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On the approximate solution of a coefficient inverse problem for the kinetic equation[*,†]

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Abstract. In this paper, the existence, uniqueness and stability of the solution of a coefficient inverse problem (IP) for the kinetic equation (KE) are proven. The approximate solution of this IP for one-dimensional KE is investigated using two different techniques: finite difference approximation (FDA) and symbolic computation approach (SCA). A comparison among the exact solution of the problem, the numerical solution obtained from FDA and the approximate analytical solution obtained from SCA is presented.

AMS subject classifications: 35R30, 82C40, 65N21, 68W30

Key words: coefficient inverse problem, kinetic equation, finite difference approximation, symbolic computation

1. Introduction

Kinetic theory appeared in the second half of the nineteenth century with Maxwell and Boltzmann, later with Hilbert, Enskog, Chapman, Vlasov, and Grad. Searching for a form of matter which could explain Saturn's rings, Maxwell imagined that they were made of rocks colliding and gravitating around the planet. The density of matter is then parametrized by the space position x and the velocity p of the rocks, the so-called phase space. A few years later, Boltzmann completely formalized the process, giving a general representation of a 'dilute gas' as particles undergoing collisions and with free motion between collisions, and he wrote the famous equation which is now named after him. Vlasov wrote another kinetic equation (KE) for plasmas of charged particles. There, each particle undergoes a collective Coulombic attraction from others. Nowadays kinetic equations (KEs) appear in a variety of sciences and applications such as astrophysics, aerospace engineering, nuclear engineering, particle fuid interactions, semi-conductor technology, social sciences or in biology like chemotaxis and immunology. The common feature of these models is that the underlying Partial Differential Equation is posed in the phase space $(x, p) \in \mathbb{R}^{2n}$, $n \geq 1$, [15].

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[†]For interpretation of color in all figures, the reader is referred to the web version of this article available at www.mathos.hr/mc.

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In this study, the solvability of a coefficient inverse problem (IP) for the KE is proven in the case where the values of the solution are known on the boundary of a domain. A numerical scheme based on the Finite Difference Approximation (FDA) is described to obtain the approximate solution of the problem. In order to evaluate the effectiveness and stability of the proposed method, a random noise is added to the exact data and several computational experiments are performed. The obtained numerical solutions are compared with the exact solution and with the approximate analytical solution computed from Symbolic Computation Approach (SCA) which is based on the Galerkin method.

2. Statement of the problem

Let Ω be a domain in the Euclidean space $\mathbb{R}^{2n}$, $n \geq 1$. For the variables $(x,p) \in \Omega$, it is assumed that $x \in D$, $p \in G$, where D and G are domains in $\mathbb{R}^n$ with boundaries of class C^2, $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\Gamma_1 = \partial D \times G$, $\Gamma_2 = D \times \partial G$.

We consider the following KE in the domain Ω

$$\{u, H\} - \sigma(x)\, u = 0, \tag{1}$$

where $H(x,p)$ is the Hamiltonian function, $\{.,.\}$ is the Poisson bracket of u and H defined by

$$\{u, H\} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial u}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial u}{\partial p_i} \right).$$

In applications, u represents the number (or the mass) of particles in the unit volume element of the phase space in the neighbourhood of the point (x,p), $\nabla_x H$ is the force acting on a particle and σ is the absorption term. The Hamiltonian function is often of the form $H(x,p) = \frac{1}{2}|p|^2 + \phi(x)$, where $\frac{1}{2}|p|^2$ is the kinetic energy and $\phi(x)$ is the potential energy of the physical system under consideration, [1, 3, 6].

Problem 1. *Determine the functions $u(x,p)$ and $\sigma(x)$ that satisfy equation (1), provided that $u(x,p) > 0$, the $H(x,p) \in C^2\left(\overline{\Omega}\right)$ is given and the trace of the solution of equation (1) on the boundary $\partial\Omega$ exists and is known: $u|_{\partial\Omega} = u_0$.*

IP for a differential equation is the problem of determining the coefficients, the right-hand side, initial conditions or boundary conditions of the equation from some additional data on the solution of the equation. If a differential equation describes a physical process (physical field), its coefficients describe the characteristics (parameters) of the medium in which the process (field) is considered. The right-hand side of the equation describes the sources of the process. Therefore, from the physical point of view, inverse problems (IPs) are concerned with determining the characteristics of the medium and (or) the sources of the physical field by using some information on the physical field (solution of the direct problem). In IPs, it is often required to find these characteristics and (or) sources of the field inside a certain domain, and the information is given only at the boundary of this domain. Direct problems consist of finding the physical field in the domain under consideration if the characteristics of the medium and the sources are given, [12]. IPs, generally speaking, are nonlinear and most of them are not well-posed in the sense of Hadamard. The general theory of ill-posed problems and their applications are developed by A. N. Tikhonov, V. K. Ivanov, M. M. Lavrent'ev and their students [9, 10, 16].

The theory and applications of IPs have long made a silent imprint in science and engineering as a critical tool in establishing the link between the model and observations. In recent times, however, IPs have taken the center stage in many disciplines, a trend spurred not only by the advances in sensor technologies, wireless communications, and signal processing, but also by the necessity to obtain physically relevant parameters and input for computational models with ever-growing complexity and sophistication. Examples of such disciplines include seismic and medical imaging, non-destructive material characterization, and structural health monitoring, [7, 14].

IPs for KE are important both from theoretical and practical points of view. Interesting results in this field are presented in [2-6, 8].

3. Definitions and notations

In this section, we give some necessary definitions and notations used throughout the paper. For a bounded domain G, $C^m(G)$ is the Banach space of functions that are m times continuously differentiable in G; $C^\infty(G)$ is the set of functions that belong to $C^m(G)$ for all $m \geq 0$; $C_0^\infty(G)$ is the set of functions which have compact support in G and belong to $C^\infty(G)$; $L_2(G)$ is the space of measurable functions that are square integrable in G, $H^k(G)$ is the Sobolev space and $\overset{\circ}{H}{}^k(G)$ is the closure of $C_0^\infty(G)$ with respect to the norm of $H^k(G)$. These standard spaces are described in detail, for example, in [11, 13].

The following definitions and notations are based on [3].

Let $\widetilde{C}_0^3 = \left\{\varphi : \varphi \in C^3(\Omega),\ \varphi = 0 \text{ on } \partial\Omega\right\}$ and select a set $\{w_1, w_2, ...\} \subset \widetilde{C}_0^3(\Omega)$, which is a complete and orthonormal set in $L_2(\Omega)$. We may assume here that the linear span of this set is everywhere dense in $\overset{\circ}{H}_{1,2}(\Omega)$, where $\overset{\circ}{H}_{1,2}(\Omega)$ is the set of all real-valued functions $u(x,p) \in L_2(\Omega)$ that have generalized derivatives u_{x_i}, u_{p_i}, $u_{x_i p_j}$, $u_{p_i p_j}$ $(i,\ j = 1, 2, ..., n)$, which belong to $L_2(\Omega)$ and whose trace on $\partial\Omega$ is zero. Indeed, the space $\overset{\circ}{H}_{1,2}(\Omega)$ being separable, there exists a countable set $\{\varphi_i\}_{i=1}^\infty \subset \widetilde{C}_0^3(\Omega)$ which is everywhere dense in this space. If necessary, this set up can be extended to a set which is everywhere dense in $L_2(\Omega)$. Orthonormalizing the latter in $L_2(\Omega)$, we obtain $\{w_i\}_{i=1}^\infty$. The orthogonal projector of $L_2(\Omega)$ onto $\mathcal{M}_n$ is denoted by $\mathcal{P}_n$, where $\mathcal{M}_n$ is the linear span of $\{w_1, w_2, ...w_n\}$. By $\Gamma(A)$ we denote the set of all functions $u \in L_2(\Omega)$ with the following properties:

i) For any $u \in \Gamma(A)$ there exists a function $v \in L_2(\Omega)$ such that $\langle u, A^*\eta\rangle = \langle v, \eta\rangle$ holds for all $\eta \in C_0^\infty(\Omega)$, where

$$Au = \widehat{L}Lu,\ L = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial p_i}\right),\ \widehat{L} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i \partial p_i}$$

and A^* is the differential expression conjugate to A in the sense of Lagrange, and $\langle .,. \rangle$ is the scalar product in $L_2(\Omega)$.

ii) There exists a sequence $\{u_k\} \subset \tilde{C}_0^3(\Omega)$ such that $u_k \to u$ weakly in $L_2(\Omega)$ and $\langle Au_k, u_k\rangle \to \langle Au, u\rangle$ as $k \to \infty$.

4. Solvability of the problem

By introducing a new unknown function $\ln u = y$, Problem 1 can be reduced to the following problem.

Problem 2. *Find a pair of functions (y, σ) satisfying the equation*

$$Ly \equiv \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial y}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial y}{\partial p_i} \right) = \sigma(x), \tag{2}$$

provided that the Hamiltonian $H(x,p) \in C^2\left(\overline{\Omega}\right)$ is given and y is known on $\partial\Omega$: $y|_{\partial\Omega} = \ln u_0 = \mathrm{y}_0$, where $u_0 > 0$.

The main difficulty in studying the solvability of such IPs for KEs is their overdeterminancy. In the theory of IPs, if the number of free variables in the additional data exceeds the number of free variables in the unknown coefficient $(\sigma(x))$ or right-hand side of the equation, then the problem is called overdetermined, but this is not the case for $n = 1$ here. However, for dimension $n \geq 2$, Problem 2 is overdetermined in this sense. On the other hand, inverse problems for KE and integral geometry problems (IGP) are closely connected, i.e., many problems of integral geometry can be reduced to the corresponding IP for KE, and vice versa [3]. And here, the underlying operator of the related IGP for Problem 2 is compact and its inverse operator is unbounded. Therefore, it is impossible to prove general existence results. So, the initial data for these problems cannot be arbitrary; they should satisfy some "solvability conditions" which are difficult to establish, [3]. In this paper, we use the term "overdetermined problem" in this meaning.

On using some extension of the class of unknown functions σ, overdetermined Problem 2 is replaced by a determined one. This is achieved by assuming the unknown function σ depends not only upon the space variables x, but also upon the direction p in some special manner, i.e., consider $\sigma(x,p)$. The dependence upon p of $\sigma(x,p)$ cannot be arbitrary, because the problem would be underdetermined in this case. This special dependence means that $\sigma(x,p)$ satisfies a certain differential equation $\left(\widehat{L}\sigma = 0\right)$ with the following properties:

1) Problem 2 with the function $\sigma(x,p)$ becomes a determined one. In other words, the class of unknown functions σ is extended so that Problem 2 becomes a well-posed problem for the new class.

2) The sufficiently smooth functions σ depending only on x satisfy this equation.

Suppose that we have found a differential equation for $\sigma(x,p)$ satisfying properties 1 and 2. Suppose also that we know a priori that the function y_0^e represents the exact data of Problem 2 related to a function σ depending only on x. Then, utilizing y_0^e, a solution $\tilde{\sigma}$ to Problem 2 can be constructed. By the uniqueness, $\tilde{\sigma}$ and $\sigma(x)$ coincide. At the same time, knowing the approximate data y_0^a with $\|y_0^e - y_0^a\|_{H^3(\partial\Omega)} \leq \varepsilon$, an approximate solution $\sigma^a(x,p)$ can be constructed such that

$$\|\sigma - \sigma^a\|_{L_2(\Omega)} \leq C\varepsilon.$$

Recall that, if σ depends only on x and y_0^a does not satisfy the "solvability conditions", the solution σ^a depending only on x does not exist. In other words, a regularising procedure is constructed for Problem 2.

Application of this method of solvability of Problem 2 leads to a Dirichlet problem for the third order equation of the form

$$Au \equiv \widehat{L}Lu = \mathcal{F}.$$

The method was firstly proposed by Amirov [2] for the transport equation.

Problem 3. *Find a pair of functions $(y(x,p), \sigma(x,p))$ defined in Ω from equation (2), provided that $\sigma(x,p)$ satisfies the equation*

$$\left\langle \sigma, \widehat{L}\eta \right\rangle = 0,\ \widehat{L} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial p_i} \tag{3}$$

for any $\eta \in C_0^\infty(\Omega)$, the trace of $y(x,p)$ on $\partial\Omega$ exists and given is $y|_{\partial\Omega} = \mathrm{y}_0$, where $\langle .,. \rangle$ denotes the inner product in $L_2(\Omega)$. Equation (3) is satisfied in the generalized function sense.

Theorem 1. *Assume that $H \in C^2\left(\overline{\Omega}\right)$ and the following inequalities hold for all $\xi \in \mathbb{R}^n$, $(x,p) \in \overline{\Omega}$:*

$$\sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial p_i \partial p_j} \xi^i \xi^j \geq \alpha \left|\xi\right|^2,\ \sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j} \xi^i \xi^j \leq 0, \tag{4}$$

where α is a positive number. Then Problem 3 has at most one solution (y, σ) such that $y \in \Gamma(A)$ and $\sigma \in L_2(\Omega)$.

Proof. Since Problem 3 is linear, in order to prove the uniqueness of the solution of the problem, it is sufficient to establish that the homogeneous version of the problem has only a trivial solution.

Let (y, σ) be a solution to Problem 3 such that $y = 0$ on $\partial\Omega$ and $y \in \Gamma(A)$. Equation (2) and condition (3) imply $Ay = 0$. Since $y \in \Gamma(A)$, there exists a sequence $\{y_k\} \subset \widetilde{C}_0^3$ such that $y_k \to y$ weakly in $L_2(\Omega)$ and $\langle Ay_k, y_k \rangle \to 0$ as $k \to \infty$. Observing that $y_k = 0$ on $\partial\Omega$, we get

$$-\langle Ay_k, y_k \rangle = -\left\langle \widehat{L}Ly_k, y_k \right\rangle = \sum_{i=1}^{n} \left\langle \frac{\partial}{\partial p_i}(Ly_k), \frac{\partial y_k}{\partial x_i} \right\rangle. \tag{5}$$

We consider the identity

$$\sum_{j=1}^{n}\frac{\partial y_k}{\partial x_j}\frac{\partial}{\partial p_j}\left(Ly_k\right)=\sum_{j=1}^{n}\frac{\partial y_k}{\partial x_j}\frac{\partial}{\partial p_j}\left(\sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial y_k}{\partial x_i}-\frac{\partial H}{\partial x_i}\frac{\partial y_k}{\partial p_i}\right)\right)$$
$$=\frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial^2 H}{\partial p_i\partial p_j}\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j}-\frac{\partial^2 H}{\partial x_i\partial x_j}\frac{\partial y_k}{\partial p_i}\frac{\partial y_k}{\partial p_j}\right)$$
$$+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial p_j}\left[\frac{\partial y_k}{\partial x_j}\left(\frac{\partial y_k}{\partial x_i}\frac{\partial H}{\partial p_i}-\frac{\partial y_k}{\partial p_i}\frac{\partial H}{\partial x_i}\right)\right]$$
$$-\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left[\frac{\partial y_k}{\partial p_j}\left(\frac{\partial y_k}{\partial x_i}\frac{\partial H}{\partial p_i}-\frac{\partial y_k}{\partial p_i}\frac{\partial H}{\partial x_i}\right)\right]$$
$$+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial p_i}\frac{\partial y_k}{\partial x_j}\frac{\partial y_k}{\partial p_j}\right)-\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial x_i}\frac{\partial y_k}{\partial x_j}\frac{\partial y_k}{\partial p_j}\right).(6)$$

If the geometry of the domain Ω and the condition $y_k=0$ on $\partial\Omega$ are taken into account, then $\frac{\partial y_k}{\partial p_i}=0$ on Γ_1 and $\frac{\partial y_k}{\partial x_i}=0$ on Γ_2, $i=\overline{1,n}$. Therefore the divergent terms will disappear in (6), so from (5) we obtain

$$-\left\langle Ay_k,y_k\right\rangle=J\left(y_k\right), \tag{7}$$

where

$$J\left(y_k\right)\equiv\frac{1}{2}\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial^2 H}{\partial p_i\partial p_j}\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j}-\frac{\partial^2 H}{\partial x_i\partial x_j}\frac{\partial y_k}{\partial p_i}\frac{\partial y_k}{\partial p_j}\right)d\Omega. \tag{8}$$

From (4) and (8), we have

$$J\left(y_k\right)\geq\frac{1}{2}\sum_{i,j=1}^{n}\int_{\Omega}\frac{\partial^2 H}{\partial p_i\partial p_j}\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j}d\Omega\geq\frac{\alpha}{2}\int_{\Omega}\left|\nabla_x y_k\right|^2 d\Omega.$$

The use of Poincaré inequality yields

$$\int_{\Omega}y_k^2 d\Omega\leq C_0\int_{\Omega}\left|\nabla_x y_k\right|^2 d\Omega\leq CJ\left(y_k\right),$$

where $C_0>0$ depends on the Lebesgue measure of the domain Ω and does not depend on k, $C=2\alpha^{-1}C_0$ and $\nabla_x y_k=\left(y_{k_{x_1}},y_{k_{x_2}},...,y_{k_{x_n}}\right)$. Thus from (7) and the definition of $\Gamma\left(A\right)$, we get

$$\int_{\Omega}y^2 d\Omega\leq\underline{\lim}_{k\to\infty}\left\|y_k\right\|^2\leq C\,\underline{\lim}_{k\to\infty}J\left(y_k\right)=-C\,\underline{\lim}_{k\to\infty}\left\langle Ay_k,y_k\right\rangle=0,$$

i.e., $y=0$ in Ω. Then (2) implies $\sigma\left(x,p\right)=0$. Hence uniqueness of the solution of the problem is proven. □

It is worth to note that, if $y|_{\partial\Omega} = y_0 \neq 0$, then Problem 3 will not have a solution from $\Gamma(A)$. Because as can be seen from the proof of Theorem 1, if a pair (y, σ) satisfies equation (2), condition (3) and $y \in \Gamma(A)$, then $y = 0$ on Ω. So in the case when $y_0 \neq 0$ the solution would not satisfy the boundary condition.

Remark 1. *The geometry of the domain Ω is essential for the uniqueness of the solution of Problem 3. More precisely, it is important that Ω can be represented in form of a direct product of two domains in the spaces of x and p, correspondingly. Indeed, the assertion of the theorem does not hold if for Ω we take a ball of the form $\widetilde{\Omega} = \left\{(x,p) \in \mathbb{R}^1 \times \mathbb{R}^1 : x^2 + (2-p)^2 < 1\right\}$. To prove this, consider the equation $pu_x = \sigma(x,p)$ in $\widetilde{\Omega}$. It is obvious that condition (4) is satisfied, since $H = \frac{1}{2}p^2$. Also, it can be directly verified that the pair of functions $u = \frac{1}{2p}\left(x^2 + (2-p)^2 - 1\right)$ and $\sigma = x$ satisfies the equation $pu_x = \sigma$, σ satisfies condition (3) and $u = 0$ on $\partial\widetilde{\Omega}$.*

If $y_0 \in C^3(\partial\Omega)$ and $\partial D \in C^3$, $\partial G \in C^3$, then Problem 3 can be reduced to the following problem.

Problem 4. *Find a pair of functions (y,σ) satisfying the equation*

$$Ly = \sigma + F \tag{9}$$

provided that F is a known function in $H^2(\Omega)$, the trace of the solution y on $\partial\Omega$ exists and is zero, and σ satisfies condition (3).

In this reduction, we simply consider a new unknown function $\overline{y} = y - \Phi$, where Φ is a function such that $\Phi|_{\partial\Omega} = y_0$ and $\Phi \in C^3(\Omega)$. Since $y_0 \in C^3(\partial\Omega)$ and $\partial D \in C^3$, $\partial G \in C^3$ the existence of the function Φ follows from Theorem 2 in [13, p. 130]. Finally, if we again denote $\overline{y}$ by y, we can obtain (9) and the condition $y|_{\partial\Omega} = 0$, where $F = -L\Phi$.

The following theorem establishes the existence and stability of the solution of the problem.

Theorem 2. *If $H \in C^2\left(\overline{\Omega}\right)$, $F \in H^2(\Omega)$ and the inequalities*

$$\sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial p_i \partial p_j}\xi^i\xi^j \geq \alpha_1 |\xi|^2, \quad \sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j}\xi^i\xi^j \leq -\alpha_2 |\xi|^2 \tag{10}$$

hold for all $(x,p) \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$, then there exists a solution (y,σ) of Problem 4 such that $y \in \Gamma(A) \cap \mathring{H}^1(\Omega)$, $\sigma \in L_2(\Omega)$ and the inequality

$$\|y\|_{\mathring{H}^1(\Omega)} + \|\sigma\|_{L_2(\Omega)} \leq C\, \||\nabla_p F|\|_{L_2(\Omega)} + \|F\|_{L_2(\Omega)}$$

holds, where C depends on the given functions and the Lebesgue measure of the domain Ω, $\nabla_p F = (F_{p_1}, ..., F_{p_n})$. In (10), α_1 and α_2 are some positive numbers.

Assumption (10) of the theorem has a physical meaning: The family of rays corresponding to the Hamiltonian $H(x,p)$ is regular, which implies the absence of waveguides in the domain Ω.

Proof. Let us consider the following auxiliary problem

$$Ay = \mathcal{F}, \tag{11}$$

$$y|_{\partial\Omega} = 0, \tag{12}$$

where $\mathcal{F} = \widehat{L}F$.

An approximate solution of problem (11)-(12) is sought in the form

$$y_N = \sum_{i=1}^{N} \alpha_{N_i} w_i; \;\; \alpha_N = (\alpha_{N_1}, \alpha_{N_2}, ..., \alpha_{N_N}) \in \mathbb{R}^N,$$

where the unknown vector α_N is determined from the following system of linear algebraic equations:

$$\langle Ay_N - \mathcal{F}, w_i \rangle = 0, \qquad i = 1, 2, ..., N. \tag{13}$$

We shall prove that there exists a unique solution α_N of system (13) for any $F \in H^2(\Omega)$ under the hypotheses of the theorem. For this purpose, the i-th equation of the homogeneous system ($\mathcal{F} = 0$) is multiplied by $-2\alpha_{N_i}$ and the sum from 1 to N with respect to i. Hence

$$-2 \langle Ay_N, y_N \rangle = 0$$

is obtained. If the identity $-\langle Ay_N, y_N \rangle = J(y_N)$ is considered, then the assumptions of the theorem imply $\nabla y_N = 0$, where $\nabla y_N = \left(y_{N_{x_1}}, ..., y_{N_{x_n}}, y_{N_{p_1}}, ..., y_{N_{p_n}},\right)$. As a result of the condition $y_N|_{\partial\Omega} = 0$, we have $y_N = 0$ in Ω. Since the system $\{w_i\}$ is linearly independent, we get $\alpha_{N_i} = 0$, $i = 1, 2, ..., N$. Thus the homogeneous version of system (13) has only a trivial solution. Therefore, the original inhomogeneous system (13) has a unique solution $\alpha_N = (\alpha_{N_i})$, $i = 1, ..., N$ for any function $F \in H^2(\Omega)$.

Now we estimate the solution y_N in terms of F. We multiply the i-th equation of the system by $-2\alpha_{N_i}$ and the sum from 1 to N with respect to i. Since $\mathcal{F} = \widehat{L}F$, we obtain

$$-2 \langle Ay_N, y_N \rangle = -2 \left\langle \widehat{L}F, y_N \right\rangle. \tag{14}$$

Observing that $y_N = 0$ on $\partial\Omega$, the right-hand side of (14) can be estimated as follows:

$$\begin{aligned} -2 \left\langle \widehat{L}F, y_N \right\rangle &= 2 \int_{\Omega} \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial y_N}{\partial x_i} d\Omega \\ &\leq \beta \int_{\Omega} |\nabla_p F|^2 \, d\Omega + \beta^{-1} \int_{\Omega} |\nabla_x y_N|^2 \, d\Omega, \end{aligned} \tag{15}$$

where $\beta > 0$. As can be seen from (7), the left hand-side of (14) is equal to $2J(y_N)$. Then from (14) and (15), we have

$$2J(y_N) \leq \beta \int_{\Omega} |\nabla_p F|^2 \, d\Omega + \beta^{-1} \int_{\Omega} |\nabla_x y_N|^2 \, d\Omega.$$

Recalling that Ω is bounded and $y_N|_{\partial\Omega} = 0$, the last inequality implies

$$\|y_N\|_{\mathring{H}^1(\Omega)} \leq C \, \||\nabla_p F|\|_{L_2(\Omega)}\,, \tag{16}$$

where $\beta^{-1} < \alpha_1$ and the constant $C > 0$ does not depend on N.

This implies that the set of functions y_N, $N = 1, 2, 3, ...$ is bounded in $\mathring{H}^1(\Omega)$. Since $\mathring{H}^1(\Omega)$ is a Hilbert space, $\{y_N\}$ is weakly compact in it. Hence, there exists a subsequence in this set that is denoted again for simplicity by $\{y_N\}$ converging weakly in $\mathring{H}^1(\Omega)$ to a certain function $y \in \mathring{H}^1(\Omega)$. From inequality (16) and weak convergence of $\{y_N\}$ to y in $\mathring{H}^1(\Omega)$, it follows that

$$\|y\|_{\mathring{H}^1(\Omega)} \leq \varliminf_{N\to\infty} \|y_N\|_{\mathring{H}^1(\Omega)} \leq C \, \||\nabla_p F|\|_{L_2(\Omega)}\,. \tag{17}$$

On the other hand, from estimate (16), it is easy to prove that there exists a subsequence of $\{y_N\}$ and using (13), we have

$$\left\langle Ly_N - F, \widehat{L}w_i \right\rangle = 0. \tag{18}$$

Since the linear span of the functions w_i, $i = 1, 2, 3, ...$ is everywhere dense in $\mathring{H}_{1,2}(\Omega)$, passing to the limit as $N \to \infty$ in (18) yields

$$\left\langle Ly - F, \widehat{L}\eta \right\rangle = 0, \tag{19}$$

for any $\eta \in \mathring{H}_{1,2}(\Omega)$. Setting $\sigma = Ly - F$, from (19) we see that σ satisfies condition (3) for any $\eta \in C_0^\infty(\Omega) \subset \mathring{H}_{1,2}(\Omega)$ and the following estimate is valid:

$$\|\sigma\|_{L_2(\Omega)} \leq C \, \|y\|_{\mathring{H}^1(\Omega)} + \|F\|_{L_2(\Omega)}\,. \tag{20}$$

Consequently, by using inequality (17), we obtain

$$\|y\|_{\mathring{H}^1(\Omega)} + \|\sigma\|_{L_2(\Omega)} \leq C \, \||\nabla_p F|\|_{L_2(\Omega)} + \|F\|_{L_2(\Omega)}\,. \tag{21}$$

In expression (21), C stands for different constant that depend only on the given functions and the size of the domain Ω. Thus we have found a solution (y, σ) to Problem 4, where $y \in \mathring{H}^1(\Omega)$ and $\sigma \in L_2(\Omega)$. Now it will be proven that $y \in \Gamma(A)$. Since $y \in L_2(\Omega)$ and $F \in H^2(\Omega)$, it follows that $\mathcal{F} = Ay \in L_2(\Omega)$ in the generalized sense. Indeed, for any $\eta \in C_0^\infty(\Omega)$ we have

$$\langle y, A^*\eta \rangle = \left\langle y, L^* \left(\widehat{L}\right)^* \eta \right\rangle = \left\langle Ly, \left(\widehat{L}\right)^* \eta \right\rangle = \left\langle F, \left(\widehat{L}\right)^* \eta \right\rangle = \langle \mathcal{F}, \eta \rangle\,.$$

Here $\mathcal{F} = \widehat{L}F \in L_2(\Omega)$.

To complete the proof, it remains to show the convergence

$$\langle Ay_N, y_N \rangle \to \langle Ay, y \rangle \quad \text{as } N \to \infty.$$

From (13), it follows that $\mathcal{P}_N A y_N = \mathcal{P}_N \mathcal{F}$. Since $\mathcal{P}_N$ is an orthogonal projector, $\mathcal{P}_N \mathcal{F}$ strongly converges to $\mathcal{F}$ in $L_2(\Omega)$ as $N \to \infty$, i.e., $\mathcal{P}_N A y_N \to \mathcal{F} = Ay$ strongly in $L_2(\Omega)$ as $N \to \infty$. Then we have $\langle \mathcal{P}_N A y_N, y_N \rangle \to \langle Ay, y \rangle$ as $N \to \infty$ because $\{y_N\}$ weakly converges to y and $\{\mathcal{P}_N A y_N\}$ strongly converges to Ay in $L_2(\Omega)$ as $N \to \infty$. By the definition of $\mathcal{P}_N$ and y_N (since the operator $\mathcal{P}_N$ is self adjoint in L_2),

$$\langle A y_N, y_N \rangle = \langle A y_N, \mathcal{P}_N y_N \rangle = \langle \mathcal{P}_N A y_N, y_N \rangle .$$

Hence $\langle A y_N, y_N \rangle \to \langle Ay, y \rangle$ as $N \to \infty$, which completes the proof. □

5. Approximate solution of the problem

In this section, we present and compare two different approaches for the approximate solution of Problem 1 for $n = 1$. The first one is "finite difference approximation (FDA)" which is based on the finite difference method and the second one is "symbolic computation approach (SCA)" which is based on the Galerkin method.

5.1. The finite difference approximation

We shall consider the following auxiliary Dirichlet problem with homogeneous boundary data:

Problem 5. *Find a function y which satisfies the following third order partial differential equation*

$$Ay \equiv y_{xpx}H_p - y_{ppx}H_x + y_{xx}H_{pp} - y_{pp}H_{xx} + y_{xp}H_{px} - y_{px}H_{xp} + y_x H_{ppx} - y_p H_{xpx} = \mathcal{F}, \tag{22}$$

and the boundary condition

$$y|_{\partial\Omega} = 0, \tag{23}$$

where $Ay = \widehat{L}Ly$ and $\mathcal{F} = \widehat{L}F$.

Equation (22) can be derived by applying the operator $\widehat{L}$ to equation (9) in Problem 4. We establish the FDA to the solution of Problem 5 on $\Omega = (a, b) \times (c, d)$, where $a, b, c, d \in \mathbb{R}$. Application of central FDA to Problem 5 yields the following system of linear algebraic equations:

$$\begin{aligned}
&(-k_1 + k_2)\, \tilde{y}_{i-1,j-1} + (2k_1 - k_4 + k_6)\, \tilde{y}_{i,j-1} + (-k_1 - k_2)\, \tilde{y}_{i+1,j-1} \\
&+ (-2k_2 + k_3 - k_5)\, \tilde{y}_{i-1,j} + (-2k_3 + 2k_4)\, \tilde{y}_{i,j} + (2k_2 + k_3 + k_5)\, \tilde{y}_{i+1,j} \\
&+ (k_1 + k_2)\, \tilde{y}_{i-1,j+1} + (-2k_1 - k_4 - k_6)\, \tilde{y}_{i,j+1} + (k_1 - k_2)\, \tilde{y}_{i+1,j+1} \\
&\quad = \mathcal{F}_{i,j}, \;\; i = 1, ..., I,\; j = 1, ..., J,
\end{aligned} \tag{24}$$

where

$$k_1 = \frac{h_{i,j+1} - h_{i,j-1}}{4\left(\Delta x\right)^2\left(\Delta p\right)^2},\ k_2 = \frac{h_{i+1,j} - h_{i-1,j}}{4\left(\Delta x\right)^2\left(\Delta p\right)^2},$$
$$k_3 = \frac{h_{i,j+1} - 2h_{i,j} + h_{i,j-1}}{\left(\Delta x\right)^2\left(\Delta p\right)^2},\ k_4 = \frac{h_{i+1,j} - 2h_{i,j} + h_{i-1,j}}{\left(\Delta x\right)^2\left(\Delta p\right)^2},$$
$$k_5 = \frac{h_{i+1,j+1} - 2h_{i+1,j} + h_{i+1,j-1} - h_{i-1,j+1} + 2h_{i-1,j} - h_{i-1,j-1}}{4\left(\Delta x\right)^2\left(\Delta p\right)^2},$$
$$k_6 = \frac{h_{i+1,j+1} - 2h_{i,j+1} + h_{i-1,j+1} - h_{i+1,j-1} + 2h_{i,j-1} - h_{i-1,j-1}}{4\left(\Delta x\right)^2\left(\Delta p\right)^2}.$$

In the above equations, I, J are positive integers, $\Delta x = \frac{(b-a)}{(I+1)}$ and $\Delta p = \frac{(d-c)}{(J+1)}$ are step sizes in the directions x, p, respectively. The notation $\tilde{y}_{i,j}$ denotes the finite difference approximation to the solution $y(x_i, p_j) = y(a + i\Delta x, c + j\Delta p)$ and $h_{i,j}$ is the approximation to $H(x_i, p_j) = H(a + i\Delta x, c + j\Delta p)$. The condition $y|_{\partial\Omega} = 0$ in Problem 5 is discretized as

$$\tilde{y}_{0,j} = \tilde{y}_{I+1,j} = \tilde{y}_{i,0} = \tilde{y}_{i,J+1} = 0,\ (i = 0, 1, ..., I+1,\ j = 0, 1, ..., J+1).$$

System (24) can be written in the matrix form as follows

$$\mathbf{T}\ \widetilde{\mathbf{y}} = \mathbf{b}. \tag{25}$$

Here $\mathbf{T}$ is a block tridiagonal matrix of order $I \times J$ with tridiagonal blocks of order I and it consists of the coefficients of system (24), $\widetilde{\mathbf{y}}$ is the column matrix:

$$\widetilde{\mathbf{y}} = \left[\tilde{y}_{1,1}, \tilde{y}_{2,1},, ..., \tilde{y}_{I,1}, \tilde{y}_{1,2}, \tilde{y}_{2,2}, ..., \tilde{y}_{I,2}, ..., \tilde{y}_{1,J}, \tilde{y}_{2,J}, ..., \tilde{y}_{I,J}\right]^T,$$

and $\mathbf{b}$ is the column matrix, which consists of the values $\mathcal{F}_{i,j}$. By solving the matrix equation (25), we obtain the approximate values $\tilde{y}_{i,j}$ at $I \times J$ mesh points of Ω. Finally, by taking into account the relation $\ln u = y$ and the reduction of Problem 3 to Problem 4, the approximate values of u can be easily obtained by setting $\tilde{u}_{i,j} = \exp(\tilde{y}_{i,j} + \Phi_{i,j})$.

Numerical solution for σ can be obtained using the approximate values $\tilde{y}_{i,j}$ from the difference equation

$$\Delta x \Delta p\left[k_1 \tilde{y}_{i+1,j} - k_1 \tilde{y}_{i-1,j} - k_2 \tilde{y}_{i,j+1} + k_2 \tilde{y}_{i,j-1}\right] = \tilde{\sigma}_{i,j}, \tag{26}$$

which is a discrete form of equation (2) for $n = 1$, $i = 1, 2, ..., I$, $j = 1, 2, ..., J$. Here $\tilde{\sigma}_{i,j}$ is the finite difference approximation to the unknown coefficient $\sigma(x_i, p_j) = \sigma(a + i\Delta x, c + j\Delta p)$.

5.2. The symbolic computation approach

The approximate analytical solution of Problem 5 will be sought in the following form:

$$Y_N = \sum_{i,j=0}^{N-1} \alpha_{i,j} w_{i,j} \zeta\left(x\right) \xi\left(p\right), \tag{27}$$

where the functions $\zeta(x)$, $\xi(p)$ are selected such that they vanish on the boundary and outside of the corresponding domains. In (27), $w_{i,j}=x^i p^j$ and $\{x^i\}_{i=0}^{\infty}$, $\{p^j\}_{j=0}^{\infty}$ are complete systems in $L_2(D)$ and $L_2(G)$, respectively. The unknown coefficients $\alpha_{i,j}$ $(i,j=0,...,N-1)$ are determined from the following system of linear algebraic equations:

$$\sum_{i,j=0}^{N-1}\left(A\left(\alpha_{i,j}w_{i,j}\right)\zeta(x)\,\xi(p)\,,w_{i',j'}\zeta(x)\,\xi(p)\right)_{L_2(\Omega)}=\left(\mathcal{F},w_{i',j'}\zeta(x)\,\xi(p)\right)_{L_2(\Omega)},$$

where $i',j'=0,...,N-1$. Finally, we obtain the approximate u by setting $U_N=\exp(Y_N+\Phi)$, where Φ is the function used in the reduction of Problem 3 to Problem 4. The unknown coefficient σ can be computed approximately from (9) with the help of Y_N, [4].

6. Numerical experiments

The proposed methods have been implemented and evaluated on various IPs. Two examples are presented below. We test the robustness of the FDA by using noisy data in the experiments. For this aim, we added multiplicative random noise to the exact boundary data u_b as follows:

$$u_{noisy}(x_i,p_j)=u_b(x_i,p_j)\left[1+\frac{\alpha\left(u_{\max}-u_{\min}\right)\gamma}{100}\right].$$

Here, (x_i,p_j) is a mesh point at the boundary $\partial\Omega$, α is a random number in the interval $[-1;1]$, $u_{\max}$ and $u_{\min}$ are maximal and minimal values of the boundary data u_b, respectively, and γ is the noise level in percents. We compared the result of FDA with the exact solution of the problem and with the result obtained by the SCA which was developed by the authors in [4]. The maximum absolute percentage error δ is calculated as follows:

$$\delta=\frac{|u_{\text{exact}}-u_{\text{approximate}}|}{|u_{\text{exact}}|}\times 100\%.$$

Example 1. *Let us consider the problem of finding (u,σ) in $\Omega=(-1,1)\times(1,2)$ from the equation*

$$H_p(x,p)\,u_x(x,p)-H_x(x,p)\,u_p(x,p)-\sigma(x,p)\,u=0, \tag{28}$$

provided that $H(x,p)=x-\ln(p)$, $\widehat{L}\sigma=0$, and the boundary conditions

$$\begin{aligned}
u(-1,p)&=\exp(-p+\frac{p}{8(4+p^2)}+\frac{arc\tan(p/2)}{16}),\\
u(1,p)&=\exp(p+\frac{p}{8(4+p^2)}+\frac{arc\tan(p/2)}{16}),\\
u(x,1)&=\exp(x^3+\frac{1}{40}+\frac{arc\tan(1/2)}{16}),\\
u(x,2)&=\exp(2x^3+\frac{1}{32}+\frac{arc\tan(1)}{16}),
\end{aligned}$$

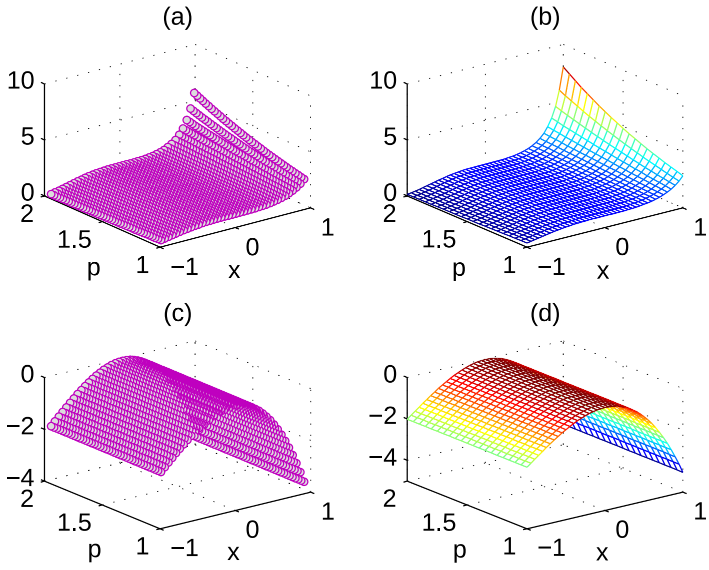

Figure 1: *(a) Computed u, (b) Exact u, (c) Computed σ, (d) Exact σ*

are given. The exact solution pair of the problem is $u(x,p) = \exp(x^3 p + \dfrac{p}{8(4+p^2)} + \dfrac{arc\tan(p/2)}{16})$, $\sigma(x,p) = -3x^2 - x^3 - \dfrac{1}{(4+p^2)^2}$.

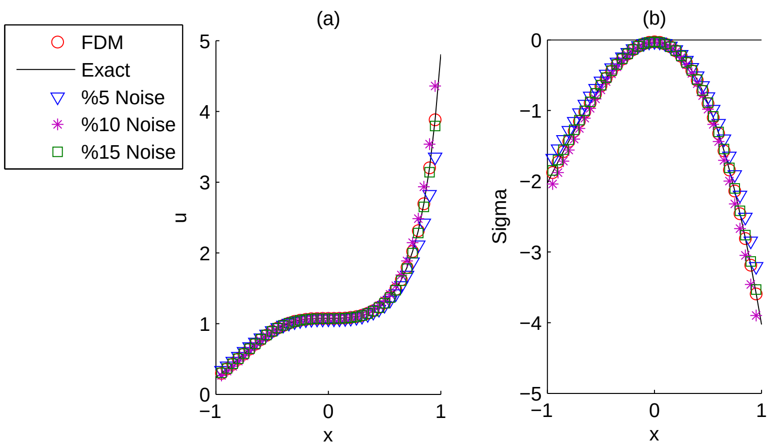

Figure 2: *The exact solution and numerical solution for different noise levels*

In Figure 1 above, the exact solution and a finite difference solution of the problem are given for $I = 24$, $J = 199$. Figure 2 displays one-dimensional cross sections ($p = 1.5$) of computed approximate solutions with different noise levels superimposed with the exact solution of the IP. In Figure 3, we present the results obtained from SCA for the approximate analytical solution of the same problem.

The maximum absolute percentage error in the finite difference solution is $\delta = 0.4 \times 10^{-4}\%$ and in symbolic computation $\delta = 0.6 \times 10^{-14}\%$.

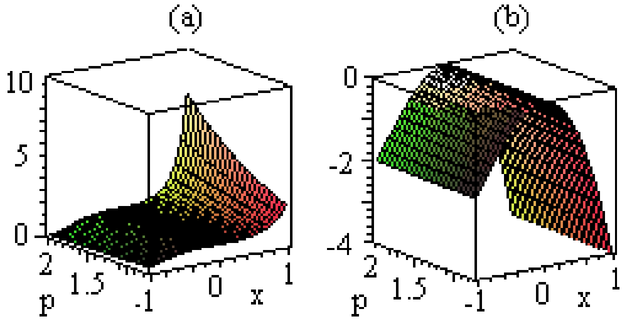

Figure 3: *A symbolic computational approach: (a) for u, (b) for σ*

Example 2. *Determine a pair of functions (u, σ) defined in $\Omega = (-1,1) \times (-1,1)$ that satisfies equation (28), $\widehat{L}\sigma = 0$ and the boundary conditions*

$$u(-1,p) = \exp((p-1)^3 e^{5p}),\ u(1,p) = \exp(4 + (p-1)^3 e^{5p}),$$
$$u(x,-1) = \exp(2(x+1) - 8e^{-5}),\ u(x,1) = \exp(2(x+1)),$$

where $H(x,p) = -x + p^2$ is given. The exact solution of the problem is $u(x,p) = \exp(2(x+1) + (p-1)^3 e^{5p})$, $\sigma(x,p) = 4p + (3(p-1)^2 + 5(p-1)^3)e^{5p}$.

In Figure 4, the numerical solution and the exact solution of the problem are shown for $I = J = 39$.

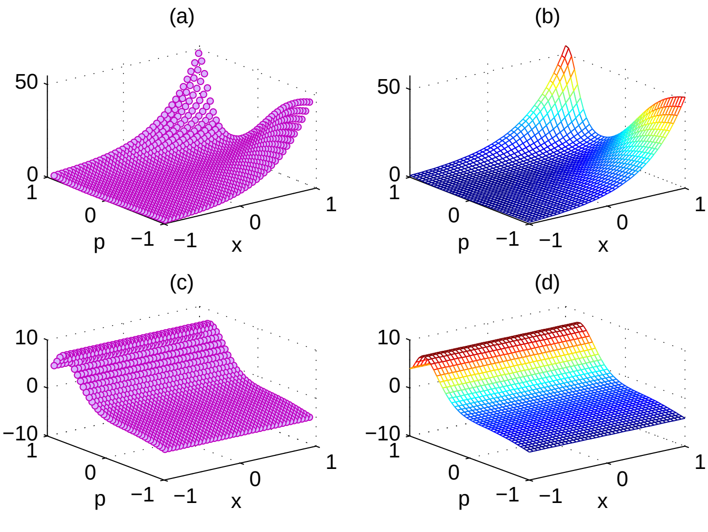

Figure 4: *(a) Computed u, (b) Exact u, (c) Computed σ, (d) Exact σ*

In Figure 5, a comparison between the exact solution and the approximate solution of the inverse problem for different noise levels is presented by one-dimensional cross sections ($p = 0$).

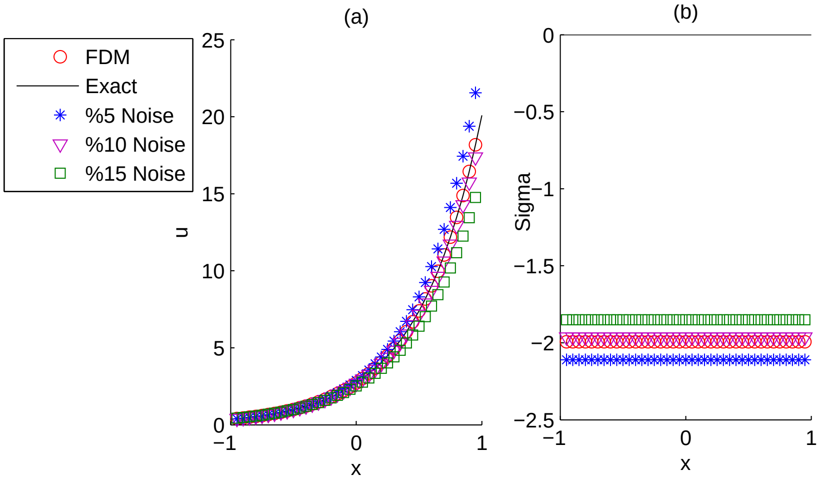

Figure 5: *The exact solution and finite difference solutions with different noise levels*

Figure 6 shows the results obtained from SCA for the approximate analytical solution of the problem.

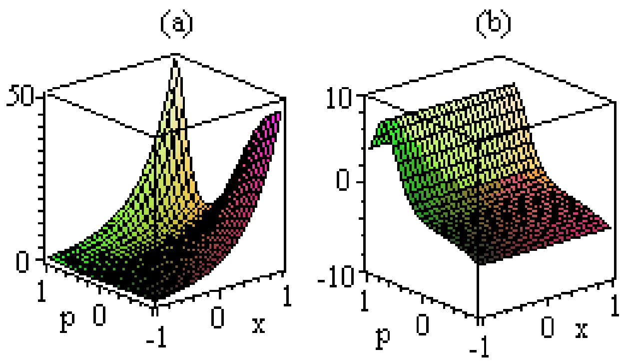

Figure 6: *Approximate analytical solution of the problem using SCA: (a) for u, (b) for σ*

Maximum absolute percentage error of the FDA is $\delta = 0.1 \times 10^{-10}\%$ and in symbolic computation the error is $\delta = 0.9 \times 10^{-14}\%$. In computations, matrix equation (25) is solved using a Matlab program and symbolic computations are performed using a Maple program on a PC with Intel Core 2 T7200 2 GHz. Computational experiments show that proposed methods provide highly accurate numerical solutions and they are robust against the data noises.

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