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The velocity averaging for a heterogeneous heat type equation*

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Abstract. We prove that the sequence of averaged quantities $\int_{\mathbf{R}^m} u_n(t,\mathbf{x},\mathbf{y})v(\mathbf{y})d\mathbf{y}$ is strongly precompact in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^{1+d})$, where $v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^m)$, and $u_n \in \mathrm{L}^2(\mathbf{R}^{1+d}\times\mathbf{R}^m)$ are solutions to strictly parabolic transport equations with flux explicitly depending on space and time. In order to obtain the result, we use a recently introduced parabolic variant of H-measures.
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1. Introduction

In the paper we work in a $(1+d)$-dimensional space $\mathbf{R}^{1+d}$, where we distinguish between the first variable corresponding to time (t in the physical or τ in the dual space), and the second one corresponding to *space* ($\mathbf{x}$ in the physical or $\boldsymbol{\xi}$ in the dual space). We consider a parabolic differential operator P of the form

$$Pu(t,\mathbf{x},\mathbf{y}) = \partial_t(b(t,\mathbf{x},\mathbf{y})u(t,\mathbf{x},\mathbf{y})) - \sum_{|\boldsymbol{\alpha}|\le 2} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\big(a_\alpha(t,\mathbf{x},\mathbf{y})u(t,\mathbf{x},\mathbf{y})\big),$$

where b, a_α are real, continuous coefficients such that $b>0$ (or $b<0$), and the principal symbol of P satisfies the ellipticity condition

$$\big(\forall\,(t,\mathbf{x},\mathbf{y}) \in \mathbf{R}^{1+d}\times\mathbf{R}^m\big)\ \big(\forall\,\boldsymbol{\xi}\in \mathrm{S}^{d-1}\big) \quad \sum_{|\boldsymbol{\alpha}|=2} \boldsymbol{\xi}^{\boldsymbol{\alpha}} a_\alpha(t,\mathbf{x},\mathbf{y}) \neq 0. \tag{1}$$

Operator P involves the time/space variables $(t,\mathbf{x}) \in \mathbf{R}^{1+d}$, with respect to which we have derivatives, and the parameter $\mathbf{y}\in\mathbf{R}^m$, which is usually called the velocity

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variable. Our aim is to inspect conditions on (u_n) and (Pu_n) under which a sequence of averaged quantities

$$\int u_n(t,\mathbf{x},\mathbf{y})v(\mathbf{y})d\mathbf{y}, \quad v \in \mathrm{C}_c^\infty(\mathbf{R}^m)$$

is relatively compact in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^{1+d})$.

Analogously to standard Sobolev spaces H^s, for $s \in \mathbf{R}$ we consider anisotropic function spaces

$$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) := \{u \in \mathcal{S}' : (1+\rho^4)^{s/4}\,\hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d})\},$$

where $\rho^4(\tau,\boldsymbol{\xi}) := (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4$. With the scalar product

$$\langle\, u \mid v\,\rangle_{\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})} := \Big\langle\, (1+\rho^4)^{s/4}\,\hat{u} \,\Big|\, (1+\rho^4)^{s/4}\,\hat{v}\,\Big\rangle_{\mathrm{L}^2(\mathbf{R}^{1+d})}$$

$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})$ is a Hilbert space.

It is easily checked that for $s \in \mathbf{R}^+$, $\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})$ is continuously embedded into $\mathrm{H}^{\frac{s}{2}}(\mathbf{R}^{1+d})$, while its dual is the anisotropic space $\mathrm{H}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d})$. Indeed, $\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})$ is isomorphic to the $\mathrm{L}^2(\mathbf{R}^{1+d})$ weighted by the function $\omega = (1+\rho^4)^{s/2}$ - denote it by $\mathrm{L}^2_\omega(\mathbf{R}^{1+d})$. Thus, its dual is isomorphic to $\mathrm{L}^2_{\omega^{-1}}(\mathbf{R}^{1+d})$ which is, in turn, isomorphic to $\mathrm{H}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d})$ (the details can be found in [17]).

The following theorem is the main result of the paper.

Theorem 1. *Let $u_n \rightharpoonup 0$ weakly in $\mathrm{L}^2(\mathbf{R}^{1+d}\times\mathbf{R}^m)$. Assume that, for an $l \in \mathbf{N}$,*

$$\partial_t(bu_n) - \sum_{|\boldsymbol{\alpha}|\leq 2} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(a_\alpha u_n) = \sum_{|\boldsymbol{\beta}|\leq l} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} F_n^{\boldsymbol{\beta}}, \tag{2}$$

where $F_n^{\boldsymbol{\beta}} \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^m;\mathrm{H}^{-1,-2}(\mathbf{R}^{1+d}))$, while b, a_α satisfy the above assumptions.

Then for every $v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^m)$, the sequence of averaged quantities $(\int_{\mathbf{R}^m} u_n(t,\mathbf{x},\mathbf{y})v(\mathbf{y})d\mathbf{y})$ is strongly precompact in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^{1+d})$.

Results similar to Theorem 1 are usually called velocity averaging lemmas. They appear to be very popular throughout the last two decades, since the question of the existence of solutions to many nonlinear problems is reduced to precompactness of a velocity averaged sequence of solutions to linear equations similar to (2) (see e.g. [2, 5, 14]).

Still, due to a lack of appropriate tools, almost all results on the velocity averaging were restricted to the homogeneous case, i.e. when coefficients of the considered equation depend only on the velocity variable (in our case that would mean $a_\alpha(t,\mathbf{x},\mathbf{y}) = a_\alpha(\mathbf{y})$ and $b(t,\mathbf{x},\mathbf{y}) = b(\mathbf{y})$; see e.g. [1, 10, 13]).

A velocity averaging result involving the non-homogeneous coefficients is [9, Theorem 2.5], in which hyperbolic equations are considered. Here, we shall extend this result to the case of parabolic equations. In order to accomplish the plan, we need to generalise a recently introduced [3, 4] parabolic variant of Tartar's H-measures (microlocal defect measures in Gérard's terminology) [15, 9]. The study of such variants H-measures was initiated by Tartar [16].

2. On the parabolic variant of H-measures

Let $\mathrm{P}^d \subset \mathbf{R}^{1+d}$ be a smooth compact hypersurface implicitly given by:

$$\mathrm{P}^d \ \dots\ \rho^4(\tau,\boldsymbol{\xi}) = (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4 = 1.$$

For any point $T = (\tau,\boldsymbol{\xi}) \in \mathbf{R}^{1+d} \setminus \{0\}$, we define its *parabolic projection* to P^d as

$$T_P = \pi_P(\tau,\boldsymbol{\xi}) = \left(\frac{\tau}{\rho^2(\tau,\boldsymbol{\xi})}, \frac{\boldsymbol{\xi}}{\rho(\tau,\boldsymbol{\xi})}\right),$$

(as $\rho^4 > 0$ on $\mathbf{R}^{1+d} \setminus \{0\}$, by choosing the positive determination of roots, this projection is uniquely defined).

In the sequel, by $\hat{\mathsf{u}}(\tau,\boldsymbol{\xi}) := \mathcal{F}\mathsf{u}(\tau,\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{-2\pi i(\tau t+\boldsymbol{\xi}\cdot\mathbf{x})}\mathsf{u}(t,\mathbf{x})\,dtd\mathbf{x}$ we denote the Fourier transform, while $\overline{\mathcal{F}}$ (or $^\vee$) denotes the inverse Fourier transform. The following theorem is in the basis of our procedure:

Theorem 2 (see [4], Theorem 4.1)**.** *If* $(\mathsf{u}_n) = ((u_n^1,\dots,u_n^r))$ *is a sequence in* $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)$ *such that* $u_n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there exists a subsequence* $(\mathsf{u}_{n'})$ *and a complex, positive semi-definite matrix Radon measure* $\boldsymbol{\mu} = \{\mu^{ij}\}_{i,j=1,\dots,d}$ *on* $\mathbf{R}^{1+d}\times\mathrm{P}^d$*, such that for all* $\varphi_1,\varphi_2 \in \mathrm{C}_0(\mathbf{R}^{1+d})$ *and* $\psi \in \mathrm{C}(\mathrm{P}^d)$*:*

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^{1+d}} &\mathcal{F}(\varphi_1 u^i_{n'})(\tau,\boldsymbol{\xi})\overline{\mathcal{F}(\varphi_2 u^j_{n'})(\tau,\boldsymbol{\xi})}\psi(\pi_P(\tau,\boldsymbol{\xi}))d\tau d\boldsymbol{\xi}\\ &= \int_{\mathbf{R}^{1+d}\times\mathrm{P}^d} \varphi_1(t,\mathbf{x})\overline{\varphi_2(t,\mathbf{x})}\psi(\tau,\boldsymbol{\xi})d\mu^{ij}(t,\mathbf{x},\tau,\boldsymbol{\xi}) \qquad (3)\\ &= \langle \mu^{ij}, \varphi_1\bar{\varphi}_2\otimes\psi\rangle, \quad i,j=1,\dots,d,\end{aligned}$$

where $\otimes$ *stands for the tensor product of functions in different variables.*

The measure $\boldsymbol{\mu}$ from the above theorem is called the parabolic variant H-measure associated to (a sub)sequence (of) (u_n).

Remark 1. *By a complex Radon measure on a locally compact Hausdorff space* X *we denote an element from the dual space* $(\mathrm{C}_0(X))'$*. Complex Radon measures form a Banach space denoted by* $\mathcal{M}_b(X)$*.*

The above theorem remains valid if a sequence u_n *is taken from* $\mathrm{L}^2_{\mathrm{loc}}$*, but in that case the corresponding variant H-measure does not have to be a (complex) Radon measure, but a distribution of order 0.*

Similarly to [11], one proves the following boundedness result on the variant H-measures.

Corollary 1. *Let* $M_i := \limsup \|u_n^i\|_{\mathrm{L}^2(\mathbf{R}^{1+d})}$*. Then*

$$\|\mu^{ij}\| \le M_i M_j, \quad i,j=1,\dots,d,$$

where $\|\mu^{ij}\| = |\mu^{ij}|(\mathbf{R}^{1+d}\times\mathrm{P}^d)$*, and* $|\mu^{ij}|$ *is the total variation of* μ^{ij}*.*

Proof. We start with the case $i = j$. Using the positivity of μ^{ii} ($\boldsymbol{\mu}$ is positive semi-definite), we conclude $\|\mu^{ii}\| = \mu^{ii}(\mathbf{R}^{1+d} \times \mathrm{P}^d)$. Next, for a positive function $\varphi \in \mathrm{C}_0(\mathbf{R}^{1+d}), \varphi \le 1$, from (3) and Plancherel's formula we obtain

$$\langle \mu^{ii}, \varphi^2 \rangle = \lim_{n'} \int_{\mathbf{R}^{1+d}} \mathcal{F}(\varphi u^i_{n'})(\tau, \boldsymbol{\xi}) \overline{\mathcal{F}(\varphi u^i_{n'})(\tau, \boldsymbol{\xi})} d\tau d\boldsymbol{\xi} \le M_i^2.$$

From the arbitrariness of φ, it follows

$$\|\mu^{ii}\| \le M_i^2. \tag{4}$$

Now, for an arbitrary $\phi \in \mathrm{C}_0(\mathbf{R}^{1+d} \times \mathrm{P}^d), |\phi| \le 1$, denote $\phi_1 = \phi/\sqrt{|\phi|}$ and $\phi_2 = \sqrt{|\phi|}$. As $\boldsymbol{\mu}$ is positive semi-definite (and hermitian), the same property holds for the matrix

$$\begin{pmatrix} \langle \mu^{ii}, \phi_1 \overline{\phi_1} \rangle & \langle \mu^{ij}, \phi_1 \overline{\phi_2} \rangle \\ \langle \mu^{ij}, \phi_1 \overline{\phi_2} \rangle & \langle \mu^{jj}, \phi_2 \overline{\phi_2} \rangle \end{pmatrix},$$

and from (4) we immediately obtain

$$|\langle \mu^{ij}, \phi \rangle| \le \big(\langle \mu^{ii}, |\phi| \rangle \langle \mu^{jj}, |\phi| \rangle \big)^{1/2} \le M_i M_j.$$

As (e.g. [7, p. 89])

$$\|\mu^{ij}\| = \sup\{ \langle \mu^{ij}, \phi \rangle : |\phi| \le 1, \phi \in \mathrm{C}_0(\mathbf{R}^{1+d} \times \mathrm{P}^d) \},$$

we conclude that the statement of the corollary holds. □

Remark 2. *In the case when $(\varphi_i \mathsf{u}_n)$ are uniformly compactly supported for $i = 1$ and 2, one can replace the parabolic homogeneous function $\psi \circ \pi_P$ from Theorem 2 by a function $\tilde{\psi} \in \mathrm{C}(\mathbf{R}^{1+d})$ being equal to a function $\psi \circ \pi_P$ outside a compact set. Indeed, due to compact supports, products $\varphi_i \mathsf{u}_n$ converge to zero weakly in $\mathrm{L}^1(\mathbf{R}^{1+d})$, thus $\mathcal{F}(\varphi_i \mathsf{u}_n) \to 0$ pointwise. The statement then follows from the Lebesgue dominated convergence theorem, with $C_1 C_2 \chi$ as a dominated function, C_i being the L^1 bound of $(\varphi_i \mathsf{u}_n)$ and $\chi := \tilde{\psi} - \psi \circ \pi_P$ the compactly supported difference (for details check [12, Remark 2]).*

Remark 3. *By using multiplier operators associated to functions defined on P^d, we can conveniently rewrite (3). More precisely, for a function $\psi \in \mathrm{C}(\mathrm{P}^d)$ we define an operator P_ψ on $\mathrm{L}^2(\mathbf{R}^{1+d})$ by $P_\psi u := ((\psi \circ \pi_P)\hat{u})^\vee$, i.e.*

$$(P_\psi u)(t, \mathbf{x}) = \int_{\mathbf{R}^{1+d}} e^{2\pi i (t\tau + \mathbf{x} \cdot \boldsymbol{\xi})} \psi\Big(\frac{\tau}{\rho^2(\tau, \boldsymbol{\xi})}, \frac{\boldsymbol{\xi}}{\rho(\tau, \boldsymbol{\xi})} \Big) \hat{u}(\tau, \boldsymbol{\xi}) \, d\tau d\boldsymbol{\xi} \,. \tag{5}$$

Clearly, P_ψ is a bounded operator, called (the Fourier) multiplier, with norm equal to $\|\psi\|_{\mathrm{L}^\infty}$.

By applying Plancherel's theorem to (3), we can rewrite it in the form:

$$\lim_{n'} \int_{\mathbf{R}^{1+d}} \Big(P_\psi \, \varphi_1 u^i_{n'} \Big)(t, \mathbf{x}) \, \overline{\varphi_2 u^j_{n'}(t, \mathbf{x})} dt d\mathbf{x} = \int_{\mathbf{R}^{1+d} \times \mathrm{P}^d} \varphi_1(t, \mathbf{x}) \overline{\varphi_2(t, \mathbf{x})} \psi(\tau, \boldsymbol{\xi}) d\mu^{ij}(t, \mathbf{x}, \tau, \boldsymbol{\xi}).$$

Notice that Theorem 2 is formulated for sequences of functions taking values in a finite dimensional Hilbert space, $\mathbf{C}^r$. In this section we shall introduce a parabolic variant of H-measures corresponding to sequences of functions indexed in an uncountable set. More precisely, instead of a sequence $(u_n(t,\mathbf{x},i)) := (u_n^i(t,\mathbf{x}))$, $i = 1,...,r$, we shall consider a sequence $(u_n(t,\mathbf{x},\mathbf{y}))_{\mathbf{y}\in E}$, where $E \subseteq \mathbf{R}^m$ is an uncountable set. Such type of extensions was first studied by Gerard [9] for classical H-measures, while Panov [12] provided it in the ultra-parabolic case, but only for H-measures corresponding to L^∞-sequences of the form $(u_n(\mathbf{x},y))_{y\in\mathbf{R}}$ which are uniformly continuous with respect to y outside a zero measure set.

Thus, let (u_n) be an arbitrary sequence of functions in variables $(t,\mathbf{x}) \in \mathbf{R}^{1+d}$ and $\mathbf{y} \in \mathbf{R}^m$, weakly converging to zero in $\mathrm{L}^2(\mathbf{R}^{1+d} \times \mathbf{R}^m)$.

Next, we introduce a regularising kernel as usual: assume that $\omega \in \mathrm{C}_c^\infty(\mathbf{R}^m)$ is a non-negative function with the total mass one (i.e. $\int_{\mathbf{R}^m} \omega(\mathbf{z})d\mathbf{z} = 1$). Denote for $\varepsilon > 0$

$$\omega_\varepsilon(\mathbf{y}) = \frac{1}{\varepsilon^m}\omega(\frac{\mathbf{y}}{\varepsilon}),$$

and for $\varepsilon_k = \frac{1}{k}$ convolute it with $(u_n(t,\mathbf{x},\mathbf{y}))$ in $\mathbf{y}$:

$$u_n^k(t,\mathbf{x},\mathbf{y}) := \Big(u_n(t,\mathbf{x},\cdot) * \omega_{\varepsilon_k}\Big)(\mathbf{y}) = \int_{\mathbf{R}^m} u_n(t,\mathbf{x},\mathbf{z})\omega_{\varepsilon_k}(\mathbf{y}-\mathbf{z})d\mathbf{z}.$$

By the Young inequality for (almost every) fixed $(t,\mathbf{x}) \in \mathbf{R}^{1+d}$

$$\|u_n^k(t,\mathbf{x},\cdot)\|_{\mathrm{L}^2(\mathbf{R}^m)} \le \|u_n(t,\mathbf{x},\cdot)\|_{\mathrm{L}^2(\mathbf{R}^m)}\|\omega_{\varepsilon_k}\|_{\mathrm{L}^1(\mathbf{R}^m)}\,.$$

As the regularised kernels are of the total mass one, we have

$$\|u_n^k\|_{\mathrm{L}^2(\mathbf{R}^{1+d+m})} = \Big\|\|u_n^k(t,\mathbf{x},\cdot)\|_{\mathrm{L}^2(\mathbf{R}^m)}\Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \le \|u_n\|_{\mathrm{L}^2(\mathbf{R}^{1+d+m})}\,,$$

implying that the sequence (u_n^k) is bounded in the space $\mathrm{L}^2(\mathbf{R}^{1+d+m})$, uniformly with respect to k. Besides, we need the following result.

Lemma 1. *For a fixed k, the sequence (u_n^k) is bounded in* $\mathrm{W}^{1,\infty}(\mathbf{R}^m;\mathrm{L}^2(\mathbf{R}^{1+d}))$.

Proof. Similarly to the above, one shows that for a fixed k

$$\|u_n^k(t,\mathbf{x},\cdot)\|_{\mathrm{L}^\infty(\mathbf{R}^m)} \le \|u_n(t,\mathbf{x},\cdot)\|_{\mathrm{L}^2(\mathbf{R}^m)}\|\omega_{\varepsilon_k}\|_{\mathrm{L}^2(\mathbf{R}^m)} \quad \text{(a.e. } (t,\mathbf{x}) \in \mathbf{R}^{1+d})\,.$$

Thus for a $\mathbf{y} \in \mathbf{R}^m$ we have

$$\begin{aligned}\int_{\mathbf{R}^{1+d}} |u_n^k(t,\mathbf{x},\mathbf{y})|^2 dtd\mathbf{x} &\le \int_{\mathbf{R}^{1+d}} \|u_n^k(t,\mathbf{x},\cdot)\|^2_{\mathrm{L}^\infty(\mathbf{R}^m)} dtd\mathbf{x}\\ &\le \int_{\mathbf{R}^{1+d}} \|u_n(t,\mathbf{x},\cdot)\|^2_{\mathrm{L}^2(\mathbf{R}^m)}\|\omega_{\varepsilon_k}\|^2_{\mathrm{L}^2(\mathbf{R}^m)} dtd\mathbf{x}\\ &\le \|u_n\|^2_{\mathrm{L}^2(\mathbf{R}^{1+d}\times\mathbf{R}^m)}\|\omega_{\varepsilon_k}\|^2_{\mathrm{L}^2(\mathbf{R}^m)},\end{aligned}$$

implying boundedness of (u_n^k) in $\mathrm{L}^\infty(\mathbf{R}^m;\mathrm{L}^2(\mathbf{R}^{1+d}))$, as well as weak convergence

$$u_n^k(\cdot,\mathbf{y}) \rightharpoonup 0 \ \text{ in } \ \mathrm{L}^2(\mathbf{R}^{1+d})$$

for every fixed $\mathbf{y}\in\mathbf{R}^m$.

On the other hand,

$$\begin{aligned}
&\|u_n^k(\cdot,\mathbf{y}_1)-u_n^k(\cdot,\mathbf{y}_2)\|^2_{\mathrm{L}^2(\mathbf{R}^{1+d})}\\
&\le \int_{\mathbf{R}^{1+d}}\left(\int_{\mathbf{R}^m}|u_n(t,\mathbf{x},\mathbf{z})||\omega_{\varepsilon_k}(\mathbf{y}_1-\mathbf{z})-\omega_{\varepsilon_k}(\mathbf{y}_2-\mathbf{z})|d\mathbf{z}\right)^2 dtd\mathbf{x}\\
&\le \|u_n\|^2_{\mathrm{L}^2(\mathbf{R}^{1+d}\times\mathbf{R}^m)}\|\omega_{\varepsilon_k}(\mathbf{y}_1-\cdot)-\omega_{\varepsilon_k}(\mathbf{y}_2-\cdot)\|^2_{\mathrm{L}^2(\mathbf{R}^m)}\,.
\end{aligned}$$

As

$$|\omega_{\varepsilon_k}(\mathbf{y}_1-\mathbf{z})-\omega_{\varepsilon_k}(\mathbf{y}_2-\mathbf{z})|\le C_\omega k^{1+m}\,|\mathbf{y}_1-\mathbf{y}_2|\,,$$

where $C_\omega=\|\omega\|_{\mathrm{W}^{1,\infty}(\mathbf{R}^m)}$, it implies that u_n^k are Lipschitz continuous as functions from $\mathbf{R}^m$ to $\mathrm{L}^2(\mathbf{R}^{1+d})$, with the Lipschitz constant independent of n. □

The following lemma associates variant H-measures to sequences of regularised functions $\left(u_n^k(\cdot,\mathbf{y})\right)$, $\mathbf{y}\in\mathbf{R}^m$.

Lemma 2. *There exists a subsequence $(u_{n'})\subseteq(u_n)$ and a family $\{\mu_k^{\mathbf{pq}}\colon \mathbf{p},\mathbf{q}\in\mathbf{R}^m\}$ of parabolic variant H-measures such that for each $k\in\mathbf{N}$, every $\varphi_1,\varphi_2\in\mathrm{C}_0(\mathbf{R}^{1+d})$, and $\psi\in\mathrm{C}(\mathrm{P}^d)$:*

$$\begin{aligned}
&\lim_{n'}\int\limits_{\mathbf{R}^{1+d}}\Big(P_\psi\,\varphi_1 u^k_{n'}(\cdot,\mathbf{p})\Big)(t,\mathbf{x})\overline{\varphi_2(t,\mathbf{x})u^k_{n'}(t,\mathbf{x},\mathbf{q})}dtd\mathbf{x}\\
&\quad=\int\limits_{\mathbf{R}^{1+d}\times\mathrm{P}^d}\varphi_1(t,\mathbf{x})\overline{\varphi_2(t,\mathbf{x})}\psi(\tau,\boldsymbol{\xi})d\mu_k^{\mathbf{pq}}(t,\mathbf{x},\tau,\boldsymbol{\xi}).
\end{aligned}\tag{6}$$

Proof. Let us first notice that it is enough to prove the Lemma for a fixed $k\in\mathbf{N}$, as an application of the diagonalisation procedure will give a subsequence such that the above relation holds for every k as well.

According to Theorem 2, for fixed $\mathbf{p},\mathbf{q}\in\mathbf{R}^m$ there exist a subsequence of (u_n) and the corresponding complex Radon measure $\mu_k^{\mathbf{pq}}$ over $\mathbf{R}^{1+d}\times\mathrm{P}^d$ such that (6) holds. Using the diagonalisation procedure, we conclude that for a countable dense subset $D\times D\subset\mathbf{R}^m\times\mathbf{R}^m$ there exists a subsequence $(u_{n'})\subset(u_n)$ such that (6) holds for every $(\mathbf{p},\mathbf{q})\in D\times D$.

Let us take now an arbitrary point $(\mathbf{p},\mathbf{q})\in\mathbf{R}^m\times\mathbf{R}^m$. Let $(\mathbf{p}_j,\mathbf{q}_j)$ be a sequence in $D\times D$ converging to $(\mathbf{p},\mathbf{q})$. It defines a sequence of variant H-measures $(\mu_k^{\mathbf{p}_j\mathbf{q}_j})$, which is bounded in $\mathcal{M}_b(\mathbf{R}^{1+d}\times\mathrm{P}^d)$, due to the bounds of (u_n^k) in $\mathrm{L}^\infty(\mathbf{R}^m;\mathrm{L}^2(\mathbf{R}^{1+d}))$ and Corollary 1.

Therefore, by the Banach-Alaoglu-Bourbaki theorem there exists a complex Radon measure $\mu_k^{\mathbf{pq}}$ such that, along a subsequence, $\mu_k^{\mathbf{p}_j\mathbf{q}_j}\rightharpoonup\mu_k^{\mathbf{pq}}$ vaguely. Thus, for arbitrary functions $\varphi = \varphi_1\bar\varphi_2, \psi$ we have

$$\begin{aligned}\int \varphi(t,\mathbf{x})\psi(\tau,\boldsymbol{\xi})\, d\mu_k^{\mathbf{pq}}(t,\mathbf{x},\tau,\boldsymbol{\xi}) &= \lim_j \int \varphi(t,\mathbf{x})\psi(\tau,\boldsymbol{\xi})\, d\mu_k^{\mathbf{p}_j\mathbf{q}_j}(t,\mathbf{x},\tau,\boldsymbol{\xi}) \\ &= \lim_j \lim_{n'} V_{n'}^k(\mathbf{p}_j,\mathbf{q}_j),\end{aligned} \tag{7}$$

where V_n^k denotes the function defined by

$$V_n^k(\mathbf{p},\mathbf{q}) := \int\limits_{\mathbf{R}^{1+d}} \Big(P_\psi\, \varphi_1 u_n^k(\cdot,\mathbf{p})\Big)(t,\mathbf{x})\, \overline{\varphi_2(t,\mathbf{x})u_n^k(t,\mathbf{x},\mathbf{q})}dtd\mathbf{x}\,. \tag{8}$$

On the other hand,

$$\begin{aligned}V_{n'}^k(\mathbf{p}_j,\mathbf{q}_j) - V_{n'}^k(\mathbf{p},\mathbf{q}) &= V_{n'}^k(\mathbf{p}_j,\mathbf{q}_j) - V_{n'}^k(\mathbf{p},\mathbf{q}_j) + V_{n'}^k(\mathbf{p},\mathbf{q}_j) - V_{n'}^k(\mathbf{p},\mathbf{q}) \\ &\le C_k\Big(|\mathbf{p}_j-\mathbf{p}| + |\mathbf{q}_j-\mathbf{q}|\Big),\end{aligned}$$

where in the last step we combined Cauchy-Schwartz inequality, boundedness of the multiplier P_ψ on $\mathrm{L}^2(\mathbf{R}^{1+d})$, and Lipschitz continuity of functions u_n^k. The constant C_k appearing above is independent of n'. Thus, we can exchange limits in (7). This actually means that the functional $\mu_k^{\mathbf{pq}}$ does not depend on the defining subsequence (i.e. it is well defined for every $\mathbf{p},\mathbf{q} \in \mathbf{R}^m$), which completes the proof. □

Using the previous assertion, we prove the existence of variant H-measures associated to functions taking values in $\mathrm{L}^2(\mathbf{R}^m)$. First, we need to recall a few basic facts about Bochner spaces $\mathrm{L}^2(\mathbf{R}^m, E)$, where E is an arbitrary Banach space with dual E'.

We say that $f : \mathbf{R}^m \to E'$ is weakly $*$ measurable if it is measurable with respect to weak $*$ $\sigma(E', E)$ topology. The same term is also used for another concept that some authors called scalarwise measurability: a function f has the latter property if for each $e \in E$ the scalar function $\mathbf{x} \mapsto \langle f(\mathbf{x}), e\rangle$ is measurable. However, in case of a separable Banach space E, two concepts coincide.

The dual of $\mathrm{L}^2(\mathbf{R}^m, E)$ corresponds to the Banach space $\mathrm{L}^2_{\mathrm{w}*}(\mathbf{R}^{2m}; E')$ of weakly $*$ measurable functions $f : \mathbf{R}^m \to E'$ such that $\int_{\mathbf{R}^m} \|f(\mathbf{x})\|^2_{E'} d\mathbf{x} < \infty$ [6, p. 606].

In our case $E = \mathrm{C}_0(\mathbf{R}^{1+d} \times \mathrm{P}^d)$ and we get a measure that belongs to the topological dual of $\mathrm{L}^2(\mathbf{R}^{2m}; \mathrm{C}_0(\mathbf{R}^{1+d} \times \mathrm{P}^d))$ corresponding to the Banach space $\mathrm{L}^2_{\mathrm{w}*}(\mathbf{R}^{2m}; \mathcal{M}_b(\mathbf{R}^{1+d} \times \mathrm{P}^d))$.

Theorem 3. *For the subsequence $(u_{n'}) \subseteq (u_n)$ extracted in Lemma 2 there exists a measure $\mu \in \mathrm{L}^2_{\mathrm{w}*}(\mathbf{R}^{2m}; \mathcal{M}_b(\mathbf{R}^{1+d} \times \mathrm{P}^d))$ such that for all $v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^{2m})$, $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^{1+d})$ and $\psi \in \mathrm{C}(\mathrm{P}^d)$:*

$$\begin{aligned}&\lim_{n'} \int\limits_{\mathbf{R}^{2m}} \int\limits_{\mathbf{R}^{1+d}} v(\mathbf{p},\mathbf{q}) \Big(P_\psi\, \varphi_1 u_{n'}(\cdot,\mathbf{p})\Big)(t,\mathbf{x})\, \overline{\varphi_2(t,\mathbf{x})u_{n'}(t,\mathbf{x},\mathbf{q})}dtd\mathbf{x}d\mathbf{p}d\mathbf{q} \\ &\quad = \int\limits_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})\, \langle \mu(\mathbf{p},\mathbf{q}), \varphi_1\bar\varphi_2 \otimes \psi\rangle d\mathbf{p}d\mathbf{q}.\end{aligned} \tag{9}$$

Proof. The estimates on (u_n^k) give us that the sequence (V_n^k), defined by (8), is uniformly bounded in $\mathrm{L}^2(\mathbf{R}^{2m})$, while for a fixed k it is bounded in $\mathrm{L}^\infty(\mathbf{R}^{2m})$. By taking an arbitrary $v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^{2m})$ we have

$$\lim_k \int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})\langle \mu_k^{\mathbf{pq}}, \varphi_1\bar\varphi_2\psi\rangle d\mathbf{p}d\mathbf{q} = \lim_k \int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q}) \lim_{n'} V_{n'}^k(\mathbf{p},\mathbf{q}) d\mathbf{p}d\mathbf{q} = \lim_k \lim_{n'} \int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q}) V_{n'}^k(\mathbf{p},\mathbf{q}) d\mathbf{p}d\mathbf{q}\,,$$

where in the last step we used the Lebesgue dominated convergence theorem.

According to Lemma 3 (below), we can exchange the limits in the last relation, providing

$$\lim_k \int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})\langle \mu_k^{\mathbf{pq}}, \varphi_1\bar\varphi_2\psi\rangle d\mathbf{p}d\mathbf{q} = \lim_{n'} \int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q}) V_{n'}(\mathbf{p},\mathbf{q}) d\mathbf{p}d\mathbf{q}\,, \tag{10}$$

where V_n is defined similarly to V_n^k, with u_n^k replaced by u_n in (8).

On the other hand, Corollary 1 and L^2-estimates on (u_n^k) enable us to define a bounded sequence of operators $\mu_k \in \mathrm{L}^2_{\mathrm{w}^*}(\mathbf{R}^{2m}; \mathcal{M}_b(\mathbf{R}^{1+d}\times \mathrm{P}^d))$:

$$\mu_k(\mathbf{p},\mathbf{q})(\phi) := \langle \mu_k^{\mathbf{pq}}, \phi\rangle, \quad \phi \in \mathrm{C}_0(\mathbf{R}^{1+d}\times \mathrm{P}^d).$$

(Notice that the mapping $(\mathbf{p},\mathbf{q}) \mapsto \langle \mu_k^{\mathbf{pq}}, \phi\rangle$ is measurable on $\mathbf{R}^{2m}$ as it is defined as a limit of measurable functions in Lemma 2).

Therefore, there exists a subsequence $(\mu_{k'}) \subseteq (\mu_k)$ such that $\mu_{k'} \xrightarrow{*} \mu$ in the space $\mathrm{L}^2_{\mathrm{w}^*}(\mathbf{R}^{2m}; \mathcal{M}_b(\mathbf{R}^{1+d}\times \mathrm{P}^d))$. By passing to the limit on the left-hand side of (10), we get relation (9).

□

Remark 4. *Notice that the last theorem remains valid in the case when the test functions φ_1, φ_2 depend on the velocity variable ($\mathbf{p}$ or $\mathbf{q}$) as well, i.e. when they are taken from the space $\mathrm{C}_0(\mathbf{R}^{1+d}\times\mathbf{R}^m)$.*

Lemma 3. *For a given $v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^{2m})$ the sequence of averaged quantities $\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q}) V_n^k(\mathbf{p},\mathbf{q}) d\mathbf{p}d\mathbf{q}$ converges to*

$$\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q}) V_n(\mathbf{p},\mathbf{q}) d\mathbf{p}d\mathbf{q} = \int_{\mathbf{R}^{2m}} \int_{\mathbf{R}^{1+d}} v(\mathbf{p},\mathbf{q}) \Big(P_\psi\, \varphi_1 u_n(\cdot,\mathbf{p})\Big)(t,\mathbf{x})\, \overline{\varphi_2(t,\mathbf{x}) u_n(t,\mathbf{x},\mathbf{q})} dt d\mathbf{x} d\mathbf{p} d\mathbf{q},$$

uniformly with respect to n.

Proof. We need to estimate

$$\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})(V_n^k - V_n)(\mathbf{p},\mathbf{q})d\mathbf{p}d\mathbf{q}$$

$$=\int_{\mathbf{R}^{2m}}\int_{\mathbf{R}^{1+d}} v(\mathbf{p},\mathbf{q})\Big(P_\psi\,\varphi_1(u_n^k - u_n)(\cdot,\mathbf{p})\Big)(t,\mathbf{x})\,\overline{\varphi_2(t,\mathbf{x})u_n^k(t,\mathbf{x},\mathbf{q})}dtd\mathbf{x}d\mathbf{p}d\mathbf{q} \quad (11)$$

$$+\int_{\mathbf{R}^{2m}}\int_{\mathbf{R}^{1+d}} v(\mathbf{p},\mathbf{q})\Big(P_\psi\,\varphi_1 u_n(\cdot,\mathbf{p})\Big)(t,\mathbf{x})\,\overline{\varphi_2(t,\mathbf{x})(u_n^k - u_n)(t,\mathbf{x},\mathbf{q})}dtd\mathbf{x}d\mathbf{p}d\mathbf{q}.$$

Obviously, it is enough to obtain the estimate for a dense set of functions (with compact support) in L^2. Furthermore, we can take v of the form $v(\mathbf{p},\mathbf{q}) = v_1(\mathbf{p})v_2(\mathbf{q})$, where v_1, v_2 belong to $\mathrm{W}^{1,\infty}_{\mathrm{c}}(\mathbf{R}^m)$. Indeed, each $v\in \mathrm{C}_c^\infty(\mathbf{R}^{2m})$ can be uniformly approximated by a finite sum of tensor products of functions of the form $v_1\otimes v_2$, where $v_1, v_2 \in \mathrm{C}_c^\infty(\mathbf{R}^m)$ (e.g. [8, p. 44]). Due to compact supports, we get an approximation by such a sum in L^2 as well.

Denote $v_1^k(\mathbf{p}) := \int_{\mathbf{R}^m} \frac{v_1(\mathbf{p}+\mathbf{z}/k)-v_1(\mathbf{p})}{|\mathbf{z}|/k}|\mathbf{z}|\omega(\mathbf{z})d\mathbf{z}$. By using a definition of functions u_n^k and a change of variables, the first integral on the right-hand side of the last relation equals

$$\frac{1}{k}\int_{\mathbf{R}^{2m}}\int_{\mathbf{R}^{1+d}}\Big(P_\psi\,\varphi_1 u_n(\cdot,\mathbf{p})\Big)(t,\mathbf{x})v_1^k(\mathbf{p})v_2(\mathbf{q})\overline{\varphi_2(t,\mathbf{x})u_n^k(t,\mathbf{x},\mathbf{q})}dtd\mathbf{x}d\mathbf{p}d\mathbf{q}$$

and it is bounded by

$$\frac{1}{k}\|\psi\|_{\mathrm{L}^\infty(\mathrm{P}^d)}\int_{\mathbf{R}^m}\|(\varphi_1 u_n)(\cdot,\mathbf{p})\|_{\mathrm{L}^2(\mathbf{R}^{1+d})}|v_1^k(\mathbf{p})|d\mathbf{p}\int_{\mathbf{R}^m}|v_2(\mathbf{q})|\|(\varphi_2 u_n^k)(\cdot,\mathbf{q})\|_{\mathrm{L}^2(\mathbf{R}^{1+d})}d\mathbf{q}\,.$$

As the functions u_n, u_n^k belong to a bounded set in $\mathrm{L}^2(\mathbf{R}^{1+d+m})$ (for all n and k), we get the required estimate.

Application of a similar procedure to the last integral in (11) completes the proof. □

3. Proof of the main result

We consider a parabolic version of the classical Riesz potential. More precisely, we introduce the multiplier operator $\mathcal{R}_2$, defined via the Fourier transform:

$$\mathcal{F}\left(\mathcal{R}_2 u\right)(\tau,\boldsymbol{\xi}) = \frac{1-\theta(\tau,\boldsymbol{\xi})}{\rho^2(\tau,\boldsymbol{\xi})}\hat{u}(\tau,\boldsymbol{\xi})\,,$$

where, let it be repeated, $\rho(\tau,\boldsymbol{\xi}) = \sqrt[4]{(2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4}$, while $\theta\in \mathrm{C}_c^\infty(\mathbf{R}^{1+d})$, such that $\theta\equiv 1$ on a neighbourhood of the origin.

In the proof of the main theorem we use the following statement.

Lemma 4. *The multiplier operator* $\mathcal{R}_2$ *is a bounded operator from* $\mathrm{L}^2(\mathbf{R}^{1+d})$ *to* $H^{1,2}(\mathbf{R}^{1+d})$.

Proof. It is enough to notice that there exists a constant $C_1 > 0$ such that for any $u \in \mathrm{L}^2(\mathbf{R}^{1+d})$:

$$\|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \mathcal{R}_2(u)\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} = \|(2\pi\boldsymbol{\xi})^{\boldsymbol{\alpha}} \frac{(1-\theta)}{\rho^2} \hat{u}\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \leq C_1 \|u\|_{\mathrm{L}^2(\mathbf{R}^{1+d})},$$

where $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ is a multiindex of length less than or equal to 2. Similarly, for a time derivative there is a constant $C_2 > 0$ such that

$$\|\partial_t \mathcal{R}_2(u)\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \leq C_2 \|u\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} .$$

The obtained estimates imply that $\|(1+\rho^4)^{1/2}\hat{u}\|_{L^2(\mathbf{R}^{1+d})} \leq C_3 \|u\|_{L^2(\mathbf{R}^{1+d})}$ for some constant $C_3 > 0$, giving that $\mathcal{R}_2 : \mathrm{L}^2(\mathbf{R}^{1+d}) \to \mathrm{H}^{1,2}(\mathbf{R}^{1+d})$ is bounded. As $\mathrm{H}^{1,2}(\mathbf{R}^{1+d})$ is continuously embedded in $\mathrm{H}^1(\mathbf{R}^{1+d})$, notice that, according to the Rellich theorem, $\mathcal{R}_2$ is a compact operator from $\mathrm{L}^2(\mathbf{R}^{1+d})$ to $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^{1+d})$. □

Proof of Theorem 1. In the first step, let us define functions

$$f_n(t,\mathbf{x},\mathbf{y}) := \int_{\mathbf{R}^m} v(\mathbf{p},\mathbf{y}) \Big((\mathcal{R}_2 \circ P_\psi)\, \varphi_1 u_n(\cdot,\mathbf{p}) \Big)(t,\mathbf{x})\, \overline{\varphi_2(t,\mathbf{x})}\, d\mathbf{p},$$

where $\varphi_1, \varphi_2 \in \mathrm{C}^2_{\mathrm{c}}(\mathbf{R}^{1+d})$, $\psi \in \mathrm{C}(\mathrm{P}^d)$, and $v \in \mathrm{C}^l_{\mathrm{c}}(\mathbf{R}^{2m})$, while l is the highest order of derivative with respect to the velocity variable $\mathbf{y}$ appearing in (2). Notice that, according to the last lemma, the sequence (f_n) is bounded in $\mathrm{H}^l(\mathbf{R}^m; \mathrm{H}^{1,2}(\mathbf{R}^{1+d}))$.

By applying f_n to (2) we get

$$\langle \sum_{|\boldsymbol{\beta}| \leq l} (-1)^{|\boldsymbol{\beta}|} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f_n, F_n^{\boldsymbol{\beta}} \rangle = \langle f_n, \partial_t (b u_n) \rangle - \langle f_n, \sum_{|\boldsymbol{\alpha}| \leq 2} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \big(a_\alpha u_n \big) \rangle,$$

where duality on $\mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^m; \mathrm{H}^{1,2}(\mathbf{R}^{1+d}))$ is considered.

Based on the estimates on (f_n) and $(F_n^{\boldsymbol{\beta}})$, the above relation can be rewritten as

$$\begin{aligned} \int_{\mathbf{R}^{2m}} \int_{\mathbf{R}^{1+d}} v(\mathbf{p},\mathbf{y}) \Bigg(& \partial_t \Big((\mathcal{R}_2 \circ P_\psi)\, \varphi_1 u_n(\cdot,\mathbf{p}) \Big)(t,\mathbf{x}) \overline{(b u_n)(t,\mathbf{x},\mathbf{y}) \varphi_2(t,\mathbf{x})} \\ & + \textstyle\sum_{|\boldsymbol{\alpha}|=2} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \Big((\mathcal{R}_2 \circ P_\psi)\, \varphi_1 u_n(\cdot,\mathbf{p}) \Big)(t,\mathbf{x}) \overline{(a_\alpha u_n)(t,\mathbf{x},\mathbf{y}) \varphi_2(t,\mathbf{x})} \Bigg) dt d\mathbf{x} d\mathbf{p} d\mathbf{y} = 0, \end{aligned} \tag{12}$$

where we have omitted terms converging to zero.

The operators $\partial_t (\mathcal{R}_2 \circ P_\psi)$, $\sum_{|\boldsymbol{\alpha}|=2} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} (\mathcal{R}_2 \circ P_\psi)$ have symbols

$$\frac{2\pi i \tau}{\rho^2(\tau,\boldsymbol{\xi})} \big(1 - \theta(\tau,\boldsymbol{\xi})\big) (\psi \circ \pi_P)(\tau,\boldsymbol{\xi}), \; -\frac{(2\pi)^2 \sum \boldsymbol{\xi}^{\boldsymbol{\alpha}}}{\rho^2(\tau,\boldsymbol{\xi})} \big(1 - \theta(\tau,\boldsymbol{\xi})\big) (\psi \circ \pi_P)(\tau,\boldsymbol{\xi}),$$

respectively, which are parabolic homogeneous outside a compact set, thus being admissible symbols for the parabolic variant H-measures according to Remark 2. Thus we can apply Theorem 3, and express the limit of (12) as

$$\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{y}) \Big\langle \big(2\pi i \tau b - (2\pi)^2 \sum_{|\boldsymbol{\alpha}|=2} \boldsymbol{\xi}^{\boldsymbol{\alpha}} a_\alpha \big) \mu(\mathbf{p},\mathbf{y}), \varphi_1 \bar{\varphi}_2 \otimes \psi \Big\rangle d\mathbf{p} d\mathbf{y} = 0\,,$$

where $\mu \in L^2_{w^*}(\mathbf{R}^{2m}; \mathcal{M}_b(\mathbf{R}^{1+d} \times P^d))$ is a variant H-measure associated to a (sub)sequence (of) (u_n).

As v is taken from a dense subset in $L^2(\mathbf{R}^{2m})$, the dual product under the above integral equals zero for almost every $(\mathbf{p}, \mathbf{y}) \in \mathbf{R}^{2m}$. On the other hand, as finite sums of tensor products $\varphi \otimes \psi, \varphi \in C_c(\mathbf{R}^{1+d}), \psi \in C(P^d)$ form a dense subset in $C_c(\mathbf{R}^{1+d} \times P^d)$, we conclude:

$$\big(2\pi i\tau b - (2\pi)^2 \sum_{|\boldsymbol{\alpha}|=2} \boldsymbol{\xi}^{\boldsymbol{\alpha}} a_\alpha\big)\mu(\mathbf{p}, \mathbf{y}) = 0 \quad \text{(a.e. } (\mathbf{p}, \mathbf{y}) \in \mathbf{R}^{2m}).$$

Taking into account the ellipticity condition (1), as well as $b > 0$ (or $b < 0$), the term multiplying μ differs from zero for $(\tau, \boldsymbol{\xi}) \in P^d$, which gives $\mu = 0$.

Finally, putting $\psi \equiv 1, \varphi_1 = \varphi_2 = \varphi \in C_0(\mathbf{R}^{1+d})$ and $v = v_1 \otimes v_1, v_1 \in L^2_c(\mathbf{R}^m)$, in Theorem 3 we get:

$$\lim_{n'} \int_{\mathbf{R}^{1+d}} \Big| \varphi(t, \mathbf{x}) \int_{\mathbf{R}^m} v_1(\mathbf{p}) u_{n'}(t, \mathbf{x}, \mathbf{p}) d\mathbf{p} \Big|^2 dt d\mathbf{x} = 0,$$

which proves the theorem. □

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