-q^4)+q^3H(q^{16})=\frac{(q^2;q^2)_\infty}{(q^8;q^8)_\infty}H(q).$$

Proof. According to the definitions of $G(q)$ and $H(q)$, we can write

$$\begin{aligned} G(q^{16})+qH(-q^4) &= \frac{1}{\langle q^{16};q^{80}\rangle_\infty}+\frac{q}{\langle q^8;-q^{20}\rangle_\infty}\\ &= \frac{1}{\langle q^8,-q^8;q^{40}\rangle_\infty}+\frac{q}{\langle q^8,-q^{28};q^{40}\rangle_\infty}\\ &= \frac{\left\langle -q^{28};q^{40}\right\rangle_\infty+q\left\langle -q^8;q^{40}\right\rangle_\infty}{\langle q^8,-q^8,-q^{28};q^{40}\rangle_\infty}. \end{aligned}$$

Putting $q\to q^{10}$ and $x\to q$ in (3), we can factorize the last numerator

$$\left\langle -q^{28};q^{40}\right\rangle_\infty+q\left\langle -q^8;q^{40}\right\rangle_\infty=\frac{(q^{10};q^{10})_\infty}{(q^{40};q^{40})_\infty}\left\langle -q;q^{10}\right\rangle_\infty$$

which leads to the first identity stated in the theorem.

Similarly, reformulating the expression

$$\begin{aligned} G(-q^4)+q^3H(q^{16}) &= \frac{1}{\langle -q^4;-q^{20}\rangle_\infty}+\frac{q^3}{\langle q^{32};q^{80}\rangle_\infty}\\ &= \frac{1}{\langle -q^4,q^{24};q^{40}\rangle_\infty}+\frac{q^3}{\langle q^{16},-q^{16};q^{40}\rangle_\infty}\\ &= \frac{\left\langle -q^{16};q^{40}\right\rangle_\infty+q^3\left\langle -q^4;q^{40}\right\rangle_\infty}{\langle -q^4,q^{16},-q^{16};q^{40}\rangle_\infty} \end{aligned}$$

and then factorizing the last numerator by letting $q\to q^{10}$ and $x\to q^3$ in (3)

$$\left\langle -q^{16};q^{40}\right\rangle_\infty+q^3\left\langle -q^4;q^{40}\right\rangle_\infty=\frac{(q^{10};q^{10})_\infty}{(q^{40};q^{40})_\infty}\left\langle -q^3;q^{10}\right\rangle_\infty$$

we derive the second identity displayed in the theorem. □

Theorem 8. *Define the function* $V(x|q):=1/\left\langle x;q^5\right\rangle_\infty$. *There hold the following six equations:*

$$V(x|q)+V(-x|q)=2\frac{(q^{20};q^{20})_\infty\left\langle -q^5x^2;q^{20}\right\rangle_\infty}{(q^5;q^5)_\infty\langle x,-x;q^5\rangle_\infty}, \tag{5a}$$

$$V(x|q)-V(-x|q)=2x\frac{(q^{20};q^{20})_\infty\left\langle -q^5/x^2;q^{20}\right\rangle_\infty}{(q^5;q^5)_\infty\langle x,-x;q^5\rangle_\infty}, \tag{5b}$$

$$V(x|q)+V(-x|-q)=2\frac{(q^{40};q^{40})_\infty\left\langle -q^{10}x^2;q^{40}\right\rangle_\infty}{(q^{10};q^{10})_\infty\langle x,-x,q^5x;q^{10}\rangle_\infty}, \tag{5c}$$

$$V(x|q)-V(-x|-q)=2x\frac{(q^{40};q^{40})_\infty\left\langle -q^{10}/x^2;q^{40}\right\rangle_\infty}{(q^{10};q^{10})_\infty\langle x,-x,q^5x;q^{10}\rangle_\infty}, \tag{5d}$$

$$V(x|q)+V(x|-q)=2\frac{(q^{40};q^{40})_\infty\left\langle -q^{20}x^2;q^{40}\right\rangle_\infty}{(q^{10};q^{10})_\infty\langle x,q^5x,-q^5x;q^{10}\rangle_\infty}, \tag{5e}$$

$$V(x|q)-V(x|-q)=2q^5x\frac{(q^{40};q^{40})_\infty\left\langle -1/x^2;q^{40}\right\rangle_\infty}{(q^{10};q^{10})_\infty\langle x,q^5x,-q^5x;q^{10}\rangle_\infty}. \tag{5f}$$

Proof. In view of $\langle x;q^5\rangle_\infty = \langle x,q^5x;q^{10}\rangle_\infty$, we can rewrite the six differences in three pairs as follows:

$$\begin{aligned}
V(x|q)\pm V(-x|q) &= \frac{[q^5,-x,-q^5/x;q^5]_\infty \pm [q^5,x,q^5/x;q^5]_\infty}{(q^5;q^5)_\infty\,\langle x,-x;q^5\rangle_\infty},\\
V(x|q)\pm V(-x|-q) &= \frac{[q^{10},-x,-q^{10}/x;q^{10}]_\infty \pm [q^{10},x,q^{10}/x;q^{10}]_\infty}{(q^{10};q^{10})_\infty\,\langle x,-x,q^5x;q^{10}\rangle_\infty},\\
V(x|q)\pm V(x|-q) &= \frac{[q^{10},-q^5x,-q^5/x;q^{10}]_\infty \pm [q^{10},q^5x,q^5/x;q^{10}]_\infty}{(q^{10};q^{10})_\infty\,\langle x,q^5x,-q^5x;q^{10}\rangle_\infty}.
\end{aligned}$$

Factorizing the right members through (4a) and (4b) leads to the identities displayed in the theorem. □

Corollary 1 (Robins [16], Eqs 5-8)**.**

$$\begin{aligned}
G(q)+G(-q) &= 2\frac{(q^{40};q^{40})_\infty\,\langle -q^{12};q^{40}\rangle_\infty}{(q^{10};q^{10})_\infty\,\langle q,-q,q^6;q^{40}\rangle_\infty},\\
G(q)-G(-q) &= 2q\frac{(q^{40};q^{40})_\infty\,\langle -q^{8};q^{40}\rangle_\infty}{(q^{10};q^{10})_\infty\,\langle q,-q,q^6;q^{40}\rangle_\infty},\\
H(q)+H(-q) &= 2\frac{(q^{40};q^{40})_\infty\,\langle -q^{16};q^{40}\rangle_\infty}{(q^{10};q^{10})_\infty\,\langle q^2,q^7,-q^7;q^{40}\rangle_\infty},\\
H(q)-H(-q) &= 2q^3\frac{(q^{40};q^{40})_\infty\,\langle -q^{4};q^{40}\rangle_\infty}{(q^{10};q^{10})_\infty\,\langle q^2,-q^7,-q^7;q^{40}\rangle_\infty}.
\end{aligned}$$

They follow from the cases $x=q$ of (5c–5d) and $x=q^2$ of (5e–5f), respectively.

Theorem 9 (Identities of Rogers–Ramanujan functions)**.**

$$\begin{aligned}
G^4(q)+G^2(q^2) &= 2\frac{(-q^5;q^5)^2_\infty}{(q^5;-q^5)^2_\infty}\,\langle -q^2;q^{10}\rangle_\infty\,G^2(q)G^2(q^2),\\
G^4(q)-G^2(q^2) &= 4q\frac{(-q^5;q^5)^2_\infty}{(q^{10};q^{20})^2_\infty}\,\langle -q^3;q^{10}\rangle_\infty\,G^2(q)G^2(q^2),\\
H^4(q)+H^2(q^2) &= 2\frac{(-q^5;q^5)^2_\infty}{(q^5;-q^5)^2_\infty}\,\langle -q^4;q^{10}\rangle_\infty\,H^2(q)H^2(q^2),\\
H^4(q)-H^2(q^2) &= 4q^2\frac{(-q^5;q^5)^2_\infty}{(q^{10};q^{20})^2_\infty}\,\langle -q;q^{10}\rangle_\infty\,H^2(q)H^2(q^2).
\end{aligned}$$

Proof. Writing the differences in terms of theta function:

$$\begin{aligned}
G^4(q)\pm G^2(q^2) &= \frac{\langle -q,-q;q^5\rangle_\infty \pm \langle q,q;q^5\rangle_\infty}{\langle q;q^5\rangle^2_\infty\,\langle q^2;q^{10}\rangle^2_\infty},\\
H^4(q)\pm H^2(q^2) &= \frac{\langle -q^2,-q^2;q^5\rangle_\infty \pm \langle q^2,q^2;q^5\rangle_\infty}{\langle q^2;q^5\rangle^2_\infty\,\langle q^4;q^{10}\rangle^2_\infty};
\end{aligned}$$

and then factorizing the resulting numerators through Lemma 1, we derive the four identities in the theorem. □

Theorem 10 (Berndt *et al.* [8], Entries 3.20 and 3.21)**.**

$$\begin{aligned} G(q)H(-q)+G(-q)H(q) &= \frac{2}{(q^2;q^4)^2_\infty}, \\ G(q)H(-q)-G(-q)H(q) &= 2q\frac{(q^{20};q^{20})_\infty}{(q^2;q^2)_\infty(q^{10};q^{20})_\infty}. \end{aligned}$$

Proof. Rewrite explicitly the differences as

$$G(q)H(-q)\pm G(-q)H(q)=\frac{\langle -q,q^7;q^{10}\rangle_\infty \pm \langle q,-q^7;q^{10}\rangle_\infty}{\langle q,-q,q^2,q^4,q^7,-q^7;q^{10}\rangle_\infty}.$$

Applying Lemma 1 to the numerator in the last line, we derive the identities appearing in the theorem. □

Theorem 11 (Robins [16], Eqs 1.25–1.26)**.**

$$\begin{aligned} G^2(q)H(q^2)+G(q^2)H^2(q) &= 2\frac{(q^{10};q^{10})^2_\infty \langle q^3,q^4;q^{10}\rangle_\infty}{(q;q)_\infty(q^2;q^2)_\infty(q^5;q^{10})_\infty}, \\ G^2(q)H(q^2)-G(q^2)H^2(q) &= 2q\frac{(q^{10};q^{10})^2_\infty \langle q,q^2;q^{10}\rangle_\infty}{(q;q)_\infty(q^2;q^2)_\infty(q^5;q^{10})_\infty}. \end{aligned}$$

Proof. Reformulating the differences

$$G^2(q)H(q^2)\pm G(q^2)H^2(q)=\frac{\langle -q,q^2;q^5\rangle_\infty \pm \langle q,-q^2;q^5\rangle_\infty}{\langle q,q^2;q^5\rangle_\infty \langle q^2,q^4;q^{10}\rangle_\infty}$$

and then applying Lemma 1 to factorize the right member of the last equation, we find the identities stated in the theorem. □

3. Ramanujan's modular equations

Recall Ramanujan's χ and ψ-functions (cf. Berndt [5], Entry 22, p. 36)

$$\chi(q)=(-q;q^2)_\infty \quad \text{and} \quad \psi(q)=\sum_{n=0}^{\infty} q^{\binom{n+1}{2}}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

This section will prove few modular equations with some of them originally belonging to Ramanujan.

Consider the linear combination

$$b(q^3;q^6)_\infty + d(q;q^2)^3_\infty = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\Big\{b[q^2,q\omega,q\omega^2;q^2]_\infty + d[q^2,q,\ q;q^2]_\infty\Big\}.$$

Expanding the triple products and then splitting the series according to the residues of the summation index k modulo 3, we may reformulate the difference inside the braces as follows:

$$\begin{aligned} b[q^2, q\omega, q\omega^2; q^2]_\infty + d[q^2, q,\ q; q^2]_\infty &= \sum_{k=-\infty}^{+\infty} (-1)^k q^{k^2} \big\{ b\omega^k + d \big\} \\ &= \sum_{\varepsilon=0,\pm1} (-1)^\varepsilon \big\{ b\omega^\varepsilon + d \big\} q^{\varepsilon^2} \\ &\quad \times \sum_{k=-\infty}^{+\infty} (-1)^k q^{9k^2+6k\varepsilon}. \end{aligned}$$

Factorizing the infinite series via Jacobi's triple product identity and then simplifying the result according to $\omega + \omega^2 = -1$, we establish the following theorem.

Theorem 12 (Theta function identity)**.**

$$b(q^3; q^6)_\infty + d(q; q^2)^3_\infty = (b + d)\, \frac{\phi(-q^9)}{\psi(q)} + (b - 2d)\, q \frac{\psi(q^9)}{\psi(q)} \chi(-q^3).$$

As special cases, this theorem contains the following interesting modular equations due to Ramanujan [6], Entry 50(i), p. 202 and [7], Eq 5.9, p. 334.

Corollary 2 ($b = 1, d = 0$ and $b = 0, d = 1$ in Theorem 12)**.**

$$\begin{aligned} \phi(-q^9) \ +\ q\chi(-q^3)\psi(q^9) &= \chi(-q^3)\psi(q), \\ \phi(-q^9) - 2q\chi(-q^3)\psi(q^9) &= \chi^3(-q)\psi(q). \end{aligned}$$

Corollary 3 ($b = 1, d = -1$ and $b = 2, d = 1$ in Theorem 12)**.**

$$\begin{aligned} 1 - \frac{\chi^3(-q)}{\chi(-q^3)} &= 3q \frac{\psi(q^9)}{\psi(q)}, \\ 1 + 2\frac{\chi(-q^3)}{\chi^3(-q)} &= 3\frac{\phi(-q^9)}{\phi(-q)}. \end{aligned}$$

Similarly, consider another linear combination

$$b(q^3; q^6)^2_\infty + d(q; q^2)^6_\infty = \frac{(q; q^2)^2_\infty}{(q^2; q^2)^2_\infty} \Big\{ b[q^2, q\omega, q\omega^2; q^2]^2_\infty + d[q^2, q,\ q; q^2]^2_\infty \Big\}.$$

Expanding the two triple products and then splitting the series according to the residues of the summation indices i and j modulo 3, we may reformulate the difference inside the braces as follows:

$$\begin{aligned} b[q^2, q\omega, q\omega^2; q^2]^2_\infty + d[q^2, q,\ q; q^2]^2_\infty &= \sum_{i,j=-\infty}^{+\infty} (-1)^{i+j} q^{i^2+j^2} \big\{ b\omega^{i+j} + d \big\} \\ &= \sum_{\varepsilon,\epsilon=0,\pm1} (-1)^{\varepsilon+\epsilon} \big\{ b\omega^{\varepsilon+\epsilon} + d \big\} q^{\varepsilon^2+\epsilon^2} \\ &\quad \times \sum_{i,j=-\infty}^{+\infty} (-1)^{i+j} q^{9i^2+9j^2+6i\varepsilon+6j\epsilon}. \end{aligned}$$

It is not hard to check that the quadruplicate sum reduces to the following three terms:

$$(b+d)\left[q^{18},q^{9},q^{9};q^{18}\right]_\infty^2, \qquad \varepsilon=\epsilon=0;$$
$$(b+4d)q^2\left[q^{18},q^{3},q^{15};q^{18}\right]_\infty^2, \qquad \varepsilon=\pm1 \quad \text{and} \quad \epsilon=\pm1;$$
$$2(b-2d)q\left[q^{18},q^{9},q^{9};q^{18}\right]_\infty\left[q^{18},q^{3},q^{15};q^{18}\right]_\infty \quad \varepsilon=0,\epsilon=\pm1 \quad \text{and} \quad \epsilon=0,\varepsilon=\pm1.$$

In terms of Ramanujan's functions, we have derived the following theorem.

Theorem 13 (Theta function identity)**.**

$$\begin{aligned}b(q^3;q^6)^2_\infty+d(q;q^2)^6_\infty &= (b+d)\frac{\phi^2(-q^9)}{\psi^2(q)}+(b+4d)q^2\frac{\psi^2(q^9)}{\psi^2(q)}\chi^2(-q^3)\\ &\quad +2(b-2d)q\frac{\phi(-q^9)\psi(q^9)}{\psi^2(q)}\chi(-q^3).\end{aligned}$$

Encouraged by the last examples, we may further consider a linear combination

$$b(q^5;q^{10})_\infty+d(q;q^2)^5_\infty=(q;q^2)_\infty\Big\{b\left\langle q\varpi,q\varpi^2;q^2\right\rangle_\infty+d\left\langle q,q;q^2\right\rangle_\infty\Big\}.$$

Expanding the two triple products and then splitting the series according to the residues of $k:=i+2j$ modulo 5, we may reformulate the difference inside the braces:

$$\begin{aligned}&(q^2;q^2)^2_\infty\Big\{b\left\langle q\varpi,q\varpi^2;q^2\right\rangle_\infty+d\left\langle q,q;q^2\right\rangle_\infty\Big\}\\ &=\sum_{i,j=-\infty}^{+\infty}(-1)^{i+j}q^{i^2+j^2}\big\{b\varpi^{i+2j}+d\big\}\\ &=\sum_{\epsilon=0,\pm1,\pm2}(-1)^\epsilon(b\varpi^\epsilon+d)q^{\epsilon^2}\sum_{k=-\infty}^{+\infty}(-1)^kq^{25k^2+10k\epsilon}\\ &\quad\times\sum_{j=-\infty}^{+\infty}(-1)^jq^{10\binom{j}{2}+(5-4\epsilon-20k)j}.\end{aligned}$$

Factorizing the two infinite series via Jacobi's triple product identity

$$\begin{aligned}&\sum_{k=-\infty}^{+\infty}(-1)^kq^{25k^2+10k\epsilon}\sum_{j=-\infty}^{+\infty}(-1)^jq^{10\binom{j}{2}+(5-4\epsilon-20k)j}\\ &=\left[q^{10},q^{5+4\epsilon},q^{5-4\epsilon};q^{10}\right]_\infty\sum_{k=-\infty}^{+\infty}(-1)^kq^{10\binom{k}{2}+2k\epsilon+5k}\\ &=\left[q^{10},q^{5+4\epsilon},q^{5-4\epsilon};q^{10}\right]_\infty\left[q^{10},q^{5+2\epsilon},q^{5-2\epsilon};q^{10}\right]_\infty\end{aligned}$$

and then simplifying the result according to $\varpi+\varpi^2+\varpi^3+\varpi^4=-1$, we establish the following theorem.

Theorem 14 (Theta function identity).

$$b(q^5;q^{10})_\infty + d(q;q^2)_\infty^5 = (b+d)\frac{\phi^2(-q^5)}{\psi^2(q)\chi(-q)} + (b-4d)q\frac{\psi^2(q^5)}{\psi^2(q)}\chi(-q^5).$$

In addition, from this theorem we can recover the following modular equations.

Corollary 4 ($b=1, d=0$ and $b=0, d=1$ in Theorem 14).

$$\frac{\phi^2(-q^5)}{\chi(-q)} + q\psi^2(q^5)\chi(-q^5) = \chi(-q^5)\psi^2(q),$$
$$\frac{\phi^2(-q^5)}{\chi(-q)} - 4q\psi^2(q^5)\chi(-q^5) = \chi^5(-q)\psi^2(q).$$

Corollary 5 ($b=1, d=-1$ and $b=4, d=1$ in Theorem 14).

$$1-\frac{\chi^5(-q)}{\chi(-q^5)} = 5q\frac{\psi^2(q^5)}{\psi^2(q)},$$
$$1+4\frac{\chi(-q^5)}{\chi^5(-q)} = 5\frac{\phi^2(-q^5)}{\phi^2(-q)}.$$

The last two modular equations are originally due to Ramanujan, whose different proofs can be found in Andrews–Berndt [2] Theorem 1.6.1, Kang [14] Theorem 2.2 and Kongsiriwong–Liu [15] Eqs 8.3 and 8.4.

References

[1] G. E. ANDREWS, *Applications of basic hypergeometric functions*, SIAM Rev. **16**(1974), 441–484.

[2] G. E. ANDREWS, B. C. BERNDT, *Ramanujans Lost Notebook, Part I*, Springer–Verlag, New York, 2005.

[3] N. D. BARUAH, B. C. BERNDT, *Partition identities and Ramanujan's modular equations*, J. Combin. Theory (Ser. A) **114**(2007), 1024–1045.

[4] N. D. BARUAH, B. C. BERNDT, *Partition identities arising from theta function identities*, Acta Math. Sinica **24**(2008), 955–970.

[5] B. C. BERNDT, *Ramanujan's Notebooks (Part III)*, Springer–Verlag, New York, 1991.

[6] B. C. BERNDT, *Ramanujan's Notebooks (Part IV)*, Springer-Verlag, New York, 1994.

[7] B. C. BERNDT, Ramanujan's Notebooks (Part V), Springer–Verlag, New York, 1998.

[8] B. C. BERNDT, G. CHOI, Y. S. CHOI, H. HAHN, B. P. YEAP, A. J. YEE, H. YESILYURT, J. YI, *Ramanujan's Forty Identities for the Rogers–Ramanujan Functions*, Mem. Amer. Math. Society **188**(2007), vi+96 pp.

[9] B. J. BIRCH, *A look back at Ramanujan's notebooks*, Math. Proc. Cambridge Philos. Soc. **78**(1975), 73–79.

[10] S. L. CHEN, S. S. HUANG, *New modular relations for the Göllnitz-Gordon functions*, J. Number Theory **93**(2002), 58–75.

[11] W. CHU, *Theta function identities and Ramanujan's congruences on the partition function*, Quart. J. Math. **56**(2005), 491–506.

[12] G. GASPER, M. RAHMAN, *Basic Hypergeometric Series* Cambridge University Press, Cambridge, 1990.

[13] C. G. J. JACOBI, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Fratrum Bornträger Regiomonti, 1829; Gesammelte Werke, Erster Band, G. Reimer, Berlin, 1881.

[14] S. Y. KANG, *Some theorems on the Rogers–Ramanujan continued fraction and associated theta function identities in Ramanujan's Lost Notebook*, The Ramanujan Journal **3**(1999), 91–111.

[15] S. KONGSIRIWONG, Z. G. LIU, *Uniform proof of q-series-product identities*, Results in Math. **44**(2003), 312–339.

[16] S. ROBINS, *Arithmetic Properties of Modular Forms*, PhD Thesis, University of California, Los Angeles, 1991.

[17] G. N. WATSON, *Proof of certain identities in combinatory analysis*, J. Indian Math. Soc. **20**(1933), 57–69.