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Complex oscillation of differential polynomials generated by analytic solutions of differential equations in the unit disc*

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Abstract. In this paper, we investigate the complex oscillation of differential polynomials generated by solutions of differential equations

$$f'' + A(z)f = 0,$$

where the coefficient A is analytic in the unit disc $\mathbb{D} = \{z : |z| < 1\}$.

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1. Introduction and main results

Complex oscillation theory of solutions of linear differential equations in the complex plane $\mathbb{C}$ was started by Bank and Laine [2, 3]. After their well-known work, many important results have been obtained on the complex oscillation theory of solutions of linear differential equations in $\mathbb{C}$, see [18, 19]. The study on value distribution theory of differential polynomials generated by solutions of complex differential equations in the case of plane, according to our knowledge, has been initiated by Bank [1]. For further results, refer to see [20, 24, 6]. In particular, some results on oscillation of fixed points of solutions of differential equations can be found in [4, 5, 11, 12, 21, 23].

Recently, Chuaqui and Stowe [13] investigated the number of times that nontrivial solutions of the equation

$$f'' + A(z)f = 0 \tag{1}$$

in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ can vanish. Cao and Yi [10] obtained some oscillation results of analytic solutions of equation (1) in $\mathbb{D}$. In [7], some results on the complex oscillation theory of analytic solutions of higher order differential

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equations in $\mathbb{D}$ were obtained. In this paper, we continue to consider this subject and investigate the complex oscillation theory of differential polynomials generated by analytic solutions of differential equations in $\mathbb{D}$.

We assume that the reader is familiar with the fundamental results and standard notations of the Nevanlinna's value distribution theory of meromorphic functions such as $T(r,f)$, $\overline{N}(r,f)$, $N(r,f)$, $m(r,f)$, see [15, 25]. Let f be an analytic function in the unit disc $\mathbb{D}=\{z:|z|<1\}$, and let $M(r,f)$ be the maximum modulus of f on the circle of radius r centered at the origin. We introduce some definitions as follows, e.g. see [7, 9, 17, 19].

Definition 1. *Defining*

$$D_M(f)=\limsup_{r\to 1^-}\frac{\log^+ M(r,f)}{-\log(1-r)},$$

we say that f is of finite degree, if $D_M(f)<\infty$, while if $D_M(f)=\infty$, we say that f is of infinite degree.

Definition 2. *For $n\in\mathbb{N}$, the iterated n-order of f is defined by*

$$\sigma_{M,n}(f)=\limsup_{r\to 1^-}\frac{\log^+_{n+1} M(r,f)}{-\log(1-r)},$$

where $\log^+_1 x=\log^+ x$, $\log^+_{n+1}=\log^+\log^+_n x$.

Definition 3. *The finiteness degree of the order of f is defined by*

$$i(f)=\begin{cases}0, & \text{if } f \text{ is of finite degree,}\\ \min\{n\in\mathbb{N}:\sigma_{M,n}(f)<\infty\}, & \text{if } f \text{ is of infinite degree,}\\ \infty, & \text{if } \sigma_{M,n}(f)=\infty \text{ for all } n\in\mathbb{N}.\end{cases}$$

Definition 4. *The iterated n-convergence exponent of the sequence of distinct zeros in $\mathbb{D}$ of f is defined by*

$$\overline{\lambda}_n(f)=\limsup_{r\to 1^-}\frac{\log^+_n \overline{N}(r,\frac{1}{f})}{-\log(1-r)}.$$

Definition 5. *The finiteness degree of the convergence exponent of the sequence of distinct zeros in $\mathbb{D}$ of f is defined by*

$$i_{\overline{\lambda}}(f)=\begin{cases}0, & \text{if } \overline{N}(r,\frac{1}{f})=O(\log\frac{1}{1-r}),\\ \min\{n\in\mathbb{N}:\lambda_n(f)<\infty\}, & \text{if some } n\in\mathbb{N} \text{ with } \lambda_n(f)<\infty \text{ exists,}\\ \infty, & \text{if } \lambda_n(f)=\infty \text{ for all } n\in\mathbb{N}.\end{cases}$$

For a function f meromorphic in $\mathbb{D}$, the iterated n-order $\sigma_n(f)$ is defined by

$$\sigma_n(f):=\limsup_{r\to 1^-}\frac{\log^+_n T(r,f)}{-\log(1-r)}.$$

Let $\mathcal{L}(G)$ denote a differential subfield of the field $\mathcal{M}(G)$ of meromorphic functions in a domain $G \subset \mathbb{C}$. Throughout this paper, we simply denote $\mathcal{L}$ instead of $\mathcal{L}(\mathbb{D})$. Special cases of such differential subfields used below are

$$\mathcal{L}_f := \{g \text{ meromorphic} : T(r,g) = S(r,f)\}$$

and

$$\mathcal{L}_{p+1,\sigma} := \{g \text{ meromorphic} : \sigma_{p+1}(g) < \sigma\}$$

where σ is a positive constant and $S(r,f) = O\left(\log^+(\frac{1}{1-r}T(r,f))\right)$ possibly outside a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} < \infty$. Note that for an analytic function f in $\mathbb{D}$, $\sigma_{M,n}(f) = \sigma_n(f)$ holds, where $n \geq 2$.

Now we show our main results as follows.

Theorem 1. *Let A be an analytic function of infinite degree and finite iterated order $\sigma_{M,p}(A) := \sigma > 0$ $(0 < p < \infty)$ in the unit disc $\mathbb{D}$, and let f be a non-zero solution of equation* (1)*. Moreover, let*

$$P[f] = P(f, f', \ldots, f^{(\nu)}) = \sum_{j=0}^{\nu} p_j f^{(j)} \tag{2}$$

be a linear differential polynomial with coefficients $p_j \in \mathcal{L}_{p+1,\sigma}$, assuming that at least one of the analytic coefficients p_j does not vanish identically. If $\varphi \in \mathcal{L}_{p+1,\sigma}$ is a non-zero analytic function in $\mathbb{D}$, and neither $P[f]$ nor $P[f]-\varphi$ vanishes identically, then we have

$$i_{\overline{\lambda}}(P[f] - \varphi) = i(f) = p + 1$$

and

$$\overline{\lambda}_{p+1}(P[f] - \varphi) = \sigma_{M,p+1}(f) = \sigma_{M,p}(A) = \sigma.$$

Theorem 2. *Let $k \geq 2$ and A be an analytic function of infinite degree and finite iterated order $\sigma_{M,p}(A) = \sigma > 0$ $(0 < p < \infty)$ in the unit disc $\mathbb{D}$. Assume that $\varphi \in \mathcal{L}_{p+1,\sigma}$ is an analytic function in $\mathbb{D}$ such that $\varphi^{(k-j)} \not\equiv 0$ $(j = 0, 1, \ldots, k)$. Then every non-zero solution f of the equation*

$$f^{(k)} + A(z)f = 0 \tag{3}$$

satisfies that for $j = 0, 1, \ldots, k$,

$$i_{\overline{\lambda}}(f^{(j)} - \varphi) = i(f^{(j)} - \varphi) = i(f) = p + 1$$

and

$$\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_{M,p+1}(f^{(j)} - \varphi) = \sigma_{M,p+1}(f) = \sigma_{M,p}(A) = \sigma.$$

Theorem 2 is an extension of Theorem 1.1 in [26] which is a result on the fixed points of analytic solutions of (3). The ideas of the proofs of Theorems 1 and 2 are from [20] and [5], respectively, with modification from the complex plane $\mathbb{C}$ to the unit disc $\mathbb{D}$. We feel that $f^{(j)}$ in Theorem 2 can be replaced by $P[f]$, but we have not been able to prove this.

2. Some lemmas

For the proofs of our main results, we need some lemmas. The first part of the following lemma is a standard result (see e. g. [15]), and the second part is due to [22].

Lemma 1. *Let f be a meromorphic function in the unit disc, and let $k \in \mathbb{N}$. Then*

$$m(r, \frac{f^{(k)}}{f}) = S(r, f),$$

where $S(r,f) = O\left(\log^{+} T(r,f)\right) + O\left(\log\left(\frac{1}{1-r}\right)\right)$, possibly outside a set $E \subset [0,1)$ with $\int\limits_{E} \frac{dr}{1-r} < \infty$. If f is of finite order of growth (namely, $\sigma_1(f) < \infty$), then

$$m(r, \frac{f^{(k)}}{f}) = O\left(\log\left(\frac{1}{1-r}\right)\right).$$

If f is non-admissible (namely, $D(f) = \limsup\limits_{r\to 1^{-}} \frac{T(r,f)}{-\log(1-r)} < \infty$), then

$$m(r, \frac{f^{'}}{f}) \le \log\frac{1}{1-r} + (2+o(1))\log\log\frac{1}{1-r}.$$

Lemma 2 (see [9])**.** *Let f be an analytic function in $\mathbb{D}$ such that $i(f) = n$ $(0 < n < \infty)$. Then there exists a set $H \subset [0,1)$ with $\int\limits_{H} \frac{dr}{1-r} = \infty$ such that for $r \in H$, given $\varepsilon > 0$, we have*

$$M(r,f) \ge \exp_n(\frac{1}{1-r})^{\sigma_{M,n}(f)-\varepsilon}.$$

Lemma 3 (see [14], Theorem 3.1)**.** *Let k and j be integers satisfying $k > j \ge 0$, and let $\varepsilon > 0$ and $d \in (0,1)$. If f is a meromorphic in D such that $f^{(j)}$ does not vanish identically, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{k-j}, \quad |z| \notin E,$$

where $E \subset [0,1)$ with finite logarithmic measure $\int_E \frac{dr}{1-r} < \infty$ and $s(|z|) = 1 - d(1 - |z|)$. Moreover, if $\sigma_1(f) < \infty$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \left(\frac{1}{(1-|z|)}\right)^{(k-j)(\sigma_1(f)+2+\varepsilon)}, \quad |z| \notin E,$$

while if $\sigma_n(f) < \infty$ for some $n \ge 2$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \exp_{n-1}\left(\left(\frac{1}{(1-|z|)}\right)^{\sigma_n(f)+\varepsilon}\right), \quad |z| \notin E.$$

Lemma 4 (see [8], Lemma 2.5). *Let A_0, A_1, …, A_{k-1} and $F \not\equiv 0$ be meromorphic functions in $\mathbb{D}$ and let f be a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = F(z), \tag{4}$$

such that $\max\{\sigma_{n+1}(F), \sigma_{n+1}(A_j)(j = 0, 1, \ldots, k-1)\} < \sigma_{n+1}(f)$. *Then* $\overline{\lambda}_{n+1}(f) = \sigma_{n+1}(f)$.

Lemma 5 (see [9, 17]). *Let A_0, A_1, …, A_{k-1} be analytic functions in $\mathbb{D}$ such that $i(A_0) = p$ $(0 < p < \infty)$ and that either* $\max\{i(A_j) : j = 1, \ldots, k-1\} < p$ *or* $\max\{\sigma_{M,p}(A_j) : j = 1, \ldots, k-1\} < \sigma_{M,p}(A_0)$. *Then every solution $f \not\equiv 0$ of the equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = 0,$$

satisfies $i(f) = p + 1$ and $\sigma_{M,p+1}(f) = \sigma_{M,p}(A_0)$.

3. Proof of Theorem 1

Since A is analytic in the unit disc $\mathbb{D}$, it is well known that f is also analytic in $\mathbb{D}$. By Lemma 5, we have $i(f) = p + 1$, and $\overline{\lambda}_{p+1}(P[f] - \varphi) \leq \sigma_{M,p+1}(f) = \sigma_{M,p}(A) = \sigma$. If the assertion is not true, then we may assume that

$$\overline{\lambda}_{p+1}(P[f] - \varphi) := \overline{\lambda}_{p+1} < \sigma. \tag{5}$$

Obviously, $A \in \mathcal{L}_{p+1,\sigma}$. We may assume that $\nu \leq 1$. Indeed, if $\nu \geq 2$, then by repeated differentiation of (1) we obtain that $f^{(k)} = q_{k,0}f + q_{k,1}f^{'}$, $q_{k,0}, q_{k,1} \in \mathcal{L}_{p+1,\sigma}$ for $k = 2, 3, \ldots, \nu$. Substituting into the form of $P[f]$ yields the required reduction. Hence, we may assume, from now on, that $P[f] = p_0 f + p_1 f^{'}$, where at least one of the coefficients $p_0, p_1 \in \mathcal{L}_{p+1,\sigma}$ does not vanish identically.

Note that

$$T(r, \frac{(P[f] - \varphi)^{'}}{P[f] - \varphi}) = m(r, \frac{(P[f] - \varphi)^{'}}{P[f] - \varphi}) + \overline{N}(r, \frac{1}{P[f] - \varphi}), \tag{6}$$

Hence, by Lemma 1, (5), (6) and the standard method of removing exceptional sets, we get that for some $\beta < \sigma$ and $r \to 1^-$, there holds

$$T(r, \frac{(P[f] - \varphi)^{'}}{P[f] - \varphi}) = O(\exp_p(\frac{1}{1-r})^{\beta}).$$

Hence, there exists a meromorphic function $h \in \mathcal{L}_{p+1,\sigma}$ such that

$$(P[f] - \varphi)^{'} = h(P[f] - \varphi). \tag{7}$$

Using the fact that $f^{''} = -Af$, we may rewrite (1) as

$$(p_0 + p_1^{'} - hp_1)f^{'} + (p_0^{'} - p_1 A - hp_0)f + h\varphi - \varphi^{'} = 0. \tag{8}$$

we denote $b_1 := p_0 + p_1' - hp_1$ and $b_0 := p_0' - p_1 A - hp_0$.

We first assume that $b_1(z) \equiv 0$ and $b_0(z) \not\equiv 0$. Then $f = \frac{\varphi' - h\varphi}{b_0}$. Hence, $f \in \mathcal{L}_{p+1,\sigma}$ and so $\sigma_{M,p+1}(f) < \sigma$, a contradiction.

Assume that $b_0(z) \equiv 0$ and $b_1(z) \not\equiv 0$. Then $f' = \frac{\varphi' - h\varphi}{b_1}$. Hence, $f' \in \mathcal{L}_{p+1,\sigma}$ and so $\sigma_{M,p+1}(f) = \sigma_{M,p+1}(f') < \sigma$, also a contradiction.

Assume that $b_0(z) \equiv 0$ and $b_1(z) \equiv 0$. Then we have $h = \frac{\varphi'}{\varphi}$ because of $\varphi(z) \not\equiv 0$. Hence, there hold

$$b_0 = p_0' - p_1 A - \frac{\varphi' p_0}{\varphi} = 0 \tag{9}$$

and

$$b_1 = p_0 + p_1' - \frac{\varphi' p_1}{\varphi} = 0. \tag{10}$$

By (9) and (10) we get

$$A = -\frac{p_1''}{p_1} + \frac{\varphi''}{\varphi} + 2\frac{\varphi'}{\varphi}\frac{p_1'}{p_1} - 2(\frac{\varphi'}{\varphi})^2.$$

this yields

$$|A(z)| \le \left|\frac{p_1''(z)}{p_1(z)}\right| + \left|\frac{\varphi''(z)}{\varphi(z)}\right| + 2\left|\frac{\varphi'(z)}{\varphi(z)}\right|\left|\frac{p_1'(z)}{p_1(z)}\right| + 2\left|\frac{\varphi'(z)}{\varphi(z)}\right|^2. \tag{11}$$

By Lemma 2 (or Lemma 2.1 in [17]), Lemma 3 and (11) we have

$$\exp_p(\frac{1}{1-r})^{\sigma-\varepsilon} \le M(r,A) \le \exp_p(\frac{1}{1-r})^{\beta+\varepsilon}, \quad r \in H \setminus E$$

for some $\beta < \sigma - 2\varepsilon$. This is a contradiction.

Therefore, we may now assume that neither b_0 nor b_1 vanishes identically. Rewrite equation (8) as

$$b_0 f + b_1 f' = \varphi' - h\varphi. \tag{12}$$

Differentiating equation (12) and making use of $f'' = -Af$, we have

$$(b_0' - b_1 A)f + (b_0 + b_1')f' = (\varphi' - h\varphi)'. \tag{13}$$

If the pair of equations (12) and (13) to determine f and f' has a nonidentically vanishing determinant, then we must have

$$(b_0^2 + b_0 b_1' - b_1 b_0' + b_1^2 A)f = -(\varphi' - h\varphi)(b_0 + b_1') + (\varphi' - h\varphi)' b_1. \tag{14}$$

Hence, we have $f \in \mathcal{L}_{p+1,\sigma}$, and thus $\sigma_{M,p+1}(f) < \sigma$, a contradiction. Hence, the determinant vanishes, and thus we have

$$b_0^2 + b_0 b_1' - b_1 b_0' + b_1^2 A = 0 \tag{15}$$

and

$$-(\varphi' - h\varphi)(b_0 + b_1') + (\varphi' - h\varphi)' b_1 = 0. \tag{16}$$

If now $\varphi'(z) - h(z)\varphi(z) \not\equiv 0$, then by an easy computation we deduce from (16) and (15) that

$$\frac{b_0}{b_1} = \frac{((\varphi' - h\varphi)/b_1)'}{(\varphi' - h\varphi)/b_1}$$

and

$$A = (\frac{b_0}{b_1})' - (\frac{b_0}{b_1})^2$$

hold, respectively. This yields

$$|A(z)| \leq \left| \left(\frac{((\varphi' - h\varphi)/b_1)'}{(\varphi' - h\varphi)/b_1} \right)' \right| + \left| \frac{((\varphi' - h\varphi)/b_1)'}{(\varphi' - h\varphi)/b_1} \right|^2. \tag{17}$$

By Lemma 2 (or Lemma 2.1 in [17]), Lemma 3 and (17) we have

$$\exp_p(\frac{1}{1-r})^{\sigma-\varepsilon} \leq M(r, A) \leq \exp_p(\frac{1}{1-r})^{\beta+\varepsilon}, \quad r \in H \setminus E$$

for some $\beta < \sigma - 2\varepsilon$. This is a contradiction. Hence, we must have $\varphi'(z) - h(z)\varphi(z) \equiv 0$, and thus $h = \frac{\varphi'}{\varphi}$. Integrating (7) we have

$$P[f] = p_0 f + p_1 f' = C\varphi, \tag{18}$$

where $C \neq 0, 1$ by assumption, while equation (12) reduces to

$$b_0 f + b_1 f' = 0. \tag{19}$$

As the determinant of the pair (18) and (19) obviously has to be nonzero, we obtain $f = \frac{C\varphi}{p_0 b_1 - b_0 p_1}$. We also obtain $f \in \mathcal{L}_{p+1,\sigma}$, and thus $\sigma_{M,p+1}(f) < \sigma$, a contradiction.

Therefore, we have

$$i(f) = i_{\overline{\lambda}}(P[f] - \varphi) = p + 1$$

and

$$\overline{\lambda}_{p+1}(P[f] - \varphi) = \sigma_{M,p+1}(f) = \sigma_{M,p}(A) = \sigma.$$

4. Proof of Theorem 2

Suppose that $f(z) \not\equiv 0$ is an analytic solution of equation (3). Set $w_j = f^{(j)} - \varphi$ ($j = 0, 1, \ldots, k$), where $\varphi \in \mathcal{L}_{p+1,\sigma}$. Then for $j = 0, 1, \ldots, k$, we deduce by Lemma 5 that $i(w_j) = i(f) = p+1$, and $\sigma_{p+1}(w_j) = \sigma_{M,p+1}(f) = \sigma_{M,p}(A) = \sigma$. Differentiating both sides of $w_j = f^{(j)} - \varphi$ and replacing $f^{(k)}$ with $f^{(k)} = -Af$, we obtain that

$$w_j^{(k-j)} = -Af - \varphi^{(k-j)}, \quad j = 0, 1, \ldots, k.$$

Thus we have

$$f = -\frac{w_j^{(k-j)} + \varphi^{(k-j)}}{A}. \tag{20}$$

Combining (3) and (20) we obtain

$$\left(\frac{w_j^{(k-j)}}{A}\right)^{(k)} + w_j^{(k-j)} = -\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + \varphi^{(k-j)}\right),$$

and thus

$$\begin{aligned} &w_j^{(2k-j)} + g_{2k-j-1} w_j^{(2k-j-1)} + \ldots + g_{k-j} w_j^{(k-j)} \\ &= -A\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A(\frac{\varphi^{(k-j)}}{A})\right), \end{aligned} \tag{21}$$

where $g_{k-j}, \ldots, g_{2k-j-1} \in \mathcal{L}_{p+1,\sigma}$ $(j = 0, 1, \ldots, k)$ are meromorphic functions in $\mathbb{D}$.

Note that there holds $A \not\equiv 0$, $\varphi^{(k-j)} \not\equiv 0$ and $\frac{\varphi^{(k-j)}}{A} \in \mathcal{L}_{p+1,\sigma}$. Assume that

$$F := -A\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A(\frac{\varphi^{(k-j)}}{A})\right) \equiv 0.$$

Thus

$$\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A(\frac{\varphi^{(k-j)}}{A}) \equiv 0.$$

Then by Lemma 5 we obtain $i(\frac{\varphi^{(k-j)}}{A}) = p+1$ and $\sigma_{M,p+1}(\frac{\varphi^{(k-j)}}{A}) = \sigma_{M,p}(A) = \sigma$, a contradiction. Hence we have $F \not\equiv 0$. Obviously, there holds

$$\max\{\sigma_{p+1}(g_{k-j}), \ldots, \sigma_{p+1}(g_{2k-j-1}), \sigma_{p+1}(F)\} < \sigma \leq \sigma_{M,p+1}(w_j) = \sigma_{p+1}(w_j)$$

for $j = 0, 1, \ldots, k$. By Lemma 4 we have

$$i_{\overline{\lambda}}(w_j) = i_\lambda(w_j) = i(w_j) = p+1 \quad \text{and} \quad \overline{\lambda}_{p+1}(w_j) = \sigma_{M,p+1}(w_j),$$

where $j = 0, 1, \ldots, k$. Hence, for $j = 0, 1, \ldots, k$, we obtain our assertion that

$$i_{\overline{\lambda}}(f^{(j)} - \varphi) = i_\lambda(f^{(j)} - \varphi) = i(f^{(j)} - \varphi) = i(f) = p+1$$

and

$$\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_{M,p+1}(f^{(j)} - \varphi) = \sigma_{M,p+1}(f) = \sigma_{M,p}(A) = \sigma.$$

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