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Buffon's problem with a cluster of line segments and a lattice of parallelograms

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Abstract. A cluster $\mathcal{Z}_n$ of n line segments ($1 \le n < \infty$) is dropped at random onto two given lattices $\mathcal{R}_a$ and $\mathcal{R}_b$ of equidistant lines in the plane with angle β ($0 < \beta \le \pi/2$) between the lines of $\mathcal{R}_a$ and the lines of $\mathcal{R}_b$. Formulas for the probabibilities $p_n(i)$ of exactly i ($0 \le i \le 2n$) intersections between $\mathcal{Z}_n$ and $\mathcal{R}_{a,\,b,\,\beta} = \mathcal{R}_a \cup \mathcal{R}_b$ are derived. The limit distribution of the random variable *relative number of intersections between* $\mathcal{Z}_n$ *and* $\mathcal{R}_{a,\,b,\,\beta}$ as $n \to \infty$ is calculated.
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1. Introduction

We define two lattices $\mathcal{R}_a$ and $\mathcal{R}_b$ of parallel lines in the plane,

$$\mathcal{R}_a := \{(x,y) \in \mathbb{R}^2 \mid x \sin\beta - y\cos\beta = ka\,, k \in \mathbb{Z}\}\,,$$
$$\mathcal{R}_b := \{(x,y) \in \mathbb{R}^2 \mid y = mb\,, m \in \mathbb{Z}\}\,,$$

where a and b are positive real constants, $\beta \in \mathbb{R}$, $0 < \beta \le \pi/2$, and put $\mathcal{R}_{a,\,b,\,\beta} = \mathcal{R}_a \cup \mathcal{R}_b$ (see Figure 1). We may consider $\mathcal{R}_{a,\,b,\,\beta}$ as a lattice of parallelograms. Furthermore, we consider a *cluster* $\mathcal{Z}_n$ of n, $1 \le n < \infty$, line segments with length 1. All n line segments are fixed with one end-point in the common centre point C of $\mathcal{Z}_n$. $\mathcal{Z}_n$ is thrown at random onto $\mathcal{R}_{a,\,b,\,\beta}$ with $\min(a,b) \ge 2$. The *random throw of* $\mathcal{Z}_n$ *onto* $\mathcal{R}_{a,\,b,\,\beta}$ is defined as follows: The coordinates x and y of C are random variables uniformly distributed in $[y\cot\beta, a\csc\beta + y\cot\beta]$ and $[0,b]$ resp.; the angle ϕ_i between the x-axis and the line segment i is for $i \in \{1,\dots,n\}$ a random variable uniformly distributed in $[0,2\pi]$. All $n{+}2$ random variables are stochastically independent. There are at most $2n$ intersections between $\mathcal{Z}_n$ and $\mathcal{R}_{a,\,b,\,\beta}$.

The following intersection probabilities are already known:

- $\mathcal{Z}_1$ and $\mathcal{R}_{a,\,b,\,\beta}$ [12, p. 139], [13], [11] and [14],
- $\mathcal{Z}_n$ and $\mathcal{R}_a$ [2, pp. 82-85],
- $\mathcal{Z}_n$ and $\mathcal{R}_{a,\,b,\,\pi/2}$ [4],

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- ellipses and $\mathcal{R}_{a,\,b,\,\pi/2}$ [8],
- arbitrary convex bodies and $\mathcal{R}_{a,\,b,\,\beta}$ [1],
- $\mathcal{Z}_2$ and the lattice of regular hexagons [5].

In [7], the limit distribution of the number of intersections between a line segment (needle) of length ℓ and $\mathcal{R}_a$ as $\ell/a \to \infty$ is derived.

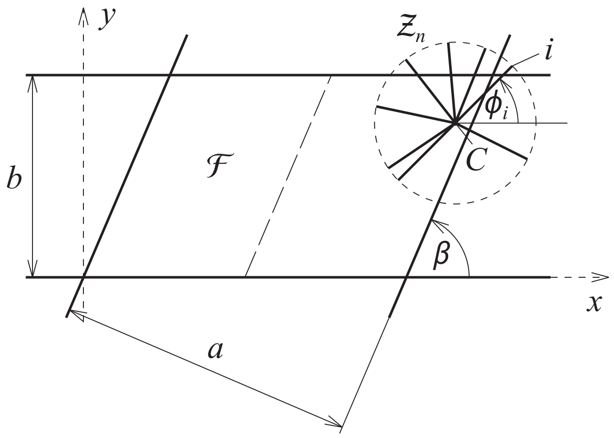

Figure 1: *Cluster $\mathcal{Z}_n$ (example $n = 8$) and lattice $\mathcal{R}_{a,\,b,\,\beta}$*

Using $\lambda := 1/a$ and $\mu := 1/b$ we define the following random variables:

$$X_{n,\,\lambda} := (\text{number of intersections between } \mathcal{Z}_n \text{ and } \mathcal{R}_a)\,/\,n\,,$$
$$X_{n,\,\mu} := (\text{number of intersections between } \mathcal{Z}_n \text{ and } \mathcal{R}_b)\,/\,n\,,$$
$$X_{n,\,\lambda,\,\mu} := (\text{number of intersections between } \mathcal{Z}_n \text{ and } \mathcal{R}_{a,\,b,\,\beta})\,/\,n\,.$$

In [2, pp. 90-93], it was shown: As $n \to \infty$, the random variables $X_{n,\,\lambda}$ converge uniformly to a random variable X_λ with distribution function

$$F_\lambda(x) = \lim_{n\to\infty} F_{n,\,\lambda}(x) = \begin{cases} 0\,, & \text{if} \quad -\infty < x < 0\,, \\ 1 - 2\lambda\cos\pi x\,, & \text{if} \quad 0 \le x < 1/2\,, \\ 1\,, & \text{if} \quad 1/2 \le x < \infty\,. \end{cases}$$

Replacing λ by μ we get the limit distribution $F_\mu = \lim_{n\to\infty} F_{n,\,\mu}$ of the random variables $X_{n,\,\mu}$.

For finite n it is not possible to calculate the distribution of $X_{n,\,\lambda,\,\mu} = X_{n,\,\lambda} + X_{n,\,\mu}$ by using the convolution of the distributions $F_{n,\,\lambda}$ and $F_{n,\,\mu}$. It follows that the random variables $X_{n,\,\lambda}$ and $X_{n,\,\mu}$ are not independent of finite n.

If the random variables X_λ and X_μ are independent, the distribution of the sum $X_{\lambda,\,\mu} = X_\lambda + X_\mu$ can be calculated with the convolution

$$F_{\lambda,\,\mu}(x) = P(X_\lambda + X_\mu \le x) = \int_{-\infty}^{\infty} F_\lambda(x-y)\,\mathrm{d}F_\mu(x), \quad \text{see [6, p. 90]}$$

and one gets

$$F_{\lambda,\,\mu}(x) = \begin{cases} 0 & \text{for } -\infty < x < 0\,, \\ 1 - 2(\lambda+\mu)\cos\pi x & \\ \quad + 2(2\cos\pi x - \pi x \sin\pi x)\lambda\mu & \text{for } \ 0 \le x < \frac{1}{2}\,, \\ 1 + 2\pi(x-1)\lambda\mu\sin\pi x & \text{for } \ \frac{1}{2} \le x < 1\,, \\ 1 & \text{for } \ 1 \le x < \infty. \end{cases} \qquad (1)$$

In [3], it was shown that the random variables $X_{n,\,\lambda,\,\mu}$ converge uniformly to $X_{\lambda,\,\mu}$ with distribution (1) as $n \to \infty$, if $\beta = \pi/2$.

2. Intersection probabilities

$p_n(i)$, $i \in \{0,\dots,2n\}$, denotes the probability of exactly i intersections between $\mathcal{Z}_n$ and $\mathcal{R}_{a,\,b,\,\beta}$.

Due to existing symmetries it is sufficient to consider only the subset

$$\mathcal{F} = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le b\,,\ y\cot\beta \le x \le (a/2)\csc\beta + y\cot\beta\}$$

of the parallelogram in Figure 1. With $p_n(i\,|\,(x,y))$ we denote the conditional probability that $\mathcal{Z}_n$ with centre point $C = (x,y) \in \mathcal{F}$ has exactly i intersections with $\mathcal{R}_{a,\,b,\,\beta}$. $q_j(x,y)$ is the conditional probability that a single line segment with point $C = (x,y) \in \mathcal{F}$ has exactly $j \in \{0,1,2\}$ intersections with $\mathcal{R}_{a,\,b,\,\beta}$. This is the case if this line segment is inside an angle or in a disjoint union of angles. $\alpha_j(x,y)$ denotes the value of this angle or the sum of the values of this disjoint union. We set $\alpha_j(x,y) = 0$, if such an angle or such a union does not exist. So we have

$$q_j(x,y) = \frac{\alpha_j(x,y)}{2\pi}\,. \qquad (2)$$

The conditional probabilities $p_n(i\,|\,(x,y))$ for the whole cluster $\mathcal{Z}_n$ are given by

$$p_n(i\,|\,(x,y)) = \sum_{j=0}^{\lfloor i/2\rfloor} \binom{n}{i-j}\binom{i-j}{j}\, q_2(x,y)^j\, q_1(x,y)^{i-2j}\, q_0(x,y)^{n-i+j}\,, \quad i \in \{0,1,\dots,2n\}\,, \qquad (3)$$

where $\lfloor\,\cdot\,\rfloor$ denotes the integer part of $\,\cdot\,$. The (total) intersection probabilities $p_n(i)$ are given by

$$p_n(i) = \iint_{\mathcal{F}} p_n(i\,|\,(x,y)) f_1(x) f_2(y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

where

$$f_1(x) = \begin{cases} (2/a)\sin\beta, & \text{for } \ y\cot\beta \le x \le (a/2)\csc\beta + y\cot\beta\,, \\ 0, & \text{elsewhere} \end{cases}$$

and

$$f_2(x) = \begin{cases} 1/b, \text{ for } & 0 \le y \le b\,, \\ 0, & \text{elsewhere} \end{cases}$$

are the density functions of x and y, respectively. Hence

$$p_n(i) = \frac{2\sin\beta}{ab} \iint_{\mathcal{F}} p_n(i\,|\,(x,y))\,\mathrm{d}x\,\mathrm{d}y\,.$$

Further calculations require to partition $\mathcal{F}$ into twelve subsets $\mathcal{F}_1, \dots, \mathcal{F}_{12}$ (see Figure 2) and to consider the cases that the centre point C is in one of these subsets.

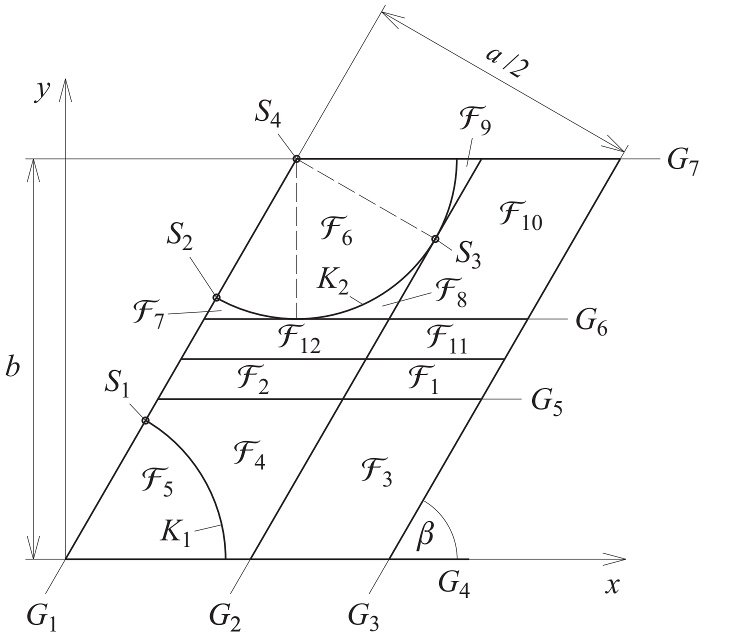

Figure 2: $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{12}$

The intersection of any two subsets of $\mathcal{F}$ is either empty or consists of a finite number of line segments and circular arcs. So we have

$$p_n(i) = \frac{2\sin\beta}{ab} \sum_{m=1}^{12} \iint_{\mathcal{F}_m} p(i\,|\,(x,y))\,\mathrm{d}x\,\mathrm{d}y\ , \quad i \in \{0, 1, \dots, 2n\}\,.$$

For abbreviation we put $I_m(i) := \iint_{\mathcal{F}_m} p(i\,|\,(x,y))\,\mathrm{d}x\,\mathrm{d}y$. Due to existing symmetries we know that $I_{10}(i) = I_3(i)$, $I_{11}(i) = I_1(i)$, $I_{12}(i) = I_2(i)$ and hence

$$p_n(i) = \frac{2\sin\beta}{ab} \Big\{ 2\big[I_1(i) + I_2(i) + I_3(i)\big] + I_4(i) + \ldots + I_9(i) \Big\}\,. \tag{4}$$

The equations of the lines $G_1, \ldots, G_7$ are given by

$$
\begin{aligned}
G_1 &:= \{(x,y) \in \mathbb{R}^2 \mid x\sin\beta - y\cos\beta = 0\}\,,\\
G_2 &:= \{(x,y) \in \mathbb{R}^2 \mid x\sin\beta - y\cos\beta = 1\}\,,\\
G_3 &:= \{(x,y) \in \mathbb{R}^2 \mid x\sin\beta - y\cos\beta = a/2\}\,,\\
G_4 &:= \{(x,y) \in \mathbb{R}^2 \mid y = 0\}\,,\\
G_5 &:= \{(x,y) \in \mathbb{R}^2 \mid y = 1\}\,,\\
G_6 &:= \{(x,y) \in \mathbb{R}^2 \mid y = b-1\}\,,\\
G_7 &:= \{(x,y) \in \mathbb{R}^2 \mid y = b\}
\end{aligned}
$$

and the equations of the circles K_1 and K_2 by

$$
\begin{aligned}
K_1 &:= \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}\,,\\
K_2 &:= \{(x,y) \in \mathbb{R}^2 \mid (x - b\cot\beta)^2 + (y-b)^2 = 1\}\,.
\end{aligned}
$$

Furthermore, we need the intersection points $S_i = (x_i, y_i)$, $i \in \{1, \ldots, 4\}$:

$$
\begin{aligned}
&S_1 \in G_1 \cap K_1 = (\cos\beta,\, \sin\beta)\ , \quad S_2 \in G_1 \cap K_2 = (b\cot\beta - \cos\beta,\, b - \sin\beta)\ ,\\
&S_3 = G_2 \cap K_2 = (b\cot\beta + \sin\beta,\, b - \cos\beta)\ , \quad S_4 = G_1 \cap G_7 = (b\cot\beta,\, b)\ .
\end{aligned}
$$

With these lines, circles and points for the subsets $\mathcal{F}_1, \ldots, \mathcal{F}_9$ one finds the following descriptions:

$$
\begin{aligned}
\mathcal{F}_1 &= \{(x,y) \in \mathbb{R}^2 \mid 1 \le y \le b/2\,,\ \csc\beta + y\cot\beta \le x \le (a/2)\csc\beta + y\cot\beta\}\,,\\
\mathcal{F}_2 &= \{(x,y) \in \mathbb{R}^2 \mid 1 \le y \le b/2\,,\ y\cot\beta \le x \le \csc\beta + y\cot\beta\}\,,\\
\mathcal{F}_3 &= \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1\,,\ \csc\beta + y\cot\beta \le x \le (a/2)\csc\beta + y\cot\beta\}\\
\mathcal{F}_4 &= \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le y_1\,,\ \sqrt{1-y^2} \le x \le \csc\beta + y\cot\beta\}\\
&\quad \cup \{(x,y) \in \mathbb{R}^2 \mid y_1 \le y \le 1\,,\ y\cot\beta \le x \le \csc\beta + y\cot\beta\}\,,\\
\mathcal{F}_5 &= \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le y_1\,,\ y\cot\beta \le x \le \sqrt{1-y^2}\,\}\,,\\
\mathcal{F}_6 &= \{(x,y) \in \mathbb{R}^2 \mid b-1 \le y \le y_2\,,\ b\cot\beta - \sqrt{1-(y-b)^2} \le x \le x_4\}\\
&\quad \cup \{(x,y) \in \mathbb{R}^2 \mid y_2 \le y \le b\,,\ y\cot\beta \le x \le x_4\}\\
&\quad \cup \{(x,y) \in \mathbb{R}^2 \mid b-1 \le y \le b\,,\ x_4 \le x \le b\cot\beta + \sqrt{1-(y-b)^2}\}\,,\\
\mathcal{F}_7 &= \{(x,y) \in \mathbb{R}^2 \mid b-1 \le y \le y_2\,,\ y\cot\beta \le x \le b\cot\beta - \sqrt{1-(y-b)^2}\}\,,\\
\mathcal{F}_8 &= \{(x,y) \in \mathbb{R}^2 \mid b-1 \le y \le y_3\,,\ b\cot\beta + \sqrt{1-(y-b)^2} \le x \le \csc\beta + y\cot\beta\}\,,\\
\mathcal{F}_9 &= \{(x,y) \in \mathbb{R}^2 \mid y_3 \le y \le b\,,\ b\cot\beta + \sqrt{1-(y-b)^2} \le x \le \csc\beta + y\cot\beta\}\,.
\end{aligned}
$$

As an example, we determine the angles $\alpha_j(x,y)$ for cluster $\mathcal{Z}_n$ with centre point $C = (x,y) \in \mathcal{F}_5$ (see figure 3): A single line segment of $\mathcal{Z}_n$ intersects $\mathcal{R}_{a,b,\beta}$ in exactly one point, if it is in one of the two angles $\alpha_{1,1} = \alpha_{1,1}(x,y)$ and $\alpha_{1,2} = \alpha_{1,2}(x,y)$. One finds

$$
\alpha_1(x,y) = \alpha_{1,1}(x,y) + \alpha_{1,2}(x,y) = 2(\pi - \beta)\,.
$$

A single line segment of $\mathcal{Z}_n$ has exactly two intersections with $\mathcal{R}_{a,b,\beta}$, if it is in the angle

$$
\alpha_2(x,y) = \arccos(x\sin\beta - y\cos\beta) + \arccos y - (\pi - \beta)\,.
$$

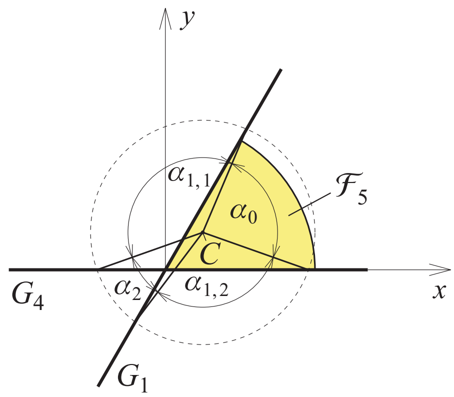

Figure 3: *Angles* $\alpha_j = \alpha_j(x, y)$ *for* $C = (x, y) \in \mathcal{F}_5$

For the angle with no intersections we get

$$\alpha_0(x, y) = 2\pi - [\arccos(x \sin\beta - y \cos\beta) + \arccos y + \pi - \beta] \, .$$

For other subsets $\mathcal{F}_1, \ldots, \mathcal{F}_4$, $\mathcal{F}_6, \ldots, \mathcal{F}_9$ one easily finds

$$\mathcal{F}_1 : \alpha_0(x, y) = 2\pi \, , \quad \alpha_1(x, y) = 0 \, , \quad \alpha_2(x, y) = 0 \, ,$$

$$\begin{aligned} \mathcal{F}_2 : \alpha_0(x, y) &= 2\pi - 2 \arccos(x \sin\beta - y \cos\beta) \, , \\ \alpha_1(x, y) &= 2 \arccos(x \sin\beta - y \cos\beta) \, , \\ \alpha_2(x, y) &= 0 \, , \end{aligned}$$

$$\begin{aligned} \mathcal{F}_3 : \alpha_0(x, y) &= 2\pi - 2 \arccos y \, , \\ \alpha_1(x, y) &= 2 \arccos y \, , \\ \alpha_2(x, y) &= 0 \, , \end{aligned}$$

$$\begin{aligned} \mathcal{F}_4 : \alpha_0(x, y) &= 2\pi - 2 \, [\arccos(x \sin\beta - y \cos\beta) + \arccos y] \, , \\ \alpha_1(x, y) &= 2 \, [\arccos(x \sin\beta - y \cos\beta) + \arccos y] \, , \\ \alpha_2(x, y) &= 0 \, , \end{aligned}$$

$$\begin{aligned} \mathcal{F}_6 : \alpha_0(x, y) &= 2\pi - [\arccos(x \sin\beta - y \cos\beta) + \arccos(b - y) + \beta] \, , \\ \alpha_1(x, y) &= 2\beta \, , \\ \alpha_2(x, y) &= \arccos(x \sin\beta - y \cos\beta) + \arccos(b - y) - \beta \, , \end{aligned}$$

$$\begin{aligned} \mathcal{F}_7 : \alpha_0(x, y) &= 2\pi - 2 \arccos(x \sin\beta - y \cos\beta) \, , \\ \alpha_1(x, y) &= 2 \, [\arccos(x \sin\beta - y \cos\beta) - \arccos(b - y)] \, , \\ \alpha_2(x, y) &= 2 \arccos(b - y) \, , \end{aligned}$$

$$\begin{aligned} \mathcal{F}_8 : \alpha_0(x, y) &= 2\pi - 2 \, [\arccos(x \sin\beta - y \cos\beta) + \arccos(b - y)] \, , \\ \alpha_1(x, y) &= 2 \, [\arccos(x \sin\beta - y \cos\beta) + \arccos(b - y)] \, , \\ \alpha_2(x, y) &= 0 \, , \end{aligned}$$

$$\begin{aligned} \mathcal{F}_9 : \alpha_0(x, y) &= 2\pi - 2 \arccos(b - y) \, , \\ \alpha_1(x, y) &= 2 \, [\arccos(b - y) - \arccos(x \sin\beta - y \cos\beta)] \, , \\ \alpha_2(x, y) &= 2 \arccos(x \sin\beta - y \cos\beta) \, . \end{aligned}$$

We summarize the results of the intersection probabilities:

Theorem 1. *A cluster $\mathcal{Z}_n$ with n $(1 \le n < \infty)$ line segments of length* 1 *is thrown at random onto a lattice $\mathcal{R}_{a,\,b,\,\beta}$ with* $\min(a,b) \ge 2$. *The probabilities $p_n(i)$ of exactly i, $i \in \{0,\ldots,2n\}$, intersections between $\mathcal{Z}_n$ and $\mathcal{R}_{a,\,b,\,\beta}$ are given by formula (4) with (3), (2) and the angles $\alpha_0(x,y)$, $\alpha_1(x,y)$, $\alpha_2(x,y)$ for the subsets $\mathcal{F}_1,\ldots,\mathcal{F}_9$.*

3. Distribution functions

For abbreviation we put $X_n := X_{n,\,\lambda,\,\mu}$, $X := X_{\lambda,\,\mu}$, $F_n := F_{n,\,\lambda,\,\mu}$ and $F := F_{\lambda,\,\mu}$ in this section.

Theorem 2. *As $n \to \infty$, the random variables*

$$X_n = \frac{\textit{number of intersections between } \mathcal{Z}_n \textit{ and } \mathcal{R}_{a,\,b,\,\beta}}{n}$$

converge weakly to the random variable X, whose distribution function $F := F_{\lambda,\,\mu}$ is given by formula (1). Moreover, there holds the uniform convergence

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| = 0\,.$$

From this theorem it directly follows that the limit distribution F is independent of the angle $\beta \in (0,\pi/2]$! It depends only on the parameters λ and μ. By calculating some examples one easily finds, that the distributions F_n are (in general) not independent of β.

Proof. The proof of the weak convergence is based on the method of moments. According to the Fréchet-Shohat theorem (see e.g. [10, pp. 81/82]), we have to show that for each $k \in \mathbb{N}$ the sequence of moments $\mathrm{E}(X_n^k) = \int_{-\infty}^{\infty} x^k\,\mathrm{d}F_n(x)$ converges to $\mathrm{E}(X^k) = \int_{-\infty}^{\infty} x^k\,\mathrm{d}F(x)$ as $n \to \infty$ and the moments $\mathrm{E}(X^k)$, $k \in \mathbb{N}$, uniquely determine F.

Since F is a distribution function that is constant outside the interval $[0,1]$, it is uniquely determined by its moments. These moments are given by

$$\begin{aligned}\mathrm{E}(X^k) = {} & [2\pi(\lambda+\mu) - 6\pi\lambda\mu] \int_0^{1/2} x^k \sin\pi x \;\mathrm{d}x \\ & -2\pi^2\lambda\mu \int_0^{1/2} x^{k+1} \cos\pi x \;\mathrm{d}x \\ & +2\pi\lambda\mu \int_{1/2}^{1} x^k\,[\sin\pi x - \pi(1-x)\cos\pi x]\,\mathrm{d}x\,,\ k\in\mathbb{N}\,.\end{aligned} \tag{5}$$

(It is not a problem to calculate the integrals in (5), but further calculations do not require to know the solutions.) For the moments $\mathrm{E}(X_n^k)$, $k \in \mathbb{N}$, we find

$$\mathrm{E}(X_n^k) = \mathrm{E}[\mathrm{E}(X_n^k \,|\, (x,y))] = 2\lambda\mu\sin\beta \iint_{\mathcal{F}} \mathrm{E}(X_n^k \,|\, (x,y))\,\mathrm{d}x\,\mathrm{d}y\,,$$

where

$$\mathrm{E}(X_n^k \,|\, (x,y)) = \sum_{i=0}^{2n} \left(\frac{i}{n}\right)^k p_n(i \,|\, (x,y)) \tag{6}$$

is the conditional k-th moment of X_n given the centre C of $\mathcal{Z}_n$ in (x,y) with $p_n(i \,|\, (x,y))$ according to formula (3). Using Lemma 1 from [9, p. 219] it can be shown that (6) converges uniformly to $[q_1(x,y) + 2\,q_2(x,y)]^k$ as $n \to \infty$ with $q_j(x,y)$ according to (2), see [3, p. 35]. Owing to the uniform convergence we can exchange the limit and the integral and get

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{E}(X_n^k) &= 2\lambda\mu \sin\beta \lim_{n\to\infty} \iint_{\mathcal{F}} \mathrm{E}(X_n^k \,|\, (x,y))\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\lambda\mu \sin\beta \iint_{\mathcal{F}} \lim_{n\to\infty} \mathrm{E}(X_n^k \,|\, (x,y))\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\lambda\mu \sin\beta \iint_{\mathcal{F}} \big[q_1(x,y) + 2q_2(x,y)\big]^k\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

Now we show that $\lim_{n\to\infty} \mathrm{E}(X_n^k) = \mathrm{E}(X^k)$ for each $k \in \mathbb{N}$. For abbreviation we put

$$L_m(k) := \iint_{\mathcal{F}_m} \lim_{n\to\infty} \mathrm{E}(X_n^k \,|\, (x,y))\,\mathrm{d}x\,\mathrm{d}y\,.$$

Due to the existing symmetries we know that $L_{10}(k) = L_3(k)$, $L_{11}(k) = L_1(k)$, $L_{12}(k) = L_2(k)$ and hence

$$\lim_{n\to\infty} \mathrm{E}(X_n^k) = 2\lambda\mu\sin\beta \Big\{2\big[L_1(k) + L_2(k) + L_3(k)\big] + L_4(k) + \ldots + L_9(k)\Big\}\,.$$

For centre point $(x,y) \in \mathcal{F}_1$ we have $\mathrm{E}(X_n^k \,|\, (x,y)) = 0$ and therefore $L_1(k) = 0$. For centre point $(x,y) \in \mathcal{F}_m$, $m \in \{2,3,4,8,10,12\}$, and $i \in \{n+1, \ldots, 2n\}$ all conditional probabilities $q_2(x,y) = 0$, hence

$$L_m(k) = \iint_{\mathcal{F}_m} q_1(x,y)^k\,\mathrm{d}x\,\mathrm{d}y\,.$$

For $(x,y) \in \mathcal{F}_2$ we have $q_1(x,y) = \frac{1}{\pi}\arccos(x\sin\beta - y\cos\beta)$, hence using the substitutions $u = x\sin\beta - y\cos\beta$ and $v = \frac{1}{\pi}\arccos u$

$$\begin{aligned}
L_2(k) &= \int_{y=1}^{b/2} \int_{x=y\cot\beta}^{\csc\beta + y\cot\beta} \left(\frac{\arccos(x\sin\beta - y\cos\beta)}{\pi}\right)^k \mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{\sin\beta} \int_{y=1}^{b/2} \int_{u=0}^{1} \left(\frac{\arccos u}{\pi}\right)^k \mathrm{d}u\,\mathrm{d}y \\
&= \frac{1}{\sin\beta} \left(\frac{b}{2} - 1\right) \int_{u=0}^{1} \left(\frac{\arccos u}{\pi}\right)^k \mathrm{d}u \\
&= \frac{1}{\sin\beta}\,\frac{1-2\mu}{2\mu}\,\pi \int_0^{1/2} v^k \sin\pi v\,\mathrm{d}v\,.
\end{aligned}$$

For $(x, y) \in \mathcal{F}_3$ we have $q_1(x, y) = \frac{1}{\pi} \arccos y$ and therefore

$$\begin{aligned} L_3(k) &= \int_{y=0}^{1} \int_{x=\csc\beta + y\cot\beta}^{(a/2)\csc\beta + y\cot\beta} \left(\frac{\arccos y}{\pi}\right)^k \mathrm{d}x\,\mathrm{d}y \\ &= \int_{y=0}^{1} \left(\frac{\arccos y}{\pi}\right)^k \mathrm{d}y \int_{x=\csc\beta + y\cot\beta}^{(a/2)\csc\beta + y\cot\beta} \mathrm{d}x \\ &= \frac{1}{\sin\beta}\left(\frac{a}{2} - 1\right) \int_{y=0}^{1} \left(\frac{\arccos y}{\pi}\right)^k \mathrm{d}y\,. \end{aligned}$$

With the substitution $v = \frac{1}{\pi} \arccos y$ we get

$$L_3(k) = \frac{1}{\sin\beta}\,\frac{1-2\lambda}{2\lambda}\,\pi \int_{0}^{1/2} v^k \sin\pi v\,\mathrm{d}v\,.$$

For $(x, y) \in \mathcal{F}_4 \cup \mathcal{F}_5$ we have

$$q_1(x, y) + 2q_2(x, y) = \frac{\arccos(x\sin\beta - y\cos\beta) + \arccos y}{\pi}$$

and therefore

$$L_4(k) + L_5(k) = \int_{y=0}^{1} \int_{x=y\cot\beta}^{\csc\beta + y\cot\beta} \left(\frac{\arccos(x\sin\beta - y\cos\beta) + \arccos y}{\pi}\right)^k \mathrm{d}x\,\mathrm{d}y\,.$$

Using the substitution $z = x\sin\beta - y\cos\beta$ yields

$$L_4(k) + L_5(k) = \frac{1}{\sin\beta} \int_{y=0}^{1} \int_{z=0}^{1} \left(\frac{\arccos z + \arccos y}{\pi}\right)^k \mathrm{d}z\,\mathrm{d}y\,.$$

With $\arccos z = \pi u$ and $\arccos y = \pi v$ ($\mathrm{d}z = -\pi \sin\pi u\ \mathrm{d}u$ and $\mathrm{d}y = -\pi\sin\pi v\ \mathrm{d}v$) it follows, that

$$L_4(k) + L_5(k) = \frac{1}{\sin\beta} \int_{0}^{1/2} \int_{0}^{1/2} (u+v)^k \sin\pi u \sin\pi v\ \mathrm{d}u\,\mathrm{d}v\,.$$

With $w = u + v$ we get $\mathrm{d}w = \mathrm{d}u$ and

$$L_4(k) + L_5(k) = \frac{1}{\sin\beta} \int_{v=0}^{1/2} \int_{w=v}^{v+1/2} w^k\ \sin\pi(w-v)\ \sin\pi v\ \mathrm{d}w\,\mathrm{d}v\,.$$

Changing the order of integrations gives

$$\begin{aligned} L_4(k) + L_5(k) = \frac{1}{\sin\beta}\Bigg[&\int_{w=0}^{1/2} w^k \int_{v=0}^{w} \sin\pi(w-v)\ \sin\pi v\ \mathrm{d}v\,\mathrm{d}w \\ &+ \int_{w=1/2}^{1} w^k \int_{v=w-1/2}^{1/2} \sin\pi(w-v)\ \sin\pi w\ \mathrm{d}v\,\mathrm{d}w \Bigg]. \end{aligned}$$

The calculation of the inner integrals yields

$$L_4(k)+L_5(k) = \frac{\pi}{2\sin\beta}\left[\int_0^{1/2} w^k\,[\sin\pi w - \pi w\cos\pi w]\,\mathrm{d}w + \int_{1/2}^1 w^k\,[\sin\pi w - \pi(1-w)\cos\pi w]\,\mathrm{d}w\right].$$

For $(x,y)\in\mathcal{F}_6\cup\ldots\cup\mathcal{F}_9$ one finds

$$q_1(x,y)+2q_2(x,y) = \frac{\arccos(x\sin\beta - y\cos\beta)+\arccos(b-y)}{\pi}$$

and therefore

$$\sum_{m=6}^{9} L_m(k) = \int_{y=b-1}^{b}\int_{x=y\cot\beta}^{\csc\beta+y\cot\beta}\left(\frac{\arccos(x\sin\beta - y\cos\beta)+\arccos(b-y)}{\pi}\right)^k \mathrm{d}x\,\mathrm{d}y\,.$$

In a similar way as for the calculation of $L_4(k)+L_5(k)$ we get the same result, hence

$$\sum_{m=6}^{9} L_m(k) = L_4(k)+L_5(k)$$

and so

$$\sum_{m=4}^{9} L_m(k) = \frac{\pi}{\sin\beta}\left[\int_0^{1/2} w^k\,[\sin\pi w - \pi w\cos\pi w]\,\mathrm{d}w + \int_{1/2}^1 w^k\,[\sin\pi w - \pi(1-w)\cos\pi w]\,\mathrm{d}w\right].$$

As a summary we get

$$\begin{aligned}\lim_{n\to\infty}\mathrm{E}(X_n^k) &= 4\lambda\mu\left\{\left(\frac{1-2\mu}{2\mu}+\frac{1-2\lambda}{2\lambda}\right)\pi\int_0^{1/2} x^k\sin\pi x\,\mathrm{d}x\right.\\ &\quad+\frac{\pi}{2}\left(\int_0^{1/2} x^k\,[\sin\pi x - \pi x\cos\pi x]\,\mathrm{d}x\right.\\ &\quad\left.\left.+\int_{1/2}^1 x^k\,[\sin\pi x - \pi(1-x)\cos\pi x]\,\mathrm{d}x\right)\right\}\\ &= [2\pi(\lambda+\mu)-6\pi\lambda\mu]\int_0^{1/2} x^k\sin\pi x\;\mathrm{d}x\\ &\quad-2\pi^2\lambda\mu\int_0^{1/2} x^{k+1}\cos\pi x\;\mathrm{d}x \qquad (7)\\ &\quad+2\pi\lambda\mu\int_{1/2}^1 x^k\,[\sin\pi x - \pi(1-x)\cos\pi x]\,.\end{aligned}$$

The comparison of (7) with (5) shows that $\lim_{n\to\infty}\mathrm{E}(X_n^k)=\mathrm{E}(X^k)$ for $k\in\mathbb{N}$. It follows that F_n converges weakly to F as $n\to\infty$.

The uniform convergence is shown in [3, p. 37]. □

4. Expectation and variance

We denote by $Z_{n,\lambda,\mu}$ the random variable $Z_{n,\lambda,\mu}$:= number of intersections between $\mathcal{Z}_n$ and $\mathcal{R}_{a,b,\beta}$. Due to the additivity of the expectation we know that $\mathrm{E}(Z_{n,\lambda,\mu}) = 2n(\lambda+\mu)/\pi$ [2, pp. 85-86]. It easily follows that $\mathrm{E}(X_{n,\lambda,\mu}) = 2(\lambda+\mu)/\pi$ and $\mathrm{E}(X_{\lambda,\mu}) = 2(\lambda+\mu)/\pi$. The result for $\mathrm{E}(X_{\lambda,\mu})$ may also be obtained with formula (5) for $k=1$. With (5) and $k=2$ we get the variance

$$\mathrm{Var}(X_{\lambda,\mu}) = \mathrm{E}(X^2_{\lambda,\mu}) - [E(X_{\lambda,\mu})]^2 = \frac{2(\pi-2)(\lambda+\mu) - 4(\lambda^2+\mu^2)}{\pi^2}\,.$$

Since X_λ und X_μ are independent, this result may also be calculated with

$$\mathrm{Var}(X_{\lambda,\mu}) = \mathrm{Var}(X_\lambda) + \mathrm{Var}(X_\mu)\,,$$

where

$$\mathrm{Var}(X_\lambda) = \frac{2(\pi-2)\lambda - 4\lambda^2}{\pi^2} \text{ and } \mathrm{Var}(X_\mu) = \frac{2(\pi-2)\mu - 4\mu^2}{\pi^2} \text{ [2, pp. 85-86].}$$

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