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Successive approximations for quasi-firmly type nonexpansive mappings*

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Abstract. In this paper, two examples of quasi-firmly type nonexpansive mappings are given to prove that the concept is different from nonexpansive mapping. Furthermore, it is studied to the convergence of the sequence of successive approximations for this class of mappings only when the super limit of iteration coefficients is less than 1. In particular, the Picard iteration $\{T^n x_0\}$ of such a mapping converges to a fixed point of T in a compact metric space.

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1. Introduction

Throughout this work, a Banach space E will always be over the real scalar field. We denote its norm by $\|\cdot\|$ and its dual space by E^*. The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x^*\rangle$ and the *normalized duality mapping* J from E into 2^{E^*} is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \quad \forall\, x \in E.$$

Let $F(T) = \{x \in E : Tx = x\}$, the set of all fixed points for a mapping T and let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of real numbers, respectively. We write $x_n \rightharpoonup x$ (respectively $x_n \overset{*}{\rightharpoonup} x$) to indicate that the sequence $\{x_n\}$ weakly (respectively weak*) converges to x; as usual $x_n \to x$ will symbolize strong convergence.

Let K be a nonempty closed convex subset of a Banach space E. A mapping $T : K \to K$ is said to be *contractive* if for some $\beta \in [0, 1)$,

$$\|Tx - Ty\| \le \beta\|x - y\| \text{ holds for all } x, y \in K;$$

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when $\beta \equiv 1$, the above inequality holds, then T is called *non-expansive*.

The method of successive approximations is very useful in determining solutions of integral, differential and algebraic equations. In the case where T is a contraction mapping, the sequence of successive approximations $x_{n+1} = Tx_n, n \in \mathbb{N}$ (which was referred to as *Picard iteration*), strongly converges to a unique fixed point of T. However, it is well-known for some time that even in a Hilbert space setting, the Picard iteration $\{T^n x_0\}$ of a nonexpansive mapping T need not actually converge to a fixed point. For example, if $T : \mathbb{R} \to \mathbb{R}$ is given by $Tx = x - 1$, then for $x_0 = 1$, $x_{n+1} = Tx_n$ gives $\{0, -1, -2, \cdots, -n, \cdots\}$. Thus, the fundamental properties of contraction mappings cannot extend to nonexpansive mappings. Consequently, considerable research efforts, within the past 60 years or so, have been devoted to studying the method of successive approximation of a nonexpansive mapping T with various types of additional conditions.

In 1953, Mann [20] considered the following method of successive approximation. Let $T : [a, b] \to [a, b]$ be a continuous mapping. Then the sequence $\{x_n\} \subset [a, b]$ given by the following iteration scheme converges to a fixed point of T.

$$x_1 = v_1 \in [a, b],\ v_n = \frac{1}{n}\sum_{k=1}^{n} x_k,\ x_{n+1} = Tv_n, n \in \mathbb{N}.$$

Obviously,

$$v_{n+1} = \frac{1}{n+1}\sum_{k=1}^{n} x_k + \frac{1}{n+1}x_{n+1} = (1 - \frac{1}{n+1})v_n + \frac{1}{n+1}Tv_n \tag{1}$$

and $\{v_n\}$ converges to the same fixed point of T as $\{x_n\}$. The iteration (1) was extended by Dotson Jr [8]. Let $A = (a_{ni})$ be an infinite real matrix satisfying: (i) $a_{ni} \geq 0$ for all $n, i \in \mathbb{N}$ and $a_{ni} = 0$ for $i > n$; (ii) $\sum_{i=1}^{n} a_{ni} = 1$ for all n; (iii) $\lim_{n\to\infty} a_{ni} = 0$ for all i; (iv) $a_{n+1,i} = (1 - a_{n+1,n+1})a_{ni}$, $i, n \in \mathbb{N}$; (v) either $a_{nn} = 1$ for all n or $a_{nn} < 1$ for all $n > 1$.

$$x_1 = v_1 \in K,\ v_n = \sum_{i=1}^{n} a_{ni}x_i,\ x_{n+1} = Tv_n, n \in \mathbb{N}.$$

Similarly, we also have (taking $\lambda_n = a_{n+1,n+1}$)

$$v_{n+1} = \sum_{i=1}^{n} a_{n+1,i}x_i + a_{n+1,n+1}x_{n+1} = (1 - \lambda_n)v_n + \lambda_n Tv_n.$$

In the framework of a uniformly convex Banach space, Krasnoselskii [18] showed the convergence of the following iteration sequence:

$$x_{n+1} = \frac{1}{2}x_n + (1 - \frac{1}{2})Tx_n, \quad n \in \mathbb{N}. \tag{2}$$

In 1966, Edelstein [9] succeeded in relaxing the condition of uniform convexity to strictly convex Banach spaces. Schaefer [27] proved the same results as Krasnoselskii's for the sequence given by

$$x_{n+1} = \alpha x_n + (1 - \alpha)Tx_n, \quad n \in \mathbb{N} \text{ and } \alpha \in (0, 1) \tag{3}$$

Diaz and Metcalf [4] proved Schaefer's results in strictly convex Banach spaces (also see Kirk [16, 17]). Petryshyn [22] extended the above results to densifying nonexpansive mappings. Genel and Lindenstrauss [11] showed that some compact condition about T or K cannot be eliminated even in Hilbert space l^2 in order to obtain the strong convergence of this iteration. Rhoades [26] unified the above iteration in a one-dimensional case and showed that if T is a continuous nondecreasing self mapping of $[a, b]$, then the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{4}$$

where $x_0 \in [a, b]$, $\alpha_0 = 1, 0 \leq \alpha_n < 1$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, converges to a fixed point of T. Reich [23] obtained that in a uniformly convex Banach space with a Fréchet differentiable norm, if a nonexpansive mapping T has a fixed point and $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$, defined by iteration (4) converges weakly to a fixed point of T.

In the sequel, we call iteration (4) *Krasnoselskii-Mann iteration.* Further improved results in various directions about such an iteration were also given by Borwein, Reich and Shafrir [2], Reinermann [24], Rhoades [25], Edelstein and O'Brien [10], Ishikawa [15], Senter and Dotson Jr.[28], Dotson Jr. [5, 6], Das, Singh and Watson [3], Atsushiba and Takahashi [1] and so on.

An iteration scheme due to Ishikawa [14], called an *Ishikawa-type iteration*, is defined as follows: $x_0 \in K$

$$\begin{cases} x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) T y_n \\ y_n &= \beta_n x_n + (1 - \beta_n) T x_n, \end{cases} \tag{5}$$

where $\alpha_n, \beta_n \in [0, 1]$. Note that the Krasnoselskii-Mann iteration is a special case of the Ishikawa-type one (corresponding to the choice $\beta_n = 1$ for all n). For comparison of the two iteration processes in the one-dimensional case, see Rhoades [26]. Recently, in the framework of a uniformly convex Banach space, Tan and Xu [30] studied strong and weak convergence of Ishikawa iteration for a nonexpansive mapping T whenever $\alpha_n, \beta_n \in [0, 1]$ satisfy the conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n(1 - \beta_n) < +\infty$;

(iii) $\liminf_{n \to \infty} \beta_n > 0$.

It is obvious that the above results do not apply when $\alpha_n = \beta_n \equiv 0$. It is very interesting to find a class of mappings, neither nonexpansive nor contraction, whose Picard iteration (weakly) converges to a fixed point of T.

In this paper we study successive approximations of quasi-firmly type nonexpansive mappings only under the assumption that the limit superior of the iteration coefficients is less than 1, thus including the case $\alpha_n = \beta_n \equiv 0$. More precisely, several examples of quasi-firmly type nonexpansive mapping T are given to prove that the concept is different from nonexpansive mapping. It is another main aim to present strong and weak convergence of Ishikawa-type iteration (5) and Krasnoselskii-Mann iteration (4) for such a mapping T only under the conditions

$$\limsup_{n\to\infty} \alpha_n < 1 \text{ and } \limsup_{n\to\infty} \beta_n < 1.$$

In particular, the Picard iteration $\{T^n x_0\}$ ($\alpha_n = \beta_n = 0$) strongly converges to a fixed point of T in a compact metric space.

2. Quasi-firmly type nonexpansive mapping and examples

Let T be a mapping with domain $D(T)$ and range $R(T)$ in Banach space E. T is called *contractive* if there exists $\beta \in [0, 1)$ for any $x, y \in D(T)$ such that $\|Tx-Ty\| \leq \beta\|x - y\|$; while T is called *nonexpansive* if the above inequality holds for $\beta = 1$. T is called *quasi-nonexpansive* if $F(T) \neq \emptyset$,

$$\|Tx - p\| \leq \|x - p\| \text{ holds for all } x \in D(T) \text{ and all } p \in F(T). \tag{6}$$

This notion, which Dotson Jr.[7] has labeled quasi-nonexpansive, was essentially introduced, along with other ideas, by Diaz and Metcalf [4]. It is obvious that a nonexpansive mapping with at least one fixed point is quasi-nonexpansive. Dotson Jr. [7] also gave an example which is continuous quasi-nonexpansive but not nonexpansive.

Recently, Song and Chai [29] introduced the notion of a firmly type nonexpansive mapping and showed that this class of mappings has better behavior than nonexpansive mapping. T is said to be *firmly type nonexpansive* if for all $x, y \in D(T)$, there exists $k \in (0, +\infty)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - k\|(x - Tx) - (y - Ty)\|^2. \tag{7}$$

Obviously, the firmly type nonexpansive mappings contain the firmly nonexpansive mappings and the resolvent of a monotone operator as a special case in Hilbert space (see [12, 13]). Thus, this class of mappings could be looked upon as one of the most important classes in nonlinear mappings. There are many examples of such mapping. For examples, a contractive mapping T with a contractive constant β is a firmly type nonexpansive mapping with a constant $k = \frac{1-\beta}{1+\beta}$; the identity operator I is firmly type nonexpansive with arbitrary constant $k \in (0, +\infty)$; in $\mathbb{R}^1$, $Tx = \frac{1}{2}x^2 (\forall x \in [0, 1])$ is firmly type nonexpansive with arbitrary constant $k \in (0, 1]$. For a detailed proof and more examples, see [29, Example 1-5].

It is now natural to introduce the following concept. T is called *quasi-firmly type nonexpansive* ([29, Remark 1]) provided T has at least one fixed point in E ($F(T) \neq \emptyset$), there exists $k \in (0, +\infty)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 - k\|x - Tx\|^2 \text{ holds for all } x \in D(T) \text{ and all } p \in F(T). \tag{8}$$

It is clear that a firmly type nonexpansive mapping with at least one fixed point is quasi-firmly type nonexpansive. A linear quasi-firmly type nonexpansive mapping on a Banach space is firmly type nonexpansive (and hence nonexpansive) on that space.

In fact, if T is a linear quasi-firmly type nonexpansive mapping, then $0 \in F(T)$, and so for all $x \in D(T)$,

$$\|Tx - 0\|^2 \leq \|x - 0\|^2 - k\|x - Tx\|^2.$$

That is,

$$\|Tx\|^2 \leq \|x\|^2 - k\|x - Tx\|^2. \tag{9}$$

Thus, for all $x, y \in D(T)$,

$$\|Tx - Ty\|^2 = \|T(x-y)\|^2 \leq \|x - y\|^2 - k\|(x - Tx) - (y - Ty)\|^2.$$

However, there exist continuous and discontinuous nonlinear quasi-firmly type nonexpansive mappings that are not nonexpansive. Now we give two examples of quasi-firmly type nonexpansive mappings which are not nonexpansive.

Example 1. *Let $E = \mathbb{R}$ be endowed with the Euclidean norm $\|\cdot\| = |\cdot|$. Assume that $K = [0, +\infty)$ and $T : K \to K$ is defined by*

$$Tx = \begin{cases} x^2, & x \in [0,1); \\ \frac{1}{2}, & x \geq 1. \end{cases}$$

Clearly, $F(T) = \{0\}$ and T is not nonexpansive since T is not continuous on 1, also see the following:

$$\|T(\frac{3}{4}) - T(\frac{1}{2})\| = \frac{9}{16} - \frac{1}{4} = \frac{5}{16} > \frac{1}{4} = \|\frac{3}{4} - \frac{1}{2}\|.$$

However, T is quasi-firmly type nonexpansive. In fact, for all $x \in [0,1)$, we have $x^2 \geq x^4$ and

$$\|x - Tx\|^2 = x^2(1-x)^2 \leq x^2(1-x)(1+x) = x^2 - x^4.$$

Then

$$\begin{aligned} \|Tx - 0\|^2 =& x^4 = \|x - 0\|^2 - (x^2 - x^4) \\ \leq& \|x - 0\|^2 - \|x - Tx\|^2. \end{aligned}$$

When $x \geq 1$, we have $\|Tx - 0\|^2 = \frac{1}{4}$ and $\|x - Tx\|^2 = (x - \frac{1}{2})^2 < (x - \frac{1}{2})(x + \frac{1}{2})$. Thus

$$\begin{aligned} \|Tx - 0\|^2 =& x^2 - (x^2 - \frac{1}{4}) = (x-0)^2 - (x - \frac{1}{2})(x + \frac{1}{2}) \\ <& (x-0)^2 - (x - \frac{1}{2})^2 = \|x - 0\|^2 - \|x - Tx\|^2. \end{aligned}$$

Hence, for all $x \in K$ and $k \in [0,1]$,

$$\|Tx - 0\|^2 \leq \|x - 0\|^2 - k\|x - Tx\|^2.$$

The following example shows that a continuous quasi-firmly type nonexpansive mapping may also not be nonexpansive.

Example 2. *Let $E = \mathbb{R}$ be endowed with the Euclidean norm $\|\cdot\| = |\cdot|$. Assume that $T : E \to E$ is defined by*

$$Tx = \begin{cases} \frac{x}{2}|\sin\frac{1}{x}|, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Obviously, T is continuous. However, T is not nonexpansive. In fact, if $x = \frac{1}{2\pi}$ and $y = \frac{2}{3\pi}$, then $\|x - y\| = \|\frac{1}{2\pi} - \frac{2}{3\pi}\| = \frac{1}{6\pi}$ and

$$\|Tx - Ty\| = \|0 - \frac{1}{3\pi}\| = \frac{1}{3\pi} > \frac{1}{6\pi} = \|x - y\|$$

T is quasi-firmly type nonexpansive. Indeed, $Tx \neq x$ for any $x \neq 0$ since if $Tx = x$, then $x = \frac{x}{2}|\sin\frac{1}{x}|$, that is, $2 = |\sin\frac{1}{x}|$ which is impossible. Thus, $F(T) = \{0\}$. Therefore, for all $x \in E$, we have

$$\begin{aligned}\|x - Tx\|^2 =&(x - \frac{x}{2}|\sin\frac{1}{x}|)^2 = \frac{x^2}{4}(2 - |\sin\frac{1}{x}|)^2 \\ \leq&\frac{x^2}{4}(2 - |\sin\frac{1}{x}|)(2 + |\sin\frac{1}{x}|).\end{aligned}$$

Then

$$\begin{aligned}\|Tx - 0\|^2 =&\frac{x^2}{4}|\sin\frac{1}{x}|^2 = \|x - 0\|^2 - (x^2 - \frac{x^2}{4}|\sin\frac{1}{x}|^2) \\ =&\|x - 0\|^2 - \frac{x^2}{4}(2 - |\sin\frac{1}{x}|)(2 + |\sin\frac{1}{x}|) \\ \leq&\|x - 0\|^2 - \|x - Tx\|^2.\end{aligned}$$

Therefore, T is a continuous quasi-firmly type nonexpansive mapping, but not nonexpansive mapping.

3. Weak convergence theorems

In the proof of main theorems in this section, we also need the following definitions and results.

A Banach space E is said to satisfy *Opial's condition* ([21]) if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \text{ with } x \neq y.$$

In particular, Opial's condition is independent of uniformly convex (smooth) since the l^p spaces satisfy this condition for $1 < p < \infty$ while it fails for the L^p $(p \neq 2)$ spaces. In fact, spaces satisfying Opial's condition need not even be isomorphic to uniformly convex spaces ([19]). A mapping T is called *demiclosed* at 0 if $x_n \rightharpoonup x$ and $Tx_n \to 0$, then $Tx = 0$.

Lemma 1. *Let E be a linear normed space. Then for all $x, y \in E$ and $t \in [0, 1]$,*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2.$$

Proof. Let $z_t = tx + (1-t)y$. Then for $j(z_t) \in J(z_t)$,

$$\begin{aligned}\|z_t\|^2 &= \langle z_t, j(z_t)\rangle = \langle tx + (1-t)y, j(z_t)\rangle \\ &= t\langle x, j(z_t)\rangle + (1-t)\langle y, j(z_t)\rangle \\ &\le t\|x\|\|j(z_t)\| + (1-t)\|y\|\|j(z_t)\| \\ &\le t\frac{\|x\|^2 + \|z_t\|^2}{2} + (1-t)\frac{\|y\|^2 + \|z_t\|^2}{2} \\ &= t\frac{\|x\|^2}{2} + (1-t)\frac{\|y\|^2}{2} + \frac{\|z_t\|^2}{2},\end{aligned}$$

and hence

$$\|tx + (1-t)y\|^2 = \|z_t\|^2 \le t\|x\|^2 + (1-t)\|y\|^2.$$

The desired result is obtained immediately from the convexity of the function $f(x) = x^2$ also defined on $\mathbb{R}$. This competes the proof. $\square$

Theorem 1. *Let E be a real reflexive Banach space that satisfies Opial's condition. Assume that K is a nonempty closed convex subset of E and $T : K \to K$ is a quasi-firmly type nonexpansive mapping. For arbitrary initial value $x_0 \in K$, define iteratively a sequence $\{x_n\}$ as follows:*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n \\ y_n \quad\; = \beta_n x_n + (1-\beta_n)Tx_n \end{cases} \tag{10}$$

Suppose that $I - T$ is demiclosed at 0 and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions

$$\limsup_{n\to\infty} \alpha_n < 1, \limsup_{n\to\infty} \beta_n < 1.$$

Then as $n \to \infty$, $\{x_n\}$ converges weakly to some fixed point x^* of T.

Proof. Take $p \in F(T)$. Then, from Lemma 1 we estimate as follows:

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(Ty_n - p)\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Ty_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\|y_n - p\|^2 - k\|y_n - Ty_n\|^2) \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\beta_n\|x_n - p\|^2 + (1-\beta_n)\|Tx_n - p)\|^2) \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 \\ &\quad + (1-\alpha_n)(1-\beta_n)(\|x_n - p)\|^2 - k\|x_n - Tx_n\|^2) \\ &= \|x_n - p\|^2 - k(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2.\end{aligned}$$

Then we have

$$k(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \tag{11}$$

and

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 \le \cdots \le \|x_0 - p\|^2. \tag{12}$$

This implies that $\{\|x_n - p\|\}$ monotonously decreases and is bounded. Therefore, the limit $\lim_{n\to\infty} \|x_n - p\|$ exists.

We claim that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{13}$$

In fact, since $\limsup\limits_{n\to\infty} \alpha_n < 1$ and $\limsup\limits_{n\to\infty} \beta_n < 1$, then there exists $a, b \in (0, 1)$ and $N \in \mathbb{N}$ such that

$$\alpha_n < a, \quad \beta_n < b \text{ for all } n \ge N.$$

Thus, following (11), for all $n \ge N$, we have

$$\begin{aligned} k(1-a)(1-b)\|x_n - Tx_n\|^2 <& k(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2 \\ \le& \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \end{aligned}$$

Therefore,

$$k(1-a)(1-b)\sum_{n=N}^{m} \|x_n - Tx_n\|^2 \le \|x_N - p\|^2 - \|x_m - p\|^2 \le \|x_N - p\|^2.$$

Then

$$\sum_{n=1}^{+\infty} \|x_n - Tx_n\|^2 < +\infty,$$

and hence

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

The reflexivity of E means that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to some point of K, say x^*. Then by the hypothesis that $I - T$ is demiclosed at 0, we have $(I - T)x^* = 0$. That is, $x^* = Tx^*$.

Next we show that $\{x_n\}$ weakly converges to x^*. Let y be another weak limit point of $\{x_n\}$ and $x^* \ne y$. Then we can choose a subsequence $\{x_{n_j}\}$ that weakly converges to y. We also have $y = Ty$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T)$, we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - x^*\| =& \limsup_{k\to\infty} \|x_{n_k} - x^*\| \\ <& \limsup_{k\to\infty} \|x_{n_k} - y\| = \lim_{n\to\infty} \|x_n - y\| \\ =& \limsup_{j\to\infty} \|x_{n_j} - y\| \\ <& \limsup_{j\to\infty} \|x_{n_j} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|, \end{aligned}$$

a contradiction, and hence $x^* = y$. The desired conclusion is proved. $\square$

Let $\beta_n \equiv 0$. The following result is obtained easily.

Corollary 1. *Let E be a real reflexive Banach space that satisfies Opial's condition. Assume that K is a nonempty closed convex subset of E and $T : K \to K$ is a quasi-firmly type nonexpansive mapping. For arbitrary initial value $x_0 \in K$, define iteratively a sequence $\{x_n\}$ as follows:*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^2 x_n. \tag{14}$$

Suppose that $I - T$ is demiclosed at 0 and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the condition

$$\limsup_{n\to\infty} \alpha_n < 1.$$

Then as $n \to \infty$, $\{x_n\}$ converges weakly to some fixed point x^* of T.

Clearly, the fact that $\beta_n \equiv 1$ cannot satisfy the conditions of Theorem 1. However, we have the following.

Theorem 2. *Let E be a real reflexive Banach space that satisfies Opial's condition. Assume that K is a nonempty closed convex subset of E and $T : K \to K$ is a quasi-firmly type nonexpansive mapping. For arbitrary initial value $x_0 \in K$, define iteratively a sequence $\{x_n\}$ as follows:*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T x_n. \tag{15}$$

Suppose that $I - T$ is demiclosed at 0 and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the condition

$$\limsup_{n\to\infty} \alpha_n < 1.$$

Then as $n \to \infty$, $\{x_n\}$ converges weakly to some fixed point x^* of T.

Proof. Take $p \in F(T)$. Using a similar technique, we also have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(Tx_n - p)\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|Tx_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\|x_n - p\|^2 - k\|x_n - Tx_n\|^2] \\
&= \|x_n - p\|^2 - k(1 - \alpha_n)\|x_n - Tx_n\|^2.
\end{aligned}$$

Then we have

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2$$

and

$$k(1 - \alpha_n)\|x_n - Tx_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Therefore, the limit $\lim_{n\to\infty} \|x_n - p\|^2$ exists, and hence,

$$\begin{aligned}
&k(1 - \limsup_{n\to\infty} \alpha_n) \limsup_{n\to\infty} \|x_n - Tx_n\|^2 \\
&\quad = k \liminf_{n\to\infty}(1 - \alpha_n) \limsup_{n\to\infty} \|x_n - Tx_n\|^2 \\
&\quad \le k \limsup_{n\to\infty}(1 - \alpha_n)\|x_n - Tx_n\|^2 \\
&\quad \le \limsup_{n\to\infty}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\
&\quad = \lim_{n\to\infty} \|x_n - p\|^2 - \lim_{n\to\infty} \|x_{n+1} - p\|^2 = 0.
\end{aligned}$$

Since $\limsup_{n\to\infty} \alpha_n < 1$, then

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

The remainder of the proof is the same as Theorem 1, so we omit it. □

Obviously, when $\alpha_n \equiv 0$, the result of Theorem 2 still holds.

Corollary 2. *Let E be a real reflexive Banach space that satisfies Opial's condition. Assume that K is a nonempty closed subset of E and $T : K \to K$ is a quasi-firmly type nonexpansive mapping. Suppose that $I - T$ is demiclosed at 0. Then for an arbitrary initial value $x_0 \in K$, $\{T^n x_0\}$ converges weakly to some fixed point x^* of T.*

4. Strong convergence theorems

Let K be a nonempty closed convex subset of a Banach space E. Recall that a mapping $T : K \to K$ is said to *satisfy Condition A* (Senter and Dotson Jr. [28]) if there exists a nondecreasing function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$\|x - Tx\| \geq f(d(x, F(T))) \text{ for all } x \in K,$$

where $d(x, F(T)) = \inf_{z\in F(T)} \|x - z\|$.

Theorem 3. *Let K be a nonempty closed and convex subset of a Banach space E and $T : K \to K$ be a continuous quasi-firmly type nonexpansive mapping. Suppose that T satisfies Condition A and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ satisfying the conditions*

$$\limsup_{n\to\infty} \alpha_n < 1, \limsup_{n\to\infty} \beta_n < 1.$$

Then as $n \to \infty$, $\{x_n\}$, given by Ishikawa-type iteration (10) converges strongly to some fixed point x^* of T.

Proof. The proof given below is different from the one of Tan and Xu [30]. It follows from the argumentation of Theorem 1 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \text{ and } \|x_{n+1} - p\| \leq \|x_n - p\| \text{ for all } p \in F(T).$$

Since T satisfies Condition A, then we have

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T)) \text{ for all } n \in \mathbb{N},$$

and hence $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. By the property of f, we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Following the fact that $\|x_{n+1} - p\| \leq \|x_n - p\|$ for each $p \in F(T)$, for $m, n \in \mathbb{N}$ (Without loss of generality, let $m > n$), we also have

$$\|x_m - x_n\| \leq \|x_m - p\| + \|p - x_n\| \leq 2\|x_n - p\|.$$

Take the infimum for $p \in F(T)$, we have

$$\|x_m - x_n\| \le 2d(x_n, F(T)),$$

and so $\lim_{n\to\infty} \|x_m - x_n\| = 0$. This shows that $\{x_n\}$ is a Cauchy sequence and hence it converges strongly to a point $x^* \in K$. By the continuity of T, we have that $F(T)$ is closed and hence $x^* = Tx^*$. This yields the desired conclusion. □

Similarly, we also have the following:

Theorem 4. *Let K be a nonempty closed and convex subset of a Banach space E and $T : K \to K$ be a continuous quasi-firmly type nonexpansive mapping. Suppose that T satisfies Condition A and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions*

$$\limsup_{n\to\infty} \alpha_n < 1.$$

Then as $n \to \infty$, $\{x_n\}$, given by Krasnoselskii-Mann iteration (15) converges strongly to some fixed point x^* of T.

From the proof technique of Theorem 3, we actually prove the following:

Theorem 5. *Let K be a nonempty closed and convex subset of a Banach space E and $T : K \to K$ a continuous quasi-firmly type nonexpansive mapping. Suppose that $\{x_n\}$ is given by Ishikawa-type iteration (10) or Krasnoselskii-Mann iteration (15) and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ satisfying the conditions*

$$\limsup_{n\to\infty} \alpha_n < 1, \limsup_{n\to\infty} \beta_n < 1.$$

Then as $n \to \infty$, $\{x_n\}$ converges strongly to some fixed point x^* of T if and only if $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$.

Theorem 6. *Let K be a compact convex subset of a Banach space E and $T : K \to K$ a continuous quasi-firmly type nonexpansive mapping. Suppose that $\{x_n\}$ is given by Ishikawa-type iteration (10) and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ satisfying the conditions*

$$\limsup_{n\to\infty} \alpha_n < 1, \limsup_{n\to\infty} \beta_n < 1.$$

Then as $n \to \infty$, $\{x_n\}$ converges strongly to some fixed point x^* of T.

Proof. It follows from the argumentation of Theorem 1 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - p\| \text{ exists for each } p \in F(T).$$

The compactness of K implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to a point $x^* \in K$. By the continuity of T, we have

$$\|x^* - Tx^*\| = \lim_{n\to\infty} \|x_{n_k} - Tx_{n_k} - (x^* - Tx^*)\| = 0,$$

and so $x^* = Tx^*$. Therefore $\lim_{n\to\infty} \|x_n - x^*\|$ exists. By $\lim_{n\to\infty} \|x_{n_k} - x^*\| = 0$, the desired conclusion is reached. □

Similarly, we also have the following.

Theorem 7. *Let K be a compact convex subset of a Banach space E and $T : K \to K$ a continuous quasi-firmly type nonexpansive mapping. Suppose that $\{x_n\}$ is given by Krasnoselskii-Mann iteration (15) and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions*

$$\limsup_{n\to\infty} \alpha_n < 1.$$

Then as $n \to \infty$, $\{x_n\}$ converges strongly to some fixed point x^* of T.

Obviously, when $\alpha_n \equiv 0$, the result of Theorem 7 still holds.

Corollary 3. *Let K be a compact subset of a Banach space E and $T : K \to K$ a continuous quasi-firmly type nonexpansive mapping. Then for an arbitrary initial value $x_0 \in K$, $\{T^n x_0\}$ converges strongly to some fixed point x^* of T.*

Also, it is evident that if $\alpha_n = 0$ for each n, the result holds in a compact metric space.

Theorem 8. *Let E be a compact metric space and $T : E \to E$ a continuous quasi-firmly type nonexpansive mapping. Then for an arbitrary initial value $x_0 \in E$, $\{T^n x_0\}$ converges strongly to some fixed point x^* of T.*

Proof. Take $p \in F(T)$, then we have

$$(d(x_{n+1}, p))^2 = (d(Tx_n, p))^2 \leq (d(x_n, p))^2 - k(d(x_n, Tx_n))^2.$$

Then we have $d(x_{n+1}, p) \leq d(x_n, p)$ and

$$k(d(x_n, Tx_n))^2 \leq (d(x_n, p))^2 - (d(x_{n+1}, p))^2.$$

Similarly, we have the limit $\lim_{n\to\infty} d(x_n, p)$ exists and

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

The compactness of K means that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to a point x^*.

By the continuity of T, we have $\lim_{n\to\infty} d(Tx_{n_k}, Tx^*) = 0$, and so

$$d(x^*, Tx^*) \leq d(x^*, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tx^*) \to 0.$$

Therefore $x^* = Tx^*$. By $\lim_{n\to\infty} d(x_{n_k}, x^*) = 0$, the desired conclusion is proved. □

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