MATHEMATICAL COMMUNICATIONS
Math. Commun. **16**(2011), 265–270.

The spectra and fine spectra of factorable matrices on c_0*

Billy E. Rhoades[1,†] and Mustafa Yildirim[2]

[1] *Department of Mathematics, Indiana University, Bloomington, Indiana–47405, U.S.A.*
[2] *Department of Mathematics, Faculty of Science, Cumhuriyet University, Sivas 58410, Turkey*

Received February 16, 2009; accepted October 14, 2010

Abstract. In a series of papers we have previously investigated the spectra and fine spectra for factorable matrices, considered as bounded operators over various sequence spaces. In this paper we examine the spectra and fine spectra of factorable matrices as operators over c_0, the space of all null sequences, normed by $\|x\| := \sup_{n\geq 0} |x_n|$.

AMS subject classifications: Primary 40G99; Secondary 47B38, 47B37, 47A10

Key words: spectrum, fine spectrum, factorable matrices, weighted mean operators, Rhaly operators, Cesáro operators

1. Introduction

Let w, c_0, c, ℓ^p, bv, and bv_0, denote, respectively, the sequence spaces of: all sequences, null sequences, convergent sequences, $\{x : \sum_n |x_n|^p < \infty\}$, $\{x : \sum_n |x_{n+1} - x_n| < \infty$, and $bv_0 := bv \cap c_0$. The symbols e and e_k denote the constant sequence of all 1's, and the coordinate sequences whose k-th terms are 1 and all other terms are 0, respectively.

An infinite matrix A is said to be conservative if it is a selfmap of c. Necessary and sufficient conditions for A to be conservative are the well-known Silverman-Toeplitz conditions

(i) $\|A\| := \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$,

(ii) $\lim_n a_{nk} = a_k$ exists for each k, and

(iii) $t := \lim_n \sum_k a_{nk}$ exists.

Associated with each conservative matrix A is a function χ defined by $\chi(A) = t - \sum a_k$. If $\chi(A) \neq 0, A$ is called coregular, and, if $\chi(A) = 0$, then A is conull. A matrix $A = (a_{nk})$ is said to be regular if $\lim_A x = \lim x$ for each $x \in c$. If $a_k = 0$ for

*The second author received partial support from the Scientific and Technical Research Council of Turkey during the preparation of this paper.
†Corresponding author. *Email addresses:* `rhoades@indiana.edu` (B. E. Rhoades), `yildirim@cumhuriyet.edu.tr` (M. Yildirim)

each k and $t = 1$ in (iii), then the operator A is called regular. A factorable matrix A is a lower triangular matrix with entries

$$a_{nk} = \begin{cases} a_n b_k, & 0 \le k \le n, \\ 0, & k > n. \end{cases} \tag{1}$$

The choices $a_n = 1/(n+1)$ and each $b_k = 1, a_n = 1/(n+1)^p (p > 1)$ and each $b_k = 1, a_n = a_n$ and each $b_k = 1$, and $a_n = 1/P_n, b_k = p_k$, where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0, P_n := \sum_{k=0}^n p_k$, generate C (the Cesàro matrix of order one), the p-Cesàro matrices, and terraced matrices, defined by Rhaly [11], and the weighted mean matrices, respectively.

The authors have calculated the spectrum and fine spectrum of factorable matrices on c and ℓ^p in [15] and [16]. It is the purpose of this paper to determine the spectra and fine spectra of factorable matrices over c_0. As corollaries we obtain the known corresponding results for weighted mean matrices, terraced matrices, and C.

In previous work the first author determined the fine spectra of certain classes of weighted mean matrices, considered as bounded linear operators over c, c_0, ℓ^p, and bv_0 (see, e.g., [3, 12, 13, 14]). The second author has considered spectral questions for certain classes of Rhaly matrices (see, e.g., [11, 17, 18, 19, 20, 21]). The spectrum of C on various spaces has been computed in [2, 4, 6, 8, 9, 10, 22].

2. Spectra

Let $B(c_0)$ denote the linear space of all bounded linear operators on c_0. We shall denote the spectrum of a matrix $A \in B(c_0)$ by $\sigma(A, c_0)$. A triangle is a lower triangular matrix with nonzero main diagonal entries. Our first result is an extension of Theorem 1 of [1] from regular to coregular matrices.

Lemma 1 (see [15], Lemma 1)**.** *Let A be a coregular triangle with inverse satisfying*

$$a_{nk}^{-1} \le 0, \quad (n < k), \quad a_{nn}^{-1} > 0 \quad (n = 0, \ldots).$$

Then $I + \alpha A$ is equivalent to convergence for $Re(\alpha) > -1/t$, where $t = \lim Ae$.

Lemma 2 (see [15], Lemma 2)**.** *Let A be a factorable lower triangular matrix, $B := A - \lambda I$, where $\lambda \in \mathbb{C}$ such that $b_{nn} \ne 0$ for each n. Then $D := B^{-1}$ exists and has nonzero entries*

$$d_{nk} = \begin{cases} \dfrac{1}{b_n a_n - \lambda}, & k = n, \\ (-1)^{n+k} \lambda^{n-k-1} a_n b_k \displaystyle\prod_{j=k}^{n} \frac{1}{a_j b_j - \lambda}, & 0 \le k < n \end{cases} \tag{2}$$

Theorem 1. *Let A be a coregular factorable triangle with nonnegative entries. Then*

$$\sigma(A, c_0)) \subseteq \Big\{ \lambda : \Big| \lambda - \frac{t}{2} \Big| \le \frac{t}{2} \Big\}.$$

Proof. The result follows from Lemma 1 and the Lemma in [13]. □

Define $\eta = \limsup a_n b_n$, and $S := \overline{\{a_n b_n : n \geq 0\}}$.

Theorem 2. *Let $A \in B(c_0)$ be a regular factorable triangle. Then*

$$\sigma(A, c_)) \supseteq \left\{\lambda : \left|\lambda - \frac{1}{2-\eta} \leq \frac{1-\eta}{2-\eta}\right\} \cup S.$$

Proof. For $\eta < 1$ the proof is identical to that of Theorem 2 in [13]. For $\eta = 1$ the proof is trivial. □

Remark 1. *Theorem 2 of [3] is a special case of Theorem 2.*

Define $\gamma = \liminf a_j b_j$.

Theorem 3. *Let A be a regular factorable triangle with nonnegative entries. Then*

$$\sigma(A, c_0) \subseteq \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| \leq \frac{1-\gamma}{2-\gamma}\right\} \cup S. \tag{3}$$

Proof. The proof is identical to that of Theorem 3 in [15]. □

We take this opportunity to point out that, in (8) of [15], $1/(1-\gamma)$ should read $1/(2-\gamma)$.

Corollary 1. *Let A be a coregular factorable triangle with nonnegative entries such that* $\lim a_n b_n$ *exists. Call the limit δ. Then*

$$\sigma(A, c_0) = \left\{\lambda : \left|\lambda - \frac{t}{2-\delta}\right| \leq \frac{t(1-\delta)}{2-\delta}\right\} \cup S.$$

Proof. Let t denote the limit of the row sums of A and define $F := (1/t)A$. Then the spectrum of F is given by Theorems 2 and 3, and the result follows by the spectral mapping theorem. □

Corollary 2. *Let A be a regular factorable triangle with nonnegative entries and with* $\lim_n a_n b_n = 0$. *Then* $\sigma(A, c_0) = \{\lambda : |\lambda - 1/2| \leq 1/2\}$.

Remark 2. *Corollary 1 of [13] is a special case of Corollary 2.*

Rhaly [11] defined a class of triangles E with nonzero entries satisfying $e_{nk} = a_n, a_n > 0$. He names these terraced matrices. Clearly, $t_n = (n+1)a_n$, so we are interested in those matrices for which $t_n \to L$.

Corollary 3. *Let A be a terraced matrix with $L > 0$. Then*

$$\sigma(A, c_0) = \left\{\lambda : \left|\lambda - \frac{L}{2}\right| \leq \frac{L}{2}\right\} \cup U, \text{where } U = \{a_n : n \geq 0\}.$$

Corollary 4 (see [10], Theorem 3)**.** $\sigma(C, c_0) = \{\lambda : \lambda - 1/2| \leq 1/2\}$.

Proof. C is a factorable matrix with $a_n = 1/(n+1)$ and $b_k = 1$. □

Note that, from Theorems 2 and 3, the spectrum of a factorable matrix in $B(c_0)$ is not determined when $\delta > \gamma$.

3. Fine spectra

We now turn our attention to obtaining the fine spectra for factorable triangles.

From Goldberg [5], if $T \in B(X), X$ a Banach space, then there are three possibilities for $R(T)$, the range of T:

(I) $R(T) = X$,

(II) $\overline{R(T)} = X$, but $R(T) \neq X$, and

(III) $\overline{R(T))} \neq X$,

and three possibilities for T^{-1}:

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists, but is discontinuous, and

(3) T^{-1} does not exist.

We shall first consider those factorable triangles for which $\delta = \lim a_n b_n$ exists.

Theorem 4. *Let A be a regular factorable triangle with nonnegative entries and assume that $\delta := \lim a_n b_n$ exists and is less than 1. If λ satisfies*

$$\left|\lambda - \frac{1}{2-\delta}\right| < \frac{1-\delta}{2-\delta} \tag{4}$$

and $\lambda \notin S$, then $\lambda \in III_1\sigma(A, c_0)$; i.e., λ is a point of the spectrum of A for which $\overline{R(T)} \neq X$ and T^{-1} exists and is continuous.

Proof. The proof is identical to that of Theorem 4 in [15]. □

Corollary 5. *Let A be a coregular factorable triangle with nonnegative entries such that δ exists and is less than 1. If λ satisfies*

$$\left|\lambda - \frac{t}{2-\delta}\right| < \frac{t(1-\delta)}{2-\delta} \tag{5}$$

and $\lambda \notin S$, then $\lambda \in III_1\sigma(A, c_0)$.

Proof. The proof is identical to that of Corollary 4 in [15]. □

Theorem 5. *Let A be a regular factorable matrix with nonnegative entries such that δ exists and $a_n b_n \geq \delta$ for all n sufficiently large. If λ satisfies*

$$\left|\lambda - \frac{1}{2-\delta}\right| = \frac{1-\delta}{2-\delta}, \quad \lambda \neq 1, \quad \frac{\delta}{1-\delta},$$

then $\lambda \in II_2\sigma(A, c_0)$.

Proof. The proof is identical to that of Theorem 6 of [15]. □

Theorem 6. *Let A be a regular factorable matrix with nonnegative entries and with $\gamma = \liminf a_n b_n$. If there exist values of n such that $0 \le a_n b_n \le \gamma/(2-\gamma)$, then $\lambda = a_n b_n$ implies that $\lambda \in III_3\sigma(A, c_0)$.*

Proof. The proof is identical to that of Theorem 8 in [15]. □

For $A = C_1$, Wenger [22] calculated the fine spectrum on c.

Let R_a denote the Rhaly terraced matrix generated by $a := \{a_n\}$. Then the following are true.

Corollary 6 (see [20], Theorem A)**.** *Let $0 < L < \infty$. If $\lambda \notin U$ and $\alpha L > 1$, then $\lambda \in III_1\sigma(R_a, c_0)$, where $\alpha = Re(1/\lambda)$.*

Corollary 7 (see [20], Theorem B)**.** *Let $0 < L < \infty$. If $\lambda \notin U$ and $\alpha L = 1$, then $\lambda \in II_2\sigma(R_a, c_0)$.*

Corollary 8 (see [20], Theorem C)**.** *Let $0 < L < \infty$. If $\lambda = a_m$ for at least one m $(m = 0, 1, \dots)$, then $\lambda = a_m \in III_3\sigma(R_a, c_0)$.*

Acknowledgement

The authors take this opportunity to thank the referee for the careful reading of the manuscript and for the suggestions for improvement.

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