GLASNIK MATEMATIČKI
Vol. 46(66)(2011), 103 – 120

FINITE p-GROUPS G WITH $p > 2$ AND $\mathrm{d}(G) > 2$ HAVING EXACTLY ONE MAXIMAL SUBGROUP WHICH IS NEITHER ABELIAN NOR MINIMAL NONABELIAN

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Abstract. We give here a complete classification (up to isomorphism) of the title groups (Theorems 1, 3 and 5). The corresponding problem for $p = 2$ was solved in [4] and for $p > 2$ with $\mathrm{d}(G) = 2$ was solved in [5]. This gives a complete solution of the problem Nr. 861 of Y. Berkovich stated in [2].

Here we determine up to isomorphism the title groups (Theorems 1, 3 and 5). It is obvious that for such groups G we have $\mathrm{d}(G) = 3$. All resulting groups will be presented in terms of generators and relations. But we shall also state all important characteristic subgroups of these groups so that the results could be useful for applications. The corresponding problem for $p = 2$ was solved in [4] and for $p > 2$ with $\mathrm{d}(G) = 2$ was solved in [5].

Our notation is standard (see [1] and [2]). In particular, $\mathrm{S}(p^3)$ denotes for $p > 2$ the nonabelian group of order p^3 and exponent p and an L_3-group is a p-group G in which $\Omega_1(G)$ is of order p^3 and exponent p and $G/\Omega_1(G)$ is cyclic of order $> p$.

In addition to known results which have been stated at the beginning of our previous paper [5], we need also the following known results which are quoted in the proof of our theorems. Moreover, if these results are quoted from the unpublished book [3], then we also give a proof.

Theorem 7.2 in [1]. If G is a regular p-group, then $\exp(\Omega_n(G)) \le p^n$ and $|\Omega_n(G)| = |G : \mho_n(G)|$.

2010 *Mathematics Subject Classification.* 20D15.
Key words and phrases. Minimal nonabelian p-groups, A_2-groups, metacyclic p-groups, Frattini subgroups, Hall-Petrescu formula, generators and relations, congruences mod p.

PROPOSITION 26.10 IN [1]. If G is a powerful p-group, $p > 2$, then each element in $\mho_1(G)$ is a p-th power.

PROPOSITION 71.1 IN [2]. Suppose that G is an A_2-group of order $> p^4$. Then $|G'| = p$ if and only if G has at least two distinct abelian maximal subgroups and in that case one of the following holds:

(a) $G = H \times \mathrm{C}_p$, where H is minimal nonabelian.

(b) $G = \langle a, b, c \rangle$, where $a^{p^m} = b^{p^n} = c^{p^2} = 1$, $m \geq n \geq 1$, $m \geq 2$, $[a, b] = d$, $c^p = d$, $[a, d] = [b, d] = [a, c] = [b, c] = 1$. Here $H = \langle a, b \rangle$ is non-metacyclic minimal nonabelian, $H' = G' = \langle d \rangle$, $G = H * \langle c \rangle$ with $H \cap \langle c \rangle = \langle d \rangle = \langle c^p \rangle$.

PROPOSITION 71.4(B) IN [2]. Suppose that G is a non-metacyclic A_2-group of order $> p^4$ possessing exactly one abelian maximal subgroup. Then $G' \cong \mathrm{E}_{p^2}$ and assume in addition that $G' \leq \mathrm{Z}(G)$. In that case $\mathrm{d}(G) = 3$, $\mathrm{Z}(G) = \Phi(G)$, and if G has a normal elementary abelian subgroup E of order p^3, then $E = \Omega_1(G)$.

EXERCISE P1 IN [3]. If a nonabelian p-group G has two distinct abelian maximal subgroups A_1, A_2, then $|G'| = p$.

PROOF. Since $A_1 \cap A_2 \leq \mathrm{Z}(G)$, we get $|G : \mathrm{Z}(G)| = p^2$. Then Lemma 1.1 in [1] implies $|G'| = p$. $\square$

EXERCISE P10 IN [3]. Let G be a p-group with $|G'| = p$. If H is a minimal nonabelian subgroup of G, then $G = H\mathrm{C}_G(H)$.

PROOF. We have $H = \langle a, b \rangle$ for some $a, b \in G$ and $H' = G'$ so that H is normal in G. Also, $\mathrm{C}_G(H) = \mathrm{C}_G(a) \cap \mathrm{C}_G(b) \trianglelefteq G$ so that $|G : \mathrm{C}_G(H)| \leq p^2$. But $H \cap \mathrm{C}_G(H) = \mathrm{Z}(H)$ and $H/\mathrm{Z}(H) \cong \mathrm{E}_{p^2}$ and so $G = H\mathrm{C}_G(H)$. $\square$

EXERCISE P11 IN [3]. All $p^2 + p + 1$ subgroups of order p^2 in an elementary abelian group $E = \langle a, b, c \rangle$ of order p^3 are: $\langle a, b \rangle$, $\langle a, b^i c \rangle$ ($p + 1$ subgroups containing $\langle a \rangle$), and $\langle a^j b, a^k c \rangle$ (p^2 subgroups not containing $\langle a \rangle$), where i, j, k are any integers modulo p.

PROOF. Proof is trivial. $\square$

We turn now to a proof of our theorems.

THEOREM 1. *Let G be a p-group, $p > 2$, which possesses exactly one maximal subgroup which is neither abelian nor minimal nonabelian. Suppose that $\mathrm{d}(G) = 3$ and G has more than one abelian maximal subgroup. Then we have one of the following possibilities:*

(a) *$G = U * Z$, where $U \cong \mathrm{S}(p^3)$, $Z \cong \mathrm{C}_{p^m}$, $m \geq 3$, and $U \cap Z = \mathrm{Z}(U)$. Here G is an L_3-group.*

(b) *$G = U \times Z$, where $U \cong \mathrm{S}(p^3)$ or $U \cong \mathrm{M}_{p^3}$ and $Z \cong \mathrm{C}_{p^m}$, $m \geq 2$.*

Conversely, groups in (a) and (b) satisfy the assumptions of our theorem.

PROOF. Our assumptions imply that $G/\mathrm{Z}(G) \cong \mathrm{E}_{p^2}$ and G has exactly $p+1$ abelian maximal subgroups (Exercise 1.6(a) in [1]). By Lemma 1.1 in [1], $|G| = p|G'||\mathrm{Z}(G)|$ gives $|G'| = p$.

Conversely, assume that G is a title group with $|G'| = p$. Since G has at most $p+1$ abelian maximal subgroups, there is a minimal nonabelian maximal subgroup H. From $|G'| = p$ follows $G = H\mathrm{C}_G(H)$, where $H \cap \mathrm{C}_G(H) = \mathrm{Z}(H) = \Phi(H) = \Phi(G)$ and $\mathrm{C}_G(H) = \mathrm{Z}(G)$. We have $G/\mathrm{Z}(G) \cong \mathrm{E}_{p^2}$ and so all $p+1$ maximal subgroups of G containing $\mathrm{Z}(G)$ are abelian.

In what follows H will denote a fixed maximal subgroup of G which is minimal nonabelian. Suppose that there is an element $c \in \mathrm{Z}(G) - H$ of order p. Then $G = H \times \langle c\rangle$ and so each maximal subgroup of G which does not contain $\langle c\rangle$ is isomorphic to $G/\langle c\rangle \cong H$ and so is minimal nonabelian, a contradiction. Hence there are no elements of order p in $\mathrm{Z}(G) - H$ which implies $\Omega_1(\mathrm{Z}(H)) = \Omega_1(\mathrm{Z}(G))$ so that $\mathrm{d}(\mathrm{Z}(H)) = \mathrm{d}(\mathrm{Z}(G))$. It follows that for each $x \in \mathrm{Z}(G) - H$, $x^p \in \mathrm{Z}(H) - \Phi(\mathrm{Z}(H))$. Obviously, $|G| \geq p^5$ since the exceptional maximal subgroup M (which is neither abelian nor minimal nonabelian) is of order $\geq p^4$.

(i) First assume that H is metacyclic. We may set:

$$\langle a, b \mid a^{p^m} = b^{p^n} = 1,\ a^b = az,\ z = a^{p^{m-1}}\rangle,$$

where $m \geq 2$, $n \geq 1$, $m+n \geq 4$, $H' = \langle z\rangle = G'$, $|H| = p^{m+n}$, and $|G| = p^{m+n+1}$. We have $\mathrm{Z}(H) = \langle a^p\rangle \times \langle b^p\rangle = \Phi(H) = \Phi(G)$ and for each $x \in \mathrm{Z}(G) - H$, $x^p \in \langle a^p, b^p\rangle - \langle a^{p^2}, b^{p^2}\rangle$ since $\langle a^{p^2}, b^{p^2}\rangle = \Phi(\mathrm{Z}(H))$.

Suppose that $n = 1$ so that $\mathrm{o}(b) = p$, $m \geq 3$, $H \cong \mathrm{M}_{p^{m+1}}$ and $\mathrm{Z}(H) = \langle a^p\rangle$. Here $\mathrm{Z}(G) \cong \mathrm{C}_{p^m}$ is cyclic and so there is $c \in \mathrm{Z}(G) - H$ such that $c^p = a^{-p}$ which gives $(ca)^p = c^p a^p = 1$ and $\mathrm{o}(ca) = p$. Since $[ca, b] = z$, we get $U = \langle ca, b\rangle \cong \mathrm{S}(p^3)$, $U \cap \mathrm{Z}(G) = \langle z\rangle$ which together with $|G : \mathrm{Z}(G)| = p^2$ gives $G = U\mathrm{Z}(G)$. We have obtained the groups stated in part (a) of our theorem. Here $M = U * \langle c^p\rangle$, all $p+1$ maximal subgroups of G containing $\langle c\rangle$ are abelian and we have to show that all subgroups $\langle c^i a, c^j b\rangle$ are minimal nonabelian maximal subgroups of G for all integers i, j mod p unless $i \equiv 1 \pmod p$ and $j \equiv 0 \pmod p$. Indeed, $[c^i a, c^j b] = [a, b] = z$ and so, by Exercise P9 in [3], $\langle c^i a, c^j b\rangle$ is minimal nonabelian and

$$\begin{aligned}\Phi(\langle c^i a, c^j b\rangle) &= \langle (c^i a)^p = c^{pi} a^p = a^{-pi} a^p = a^{p(-i+1)},\ (c^j b)^p = a^{-pj},\ z\rangle \\ &= \langle a^p\rangle = \Phi(G)\end{aligned}$$

if and only if either $i \not\equiv 1 \pmod p$ or $j \not\equiv 0 \pmod p$.

It remains to treat the case $n \geq 2$. Suppose in addition that there is an element $x \in G - H$ of order p. We know that $x \notin \mathrm{Z}(G)$ and so $[a, x] \neq 1$ or $[b, x] \neq 1$. Obviously, $\langle a, b, x\rangle = G$.

First suppose $[a,x] \neq 1$. Since $\langle a \rangle \trianglelefteq G$, $\langle a,x \rangle \cong \mathrm{M}_{p^{m+1}}$ is minimal nonabelian of order p^{m+1}. But $|G| = p^{m+n+1}$ and $n \geq 2$ which implies that $M = \Phi(G)\langle a,x \rangle = \langle a,x \rangle \times \langle b^p \rangle$ is a maximal subgroup of G which is neither abelian nor minimal nonabelian. Assume at the moment that also $[x,b] \neq 1$. In that case $\langle b \rangle \times \langle z \rangle \trianglelefteq G$ and so $\langle b,x \rangle$ is non-metacyclic minimal nonabelian of order p^{n+2}. It follows that $\langle b,x \rangle$ must be a maximal subgroup of G with $|G| = p^{n+3}$ (and so $m = 2$). But this case will be studied in part (ii) of this proof (where G possesses a non-metacyclic minimal nonabelian maximal subgroup). Hence we may assume $[x,b] = 1$. This gives $[x,ab] = [x,a] = z^r$, $r \not\equiv 0 \pmod p$ and so $\langle x,ab \rangle$ is minimal nonabelian which forces that $\langle x,ab \rangle$ must be a maximal subgroup of G. Now, $\langle ab \rangle$ covers $H/\langle a \rangle \cong \mathrm{C}_{p^n}$ and so $\mathrm{o}(ab) \geq p^n$, $n \geq 2$. We have $(ab)^{p^n} = a^{p^n} b^{p^n} = a^{p^n}$. If $n \geq m$, then $\mathrm{o}(ab) = p^n$ and $\langle ab \rangle \cap \langle a \rangle = \{1\}$. Since $\langle ab,z \rangle \trianglelefteq G$, we see that $\langle ab,x \rangle$ is a non-metacyclic minimal nonabelian subgroup of order p^{n+2}. In that case $\langle ab,x \rangle$ must be a maximal subgroup of G (with $m = 2$) and again this will be studied in part (ii) of the proof. It follows that we may assume $n < m$. In that case $\mathrm{o}(ab) = p^m$ and $\langle ab \rangle \geq \langle z \rangle$ so that $\langle ab \rangle \trianglelefteq G$. Hence $\langle ab,x \rangle$ is metacyclic minimal nonabelian of order p^{m+1} and so $\langle ab,x \rangle$ must be a maximal subgroup of G. From $|G| = p^{m+n+1}$ follows $n = 1$, contrary to our assumption.

We may assume $[a,x] = 1$ and then $[b,x] \neq 1$. Since $\langle b \rangle \times \langle z \rangle \trianglelefteq G$, $\langle b,x \rangle$ is non-metacyclic minimal nonabelian of order p^{n+2}. If $\langle b,x \rangle$ is a maximal subgroup of G, then this case will be treated in part (ii) of this proof. Thus we may assume that $\langle b,x \rangle$ is not a maximal subgroup of G and so $M = \Phi(G)\langle b,x \rangle$ with $m > 2$. It follows that $\langle ab,x \rangle$ being minimal nonabelian must be a maximal subgroup of G. Since $\langle ab \rangle$ covers $H/\langle a \rangle$, $\mathrm{o}(ab) \geq p^n$, $n \geq 2$, and $(ab)^{p^n} = a^{p^n}$. If $n \geq m$, then $\mathrm{o}(ab) = p^n$ and $\langle ab \rangle \cap \langle a \rangle = \{1\}$ and so $\langle ab,x \rangle$ is non-metacyclic minimal nonabelian of order p^{n+2}. In that case $\langle ab,x \rangle$ must be a maximal subgroup of G with $m = 2$, a contradiction. We may assume $n < m$ and then $\mathrm{o}(ab) = p^m$ with $\langle ab \rangle \geq \langle z \rangle$ and so $\langle ab,x \rangle$ is metacyclic minimal nonabelian of order p^{m+1}. But then $|G| = p^{m+n+1}$ implies $n = 1$, contrary to our assumption.

We have proved that we may assume that there are no elements of order p in $G - H$. We know that for an element $c^{-1} \in \mathrm{Z}(G) - H$, $c^{-p} \in \mathrm{Z}(H) - \Phi(\mathrm{Z}(H))$ and so c^{-p} is not a p-th power of any element in $\mathrm{Z}(H)$. But $\mho_1(H) = \mathrm{Z}(H) \geq H' = \langle z \rangle$ and so H is a powerful group. Then $c^{-p} = h^p$ for some element $h \in H - \mathrm{Z}(H)$ (see Proposition 26.10 in [1]). It follows $hc \in G - H$ and $(hc)^p = h^p c^p = 1$ and so hc is of order p, a contradiction.

(ii) It remains to consider the case where H is non-metacyclic minimal nonabelian. We may set:

$$\langle a, b \mid a^{p^m} = b^{p^n} = 1,\ [a,b] = z,\ z^p = [a,z] = [b,z] = 1 \rangle,$$

where we may assume $m \geq 2$, $n \geq 1$ since $|G| \geq p^5$. Here $H' = \langle z \rangle$, $|H| = p^{m+n+1}$, and $|G| = p^{m+n+2}$. Also, $\langle z \rangle$ is a maximal cyclic subgroup in H,

$\mathrm{Z}(H) = \langle a^p \rangle \times \langle b^p \rangle \times \langle z \rangle = \Phi(H) = \Phi(G)$ and for each $x \in \mathrm{Z}(G) - H$, $x^p \in \mathrm{Z}(H) - \Phi(\mathrm{Z}(H))$.

(ii1) First assume $n = 1$ so that $\mathrm{Z}(H) = \langle a^p \rangle \times \langle z \rangle$ and for an element $c \in \mathrm{Z}(G) - H$, $c^p = a^{pr} z^s$, where both integers r, s are not divisible by p. Suppose that $r \equiv 0 \pmod p$ and then $s \not\equiv 0 \pmod p$. Replacing c with another suitable generator of $\langle c \rangle$, we may assume that $c^p = a^{p^2 r'} z$ for some integer r'. Take the element $c' = a^{-pr'} c \in \mathrm{Z}(G) - H$ and then we get $(c')^p = a^{-p^2 r'} c^p = z$. Thus $G = H * \langle c' \rangle$ with $(c')^p = z$ and $\langle z \rangle = H'$ and so we have obtained a p-group of Proposition 71.1(ii) in [2] which is an A_2-group, a contradiction. We have proved that $c^p = a^{pr} z^s$ with $r \not\equiv 0 \pmod p$. Set $a' = a^{-r}$ so that $\mathrm{o}(a') = p^m$, $[a', b] = z^{-r}$ and $(ca')^p = c^p (a')^p = z^s$, where $\langle a', b \rangle = H$. Consider the subgroup $U = \langle ca', b \rangle$. Since $[ca', b] = z^{-r}$ and $\mathrm{o}(b) = p$, we have in case $s \not\equiv 0 \pmod p$ that $U \cong \mathrm{M}_{p^3}$ and in case $s \equiv 0$ (mod p), $U \cong \mathrm{S}(p^3)$. However, $c^p = (a')^{-p} z^s$, $G = \langle a', b, c \rangle$, $\mathrm{o}(c) = p^m$, $m > 1$, and $\langle c \rangle \cap U = \{1\}$ and so we get $G = U \times \langle c \rangle$ which are the groups stated in part (b) of our theorem. Here $M = U \times \langle c^p \rangle$ and all $p + 1$ maximal subgroups containing $\langle c \rangle$ are abelian.

We have to show that all subgroups $\langle c^i a', c^j b \rangle$ are minimal nonabelian maximal subgroups of G (not containing $\langle c \rangle$) for all integers $i, j \pmod p$ unless $i \equiv 1 \pmod p$ and $j \equiv 0 \pmod p$ holds. Indeed, $[c^i a', c^j b] = [a', b] = z^{-r}$ and

$$\begin{aligned}\Phi(\langle c^i a', c^j b \rangle) &= \langle c^{pi} (a')^p = a^{pri} z^{si} a^{-pr} = a^{pr(i-1)} z^{si}, c^{pj} = a^{prj} z^{sj}, z^{-r} \rangle \\ &= \Phi(G) = \Phi(H)\end{aligned}$$

if either $i \not\equiv 1 \pmod p$ or $j \not\equiv 0 \pmod p$.

(ii2) It remains to consider the case $n \geq 2$. In this case for an element $c \in \mathrm{Z}(G) - H$ we have $c^p = a^{pi} b^{pj} z^k$, where at least one of the integers i, j, k is $\not\equiv 0 \pmod p$.

First suppose that $i \equiv 0 \pmod p$ and $j \equiv 0 \pmod p$ so that $k \not\equiv 0$ (mod p). We may set $c^p = a^{p^2 i'} b^{p^2 j'} z^k$ for some integers i', j'. For the element $c' = a^{-pi'} b^{-pj'} c \in \mathrm{Z}(G) - H$, we get

$$(c')^p = a^{-p^2 i'} b^{-p^2 j'} c^p = z^k, \; k \not\equiv 0 \pmod p,$$

and so $G = H * \langle c' \rangle$ with $\langle c' \rangle \cap H = \langle z \rangle = H'$ and this is an A_2-group of Proposition 71.1(ii) in [2], a contradiction.

Now assume that one of the integers i, j is $\equiv 0 \pmod p$ and the other one is $\not\equiv 0 \pmod p$. Because of the symmetry, we may assume $i \not\equiv 0 \pmod p$ and $j \equiv 0 \pmod p$. We have

$$\Phi(\langle a^{-i} b^{-j} c, b \rangle) = \langle a^{-pi} b^{-pj} c^p = z^k, \, b^p, \, [a^{-i} b^{-j} c, b] = z^{-i} \neq 1 \rangle < \Phi(G)$$

and

$$\Phi(\langle ab, a^r c \rangle) = \langle a^p b^p, \, a^{pr} c^p = a^{p(i+r)} b^{pj} z^k, \, [ab, a^r c] = z^{-r} \neq 1 \rangle < \Phi(G)$$

for some suitable $r \not\equiv 0 \pmod p$ such that $i + r \equiv 0 \pmod p$. Hence $\Phi(G)\langle a^{-i}b^{-j}c, b\rangle$ and $\Phi(G)\langle ab, a^r c\rangle$ are two distinct maximal subgroups of G which are neither abelian nor minimal nonabelian, a contradiction.

Finally, we consider the case, where both i and j are $\not\equiv 0 \pmod p$. We have

$$\Phi(\langle a^{-i}b^{-j}c, b\rangle) = \langle a^{-pi}b^{-pj}c^p = z^k,\ b^p,\ [a^{-i}b^{-j}c, b] = z^{-i} \neq 1\rangle < \Phi(G)$$

and

$$\Phi(\langle a, b^r c\rangle) = \langle a^p,\ b^{pr}c^p = a^{pi}b^{p(j+r)}z^k,\ [a, b^r c] = z^r \neq 1\rangle < \Phi(G)$$

for some suitable $r \not\equiv 0 \pmod p$ such that $j + r \equiv 0 \pmod p$. Hence $\Phi(G)\langle a^{-i}b^{-j}c, b\rangle$ and $\Phi(G)\langle a, b^r c\rangle$ are two distinct maximal subgroups of G which are neither abelian nor minimal nonabelian, a contradiction. Our theorem is proved. □

Lemma 2. *Let G be a p-group, $p > 2$, which possesses exactly one maximal subgroup M which is neither abelian nor minimal nonabelian. Suppose that $\mathrm{d}(G) = 3$ and G has at most one abelian maximal subgroup. Then $\Phi(G) = \mathrm{Z}(G) = \Phi(H)$ for each minimal nonabelian maximal subgroup H of G. Also, $|G'| > p$, $|M'| = p$ and $\mathrm{d}(M) \geq 3$ which implies $|G| \geq p^5$.*

Proof. By the first two paragraphs of the proof of Theorem 1, we have $|G : \mathrm{Z}(G)| \geq p^3$ and $|G'| > p$. Let H be a maximal subgroup of G which is minimal nonabelian. Then $\Phi(H) = \mathrm{Z}(H) \leq \Phi(G)$ and $|H : \Phi(H)| = p^2$ which gives $\Phi(H) = \Phi(G)$. Let $K \neq H$ be another maximal subgroup of G which is minimal nonabelian. Then $\mathrm{Z}(K) = \Phi(K) = \Phi(G)$ which implies $\mathrm{C}_G(\Phi(G)) \geq \langle H, K\rangle = G$ and so $\Phi(G) = \mathrm{Z}(G)$. Since $|M : \Phi(G)| = p^2$ and $\Phi(G) = \mathrm{Z}(G)$, we have $M = S * \Phi(G)$, where S is minimal nonabelian and $S \cap \Phi(G) = \Phi(S) < \Phi(G) = \mathrm{Z}(M)$. This implies $M' = S' \cong \mathrm{C}_p$ and $\mathrm{d}(M) \geq 3$. □

Theorem 3. *Let G be a p-group, $p > 2$, with $\mathrm{d}(G) = 3$ which has exactly one maximal subgroup M which is neither abelian nor minimal nonabelian. Suppose that G has exactly one abelian maximal subgroup A. Then we have $\Phi(G) = \mathrm{Z}(G)$, $G' \cong \mathrm{E}_{p^2}$, $|M'| = p$, $\mathrm{d}(M) \geq 3$, $\Omega_1(G) = E \cong \mathrm{E}_{p^3}$, and $E \not\leq \Phi(G)$.*

If $E \not\leq A$, then we have the following possibilities:

(a) *$G = \langle a, b, t \mid a^{p^{m+1}} = b^{p^2} = t^p = 1,\ [b, t] = z,\ a^{p^m} = z^n,\ b^p = u,\ [t, a] = u,\ [u, a] = [u, t] = [a, b] = [z, t] = 1\rangle$, where $m \geq 2$ and $n \not\equiv 0 \pmod p$. We have $|G| = p^{m+4}$, $G' = \langle u, z\rangle \cong \mathrm{E}_{p^2}$, $\Phi(G) = \mathrm{Z}(G) = \langle a^p, u\rangle \cong \mathrm{C}_{p^m} \times \mathrm{C}_p$, $\Omega_1(G) = E = \langle u, z, t\rangle \cong \mathrm{E}_{p^3}$, $A = \langle a, b\rangle$ is abelian of type (p^{m+1}, p^2), $M = \langle b, t\rangle * \langle a^p\rangle$, where $\langle b, t\rangle$ is a non-metacyclic minimal nonabelian group of order p^4 and all other p^2+p-1 maximal subgroups of G are minimal nonabelian.*

(b) $G = \langle a, b, t \mid a^{p^2} = b^{p^2} = t^p = 1,\ a^p = z,\ u = [t, a],\ [b, t] = u^l z,\ b^p = u^n z^s,\ [u, a] = [u, t] = [a, b] = [z, t] = 1\rangle$, *where* l, n, s *are integers mod* p, $n \not\equiv 0$ *(mod* p*),* n *is a square (in GF(p)) and* $(l + s)^2 \equiv 4n$ *(mod* p*). The group* G *is a special group of order* p^5 *with* $G' = \langle u, z\rangle \cong \mathrm{E}_{p^2}$ *and* $\Omega_1(G) = E = \langle u, z, t\rangle \cong \mathrm{E}_{p^3}$*. Also,* $A = \langle a, b\rangle \cong \mathrm{C}_{p^2} \times \mathrm{C}_{p^2}$, $M = \langle a^j b, t\rangle G'$, *where* $j \equiv (1/2)(l - s)$ *(mod* p*) and all other maximal subgroups of* G *are minimal nonabelian.*

If $E \leq A$*, then we have:*

(c) $G = \langle a, b, d \mid a^{p^\alpha} = b^p = d^{p^2} = 1,\ a^{p^{\alpha-1}} = z,\ c = [a, b],\ z = [d, a],\ [d, b] = c^s,\ d^p = c^n z^r,\ c^p = [c, d] = [c, a] = [c, b] = 1\rangle$, *where* $\alpha \geq 2$, n, r, s *are integers mod* p, $n \not\equiv 0$ *(mod* p*). We have* $|G| = p^{\alpha+3}$, $G' = \langle z, c\rangle \cong \mathrm{E}_{p^2}$, $\Phi(G) = \mathrm{Z}(G) = \langle a^p, c\rangle \cong \mathrm{C}_{p^{\alpha-1}} \times \mathrm{C}_p$, *and* $A = \Phi(G)\langle b, a^{-s}d\rangle$.

If $\alpha \geq 3$*, then* $M = \Phi(G)\langle b, d\rangle$ *with* $s \not\equiv 0$ *(mod* p*).*

If $\alpha = 2$*, then* $M = \Phi(G)\langle b, a^{-r}d\rangle$ *with* $r \not\equiv s$ *(mod* p*). All other maximal subgroups of* G *(distinct from* A *and* M*) are minimal nonabelian.*

Proof. By Lemma 2, $\Phi(G) = \mathrm{Z}(G)$ (and so G is of class 2), $|M'| = p$, $\mathrm{d}(M) \geq 3$, $|G'| > p$ and $|G| \geq p^5$. By a result of A. Mann (see Exercise 1.69(a) in [1]), $|G' : (A'H')| = |G' : H'| \leq p$, where H is a minimal nonabelian maximal subgroup of G. Hence $|G'| \leq p^2$ and so $|G'| = p^2$. If $G' = \langle v\rangle \cong \mathrm{C}_{p^2}$, then all $p^2 + p + 1$ maximal subgroups of the nonabelian group $G/\langle v^2\rangle$ are abelian, a contradiction (see Exercise 1.6(a) in [1]). Hence $G' \cong \mathrm{E}_{p^2}$. We see also that each of the $p+1$ subgroups of order p in G' is the commutator group of exactly p nonabelian maximal subgroups of G.

For each $x, y \in G$, $(xy)^p = x^p y^p [y, x]^{\binom{p}{2}} = x^p y^p$ and so G is regular. By Theorem 7.2 in [1], $\Omega_1(G)$ is of exponent p and $|\Omega_1(G)| = |G : \mho_1(G)|$. Suppose that $|\Omega_1(G)| \geq p^5$. Let H be a minimal nonabelian maximal subgroup of G. Then $|H \cap \Omega_1(G)| \geq p^4$, contrary to the structure of H. We have proved that $|\Omega_1(G)| \leq p^4$.

Suppose that G has no normal elementary abelian subgroup of order p^3. Then $|\Omega_1(A)| \leq p^2$ and so A is metacyclic. If H is any minimal nonabelian maximal subgroup of G, then $|H| \geq p^4$ and the fact that $\Omega_1(H) \cong \mathrm{E}_{p^3}$ is not possible imply that H is metacyclic. In that case M (with $\mathrm{d}(M) \geq 3$) is the only maximal subgroup of G which is not metacyclic. By Proposition A.40.12 in [3] of Berkovich, G is an L_3-group. In this case $\Omega_1(G) \cong \mathrm{S}(p^3)$ (the nonabelian group of order p^3 and exponent p) and $G/\Omega_1(G)$ is cyclic of order $\geq p^2$. But then $\mathrm{E}_{p^2} \cong G' \leq \Omega_1(G)$, contrary to the fact that $G' \leq \mathrm{Z}(G)$.

We have proved that G possesses a normal elementary abelian subgroup $E \cong \mathrm{E}_{p^3}$ of order p^3. Suppose that G has an elementary abelian subgroup F of order p^4. Since $G' \cong \mathrm{E}_{p^2}$ and $G' \leq \mathrm{Z}(G)$, we have $G' \leq F$ and so $F \trianglelefteq G$. Also, $|\Omega_1(G)| \leq p^4$ implies that $F = \Omega_1(G)$. If G/F is noncyclic, then there is a minimal nonabelian maximal subgroup K of G containing F, contrary to the structure of K. Hence G/F is cyclic of order $\geq p$. Let $a \in G - F$ be such that $\langle a \rangle$ covers G/F. Then $\mathrm{o}(a) = p^m$, $m \geq 2$, which implies that $\Omega_1(\langle a \rangle) = \langle z \rangle \leq \Phi(G) = \mathrm{Z}(G)$ and so $z \in F$. On the other hand, G/G' is abelian of rank 3 which forces $z \in G'$. Set $F = G'\langle f_1, f_2 \rangle$ for some $f_1, f_2 \in F - G'$ so that $\langle a, f_1, f_2 \rangle = G$ and $\Phi(G) = G'\langle a^p \rangle$. Since $[F, \langle a \rangle] = G'$, we may choose $f_1 \in F - G'$ so that $[f_1, a] = z$ and therefore $\langle f_1, a \rangle \cong \mathrm{M}_{p^{m+1}}$. This gives $\langle f_1, a \rangle G' \cong \mathrm{C}_p \times \mathrm{M}_{p^{m+1}}$. Since $[f_1, f_2a] = z$ and $(f_2a)^p = f_2^p a^p [a, f_2]^{\binom{p}{2}} = a^p$, it follows that $\Omega_1(\langle f_2a \rangle) = \Omega_1(\langle a \rangle) = \langle z \rangle$ and $\langle f_1, f_2a \rangle \cong \mathrm{M}_{p^{m+1}}$. Hence $\langle f_1, f_2a \rangle G' \cong \mathrm{C}_p \times \mathrm{M}_{p^{m+1}}$ is another maximal subgroup of G (distinct from $\langle f_1, a \rangle G'$) which is neither abelian nor minimal nonabelian, a contradiction. We have proved that G does not possess an elementary abelian subgroup of order p^4.

Assume that $\Omega_1(G) = S$ is a nonabelian subgroup of order p^4 and exponent p. Then we have $\mathrm{Z}(S) = G'$. If G/S is noncyclic, then there is a minimal nonabelian maximal subgroup H of G containing S which contradicts the structure of H. Hence G/S is cyclic of order $\geq p$. Let $t, t' \in S - G'$ so that $S = G'\langle t, t' \rangle$ and then $1 \neq [t', t] = z \in G'$, $\langle t, t' \rangle \cong \mathrm{S}(p^3)$ and $S \cong \mathrm{S}(p^3) \times \mathrm{C}_p$. Let M be the unique maximal subgroup of G containing S so that M is neither abelian nor minimal nonabelian and let A be the unique abelian maximal subgroup of G. Then $A > G'$ and $A \cap S \cong \mathrm{E}_{p^3}$ so that we may assume that $A \cap S = G'\langle t' \rangle$. Also, A covers G/S and so if $c \in A - M$, then $\langle c \rangle$ covers $A/(A \cap S) \cong G/S$ and $\mathrm{o}(c) \geq p^2$ which implies $\Omega_1(\langle c \rangle) \leq \Phi(G) \cap S = \mathrm{Z}(G) \cap S = \mathrm{Z}(S) = G'$. We have $\Phi(G) = G'\langle c^p \rangle$ and so $G = \langle c, t, t' \rangle$ and $[c, t'] = 1$. If $[c, t] \in \langle z \rangle$, then $G/\langle z \rangle$ is abelian, a contradiction. Hence $[c, t] = u \in G' - \langle z \rangle$ so that $G' = \langle u, z \rangle$. Other $p^2 + p - 1$ maximal subgroups of G (which are distinct from A and M) are of the form $T\Phi(G)$, where T is one of the following minimal nonabelian subgroups: $\langle c, (t')^i t \rangle$ and $\langle c^j t', c^k t \rangle$, where i, j, k are integers mod p and both j and k cannot be congruent 0 (mod p) (see Exercise P9 and P11 in [3]). Here $T\Phi(G)$ must be minimal nonabelian and this will be the case if and only if $\Phi(T) \geq \Phi(G) = G'\langle c^p \rangle$. It is enough to consider $\langle c, t \rangle$ and $\langle c^j t', ct \rangle$ for any integer j mod p. We have:

$$\Phi(\langle c, t \rangle) = \langle c^p, [c, t] = u \rangle = \Phi(G)$$

if and only if $\Omega_1(\langle c \rangle) \neq \langle u \rangle$ which gives $\Omega_1(\langle c \rangle) = \langle u^l z \rangle$ for some integer l mod p. Further, we get:

$$\Phi(\langle c^j t', ct \rangle) = \langle c^{pj} (t')^p = c^{pj},\ (ct)^p = c^p t^p = c^p,\ [c^j t', ct] = u^j z \rangle = \langle c^p, u^j z \rangle.$$

But $\Omega_1(\langle c \rangle) = \Omega_1(\langle c^p \rangle) = \langle u^l z \rangle$ and so for $j = l$, $\Phi(\langle c^l t', ct \rangle) = \langle c^p \rangle < \Phi(G)$ which shows that $\langle c^l t', ct \rangle \Phi(G)$ is another maximal subgroup of G (distinct from M) which is neither abelian nor minimal nonabelian, a contradiction. We have proved that $\Omega_1(G) = E \cong \mathrm{E}_{p^3}$.

(i) Assume that $\mathrm{E}_{p^3} \cong E = \Omega_1(G) \not\leq \mathrm{Z}(G) = \Phi(G)$ and $E \not\leq A$, where A is the unique abelian maximal subgroup of G. Then $A \cap E = G'$, A covers G/E and A is metacyclic. Since G/G' is abelian of rank 3, we have $\mathrm{d}(G/E) = 2$ and so there is at least one maximal subgroup H of G which is minimal nonabelian. If H/E is noncyclic, then $E \leq \Phi(H) = \Phi(G) = \mathrm{Z}(G)$, a contradiction. Hence $H/E \cong (H \cap A)/G' \cong \mathrm{C}_{p^m}$, $m \geq 1$. Since $\mathrm{d}(G/E) = 2$, $G/E \cong A/G'$ is abelian of type (p^m, p). Let $a \in H \cap A$ be such that $\langle a \rangle$ covers $(H \cap A)/G'$. Noting that $\Omega_1(G) = E$, we have $\mathrm{o}(a) = p^{m+1}$ and $1 \neq z = a^{p^m} \in G'$. Let $t \in E - G'$ so that $[t, a] = u \in G' - \langle z \rangle$ and so $G' = \langle u, z \rangle$ since H is non-metacyclic and so $H' = \langle u \rangle$ is a maximal cyclic subgroup in H. Since $A/G' \cong \mathrm{C}_{p^m} \times \mathrm{C}_p$, there is an element $b \in A - H$ such that $1 \neq b^p \in G'$, $b^p \in G' - \langle z \rangle$ and $[a, b] = 1$. Indeed, if $b^p \in \langle z \rangle$, then the subgroup $\langle b, \Omega_2(\langle a \rangle) \rangle$ contains elements of order p in $A - H$, a contradiction. We have $\Phi(H) = \Phi(G) = \langle a^p, u \rangle$, $G = \langle a, b, t \rangle$ and $A = \langle a \rangle \times \langle b \rangle$ is abelian of type (p^{m+1}, p^2), $m \geq 1$. If $[b, t] \in \langle u \rangle$, then $G/\langle u \rangle$ would be abelian, a contradiction. Hence $[b, t] \in G' - \langle u \rangle$ and so replacing b with $b' = b^s$ (with some $s \not\equiv 0 \pmod{p}$), we may assume from the start that $[b, t] = u^l z$ for some integer l mod p.

We have $A = \langle a, b \rangle$ and so we have to check $p^2 + p$ other maximal subgroups of G which are of the form $T\Phi(G)$, where T is one of the minimal nonabelian subgroups: $\langle a, b^i t \rangle$, $\langle a^j b, a^k t \rangle$, where i, j, k are any integers mod p (see Exercise P11 in [3]).

(i1) First assume $m \geq 2$ so that $\Phi(G) > G'$. Consider $T = \langle b, t \rangle$, where

$$\Phi(\langle b, t \rangle) = \langle b^p, [b, t] \rangle \leq G' < \Phi(G)$$

and so $M = \langle b, t \rangle \Phi(G)$ must be our unique maximal subgroup of G which is neither abelian nor minimal nonabelian. All other maximal subgroups of G (which are distinct from A and M) must be minimal nonabelian. Indeed,

$$\Phi(\langle a, b^i t \rangle) = \langle a^p, b^{ip} t^p = b^{ip}, [a, b^i t] = u^{-1} \rangle = \Phi(G).$$

Further, for $j \not\equiv 0 \pmod{p}$, we have:

$$\Phi(\langle a^j b, t \rangle) = \langle a^{pj} b^p, [a^j b, t] = u^{-j} u^l z \rangle.$$

Here $\langle a^{pj} b^p \rangle$ covers $\Phi(G)/G'$ and $\Omega_1(\langle a^{pj} b^p \rangle) = \langle z \rangle$. Hence if $l \not\equiv 0 \pmod{p}$, then $\Phi(\langle a^l b, t \rangle) = \langle a^{pl} b^p \rangle < \Phi(G)$, a contradiction (since $\langle a^l b, t \rangle \Phi(G)$ will be another maximal subgroup which is neither abelian nor minimal nonabelian). Hence $l \equiv 0 \pmod{p}$ and so $[b, t] = z$. Finally, if both j and k are not congruent $0 \pmod{p}$, then we have:

$$\Phi(\langle a^j b, a^k t \rangle) = \langle a^{pj} b^p, a^{pk},\ [a^j b, a^k t] = u^{-j} z \rangle = \Phi(G).$$

We have $b^p \in G' - \langle z \rangle$ and so $b^p = u^n z^s$, where n and s are some integers mod p with $n \not\equiv 0 \pmod{p}$. Then we replace a with $a' = a^n b^{-s}$ and u with

$u' = u^n z^s$ and get:

$$(a')^{p^m} = z^n,\ b^p = u',\ [t, a'] = [t, a^n b^{-s}] = u^n z^s = u'.$$

Writing again a instead a' and u instead u', we see that we have obtained the relations stated in part (a) of our theorem.

(i2) Now assume $m = 1$ so that $|G| = p^5$ and $G' = \mathrm{Z}(G) = \Phi(G) = \langle u, z\rangle$ and therefore G is a special p-group. In this case we have:

$$a^p = z,\ [t, a] = u,\ [a, b] = 1,\ [b, t] = u^l z,\ b^p = u^n z^s,$$

where l, n, s are some integers mod p with $n \not\equiv 0 \pmod p$. Also, A is abelian of type (p^2, p^2). For all integers i mod p we have:

$$\Phi(\langle a, b^i t\rangle) \geq \langle a^p = z, [a, b^i t] = u^{-1}\rangle = G' = \Phi(G)$$

and so all subgroups $\langle a, b^i t\rangle$ are minimal nonabelian maximal subgroups of G. Hence in the rest of p^2 nonabelian maximal subgroups $\langle a^j b, a^k t\rangle G'$ (j, k are any integers mod p) exactly one is not minimal nonabelian. We have:

$$\Phi(\langle a^j b, a^k t\rangle) = \langle u^n z^{s+j}, z^k, u^{l-j} z\rangle,$$

and so if $k \not\equiv 0 \pmod p$, then $\Phi(\langle a^j b, a^k t\rangle) = \langle u, z\rangle = G'$. It remains to examine the case $k \equiv 0 \pmod p$, where we must have:

$$\Phi(\langle a^j b, t\rangle) = \langle u^n z^{s+j}, u^{l-j} z\rangle \neq G' = \langle u, z\rangle$$

for exactly one j. It follows that the quadratic congruence:

$$\begin{vmatrix} n & s+j \\ l-j & 1 \end{vmatrix} = j^2 + j(s-l) + (n - sl) \equiv 0 \pmod p$$

must have exactly one solution in j. This occurs if and only if

$$(s-l)^2 - 4(n - sl) \equiv 0 \pmod p$$

or equivalently, $(s+l)^2 \equiv 4n \pmod p$. In this case we get $j \equiv (1/2)(l-s) \pmod p$ and for that integer j the maximal subgroup $M = \langle a^j b, t\rangle G'$ is the only one which is neither abelian nor minimal nonabelian. We have obtained the groups from part (b) of our theorem.

(ii) Assume $\mathrm{E}_{p^3} \cong E = \Omega_1(G) \not\leq \mathrm{Z}(G) = \Phi(G)$ and $E \leq A$, where A is the unique abelian maximal subgroup of G. Since $\mathrm{d}(G/E) = 2$, there is (at least one) maximal subgroup H containing E which is minimal nonabelian. Then H is non-metacyclic, $H/E \neq \{1\}$ is cyclic and $\mathrm{Z}(H) \cap E = G'$. Indeed, if H/E is noncyclic, then $E \leq \Phi(H) = \Phi(G) = \mathrm{Z}(G)$, contrary to our assumption. Taking an element $a \in H - E$ such that $\langle a\rangle$ covers H/E and an element $b \in E - G'$, we have $\Omega_1(\langle a\rangle) \leq G'$ and

$$H = \langle a, b \mid a^{p^\alpha} = b^p = 1,\ c = [a, b],\ c^p = [a, c] = [b, c] = 1\rangle,$$

where $\alpha \geq 2$, $H' = \langle c\rangle$, $\mathrm{Z}(H) = \Phi(H) = \langle a^p\rangle \times \langle c\rangle$, and $|G| = p^{\alpha+3}$. Setting $a^{p^{\alpha-1}} = z$, we have $G' = \langle c, z\rangle$, $E = \langle b\rangle \times G' \cong \mathrm{E}_{p^3}$ because $\langle c\rangle$ is a maximal cyclic subgroup in H and therefore $\langle c\rangle \neq \langle z\rangle$.

Now, $\langle a, c\rangle$ is an abelian normal subgroup of G of type (p^α, p) which possesses exactly p cyclic subgroups $\langle ac^i\rangle$ (i any integer mod p) of order p^α. But $[a, b^i] = c^i$ and so $\mathrm{N}_H(\langle a\rangle) = \langle a, c\rangle$ and all subgroups $\langle ac^i\rangle$ are conjugate in H. Therefore $N = \mathrm{N}_G(\langle a\rangle)$ covers G/H and so $G = NH$ with $N \cap H = \langle a, c\rangle$. Since $N/G' \cong G/E$ is abelian of rank 2, it follows that N/G' is abelian of type (p^α, p). Hence there is an element $d \in N - H$ such that $1 \neq d^p \in G'$ and $\langle d\rangle$ normalizes $\langle a\rangle$. But N is a maximal subgroup of G which does not contain E and so N is nonabelian (noting that in our case $E \leq A$). This gives $1 \neq [d, a] \in \langle z\rangle$ and so replacing d with a suitable power d^j ($j \not\equiv 0 \pmod p$), we may assume from the start that $[d, a] = z$. If $d^p \in \langle z\rangle$, then $\langle d, a\rangle \cong \mathrm{M}_{p^{\alpha+1}}$ in which case there are elements of order p in $\langle d, a\rangle - H$, a contradiction. We have proved that $d^p \in G' - \langle z\rangle$ so that $\langle d, a\rangle$ is a metacyclic minimal nonabelian maximal subgroup of $G = \langle a, b, d\rangle$. Now, $H = \langle a, b\rangle$ is a minimal nonabelian maximal subgroup of G containing E and the other p maximal subgroups of G containing E are $\langle b, a^i d\rangle\Phi(G)$, where i is any integer mod p and $\Phi(G) = G'\langle a^p\rangle$. For exactly one i, $\langle b, a^i d\rangle\Phi(G)$ is the unique abelian maximal subgroup A of G, i.e., $[a^i d, b] = 1$ and then $c^i[d, b] = 1$ and so we may set $[d, b] = c^s$ for some integer s mod p. Since $d^p \in G' - \langle z\rangle$, we may set $d^p = c^n z^r$ for some integers n, r mod p with $n \not\equiv 0 \pmod p$.

All p^2 maximal subgroups of G which do not contain E are $\langle b^j a, b^k d\rangle\Phi(G)$, where j, k are any integers mod p. They are all metacyclic minimal nonabelian since:

$$\Phi(\langle b^j a, b^k d\rangle) = \langle a^p,\, d^p = c^n z^r,\, [b^j a, b^k d] = c^{-sj+k} z^{-1} \neq 1\rangle = \Phi(G),$$

where we have used the facts $\langle a^p\rangle \geq \langle z\rangle$ and $n \not\equiv 0 \pmod p$.

We have $A = \langle b, a^{-s} d\rangle\Phi(G)$ is the unique abelian maximal subgroup of G and in the set of $p-1$ nonabelian maximal subgroups $\langle b, a^i d\rangle\Phi(G)$ for $i \not\equiv -s \pmod p$ exactly one of them is not minimal nonabelian. We compute for all $i \not\equiv -s \pmod p$:

$$\Phi(\langle b, a^i d\rangle) = \langle b^p = 1,\, (a^i d)^p = a^{pi} c^n z^r,\, [b, a^i d] = c^{-i-s} \neq 1\rangle = \langle a^{pi} z^r, c\rangle.$$

If $\alpha \geq 3$, then $\langle z^r\rangle \leq \langle a^p\rangle$ and so in this case $\Phi(\langle b, a^i d\rangle) \neq \Phi(G)$ if and only if $i \equiv 0 \pmod p$. Then we have $M = \langle b, d\rangle\Phi(G)$ and in this case $s \not\equiv 0 \pmod p$.

If $\alpha = 2$, then $\Phi(\langle b, a^i d\rangle) = \langle z^{i+r}, c\rangle$ since $a^p = z$. Hence in this case $\Phi(\langle b, a^i d\rangle) \neq \Phi(G)$ if and only if $i \equiv -r \pmod p$. Then we have $M = \langle b, a^{-r} d\rangle\Phi(G)$ and in this case we must have $r \not\equiv s \pmod p$. We have obtained the groups stated in part (c) of our theorem.

(iii) It remains to consider the case $\mathrm{E}_{p^3} \cong E = \Omega_1(G) \leq \mathrm{Z}(G) = \Phi(G)$. We shall show that this difficult case cannot occur.

If H_i is any minimal nonabelian maximal subgroup of G, then $\Phi(H_i) = \Phi(G) \geq E \cong \mathrm{E}_{p^3}$ and so $|H_i| \geq p^5$ which implies $|G| \geq p^6$. By the first paragraph of this proof we know that there are exactly p nonabelian maximal subgroups of G whose commutator subgroup is equal $M' = \langle m \rangle$. Obviously, G/M' is an A_2-group of order $\geq p^5$ with $(G/M')' = G'/M' \cong \mathrm{C}_p$ and G/M' has exactly $p+1$ abelian maximal subgroups. By Proposition 71.1 in [2], there is a minimal nonabelian maximal subgroup H/M' of G/M' and an element $d \in G - H$ such that $1 \neq d^p \in G'$ and $[\langle d \rangle, G] = M'$. We have $H' = \langle c \rangle$ with $c \in G' - M'$ and $G' = \langle c, m \rangle$ which implies that H is a minimal nonabelian maximal subgroup of G (noting that $H \neq M$ since $H' \neq M'$).

There are exactly $p+1$ maximal subgroups X_i of G $(i = 1, 2, ..., p+1)$ containing $\langle d \rangle$. Then $X_i \cap H$ is an abelian maximal subgroup of H. We have $X_i' = [\langle d \rangle, (X_i \cap H)] \leq M'$ and so X_i is either abelian or $X_i' = \langle m \rangle$. It follows that $\{X_1, ..., X_{p+1}\} = \{A, M, H_1^*, ..., H_{p-1}^*\}$, where H_j^* $(j = 1, 2, ..., p-1)$ are minimal nonabelian with $(H_j^*)' = \langle m \rangle = M'$. But H_j^* (containing E) is non-metacyclic and so M' is a maximal cyclic subgroup in H_j^* which implies $d^p \in G' - M'$ and we may set $d^p = c^s m^t$, where s, t are some integers (mod p) with $s \not\equiv 0$ (mod p).

Consider $H^* = H_1^*$ so that H^* is a minimal nonabelian maximal subgroup of G containing $\langle d \rangle$ and $(H^*)' = \langle m \rangle$. Choose an element $a^* \in (H \cap H^*) - \Phi(G)$ so that $H^* = \langle d, a^* \rangle$ and $[d, a^*] = m$. By exercise P9 in [3],

$$\Phi(H^*) = \langle d^p, [d, a^*] = m \rangle \langle (a^*)^p \rangle = G' \langle (a^*)^p \rangle,$$

and we know that $\Phi(H^*) = \Phi(G)$. If $\Omega_1(\langle a^* \rangle) \leq G'$, then $E \not\leq \Phi(G)$, a contradiction. Hence $\Omega_1(\langle a^* \rangle) \not\leq G'$ which implies $\Phi(H^*) = \Phi(G) = \langle (a^*)^p \rangle \times G'$ and this is an abelian group of type $(p, p, p^{\gamma-1})$, where $\mathrm{o}(a^*) = p^\gamma$, $\gamma \geq 2$ and $H \cap H^* = \langle a^* \rangle \times G'$. It follows that $(H \cap H^*)/G' \cong \mathrm{C}_{p^\gamma}$ and since H/G' is noncyclic abelian, there is $b^* \in H - (H \cap H^*)$ such that $1 \neq (b^*)^p \in G'$ and $\langle a^*, b^* \rangle = H$. We may set $[a^*, b^*] = c$, where $\langle c \rangle = H'$. Also, $\langle c \rangle$ is a maximal cyclic subgroup in H which gives $(b^*)^p \in G' - \langle c \rangle$ and we have $G = \langle a^*, b^*, d \rangle$, $|G| = p^{\gamma+4}$, $\gamma \geq 2$. We may set $(b^*)^p = c^v m^w$, where v, w are some integers (mod p) with $w \not\equiv 0$ (mod p).

Suppose that $[d, b^*] = 1$ so that $A = \langle d, b^* \rangle \Phi(G)$. Also, $\langle d, a^* \rangle = H^*$ and so we investigate other maximal subgroups $\langle d, (a^*)^i b^* \rangle \Phi(G)$ containing $\langle d \rangle$, where $i \not\equiv 0$ (mod p). We get:

$$\Phi(\langle d, (a^*)^i b^* \rangle) = \langle d^p \in G' - \langle m \rangle,\ [d, (a^*)^i b^*] = m^i,\ (a^*)^{pi} (b^*)^p \rangle = \Phi(G).$$

But then there is no subgroup M in the set of maximal subgroups of G containing $\langle d \rangle$, a contradiction. Hence we have $[d, b^*] = m^r$ with $r \not\equiv 0$ (mod p).

Since the maximal subgroups A and M are contained in the set of $p+1$ maximal subgroups of G which contain $\langle d \rangle$, it follows that all p^2 maximal subgroups $\langle d^i a^*, d^j b^* \rangle \Phi(G)$ of G (i, j are any integers mod p) which do not contain $\langle d \rangle$ must be minimal nonabelian.

For each integer i mod p we must have:

$$\Phi(\langle d^i a^*, b^* \rangle) = \langle d^{pi} (a^*)^p,\ (b^*)^p = c^v m^w,\ [d^i a^*, b^*] = c m^{ri} \rangle = \Phi(G)$$

and this will be the case if and only if $\langle c^v m^w,\, cm^{ri}\rangle = G' = \langle c, m\rangle$ or equivalently:

$$\begin{vmatrix} v & w \\ 1 & ri \end{vmatrix} = vri - w \not\equiv 0 \,(\mathrm{mod}\ p).$$

But $r \not\equiv 0 \,(\mathrm{mod}\ p)$ and so if $v \not\equiv 0 \,(\mathrm{mod}\ p)$, then the congruence $vri - w \equiv 0 \,(\mathrm{mod}\ p)$ would have a solution in i, a contradiction. Hence $v \equiv 0 \,(\mathrm{mod}\ p)$ and so $(b^*)^p = m^w$.

For each integer i mod p we must have:

$$\Phi(\langle d^i a^*, db^*\rangle) = \langle d^{pi}(a^*)^p,\, d^p(b^*)^p = c^s m^{t+w},\, [d^i a^*, db^*] = cm^{ri-1}\rangle = \Phi(G)$$

and this will be the case if and only if $\langle c^s m^{t+w},\, cm^{ri-1}\rangle = G' = \langle c, m\rangle$ or equivalently:

$$\begin{vmatrix} s & t+w \\ 1 & ri-1 \end{vmatrix} = (sr)i - s - t - w \not\equiv 0 \,(\mathrm{mod}\ p).$$

But $sr \not\equiv 0 \,(\mathrm{mod}\ p)$ and so the congruence $(sr)i - s - t - w \equiv 0 \,(\mathrm{mod}\ p)$ would have a solution in i, a final contradiction. Our theorem is proved. □

LEMMA 4. *Let P be a p-group with $|P'| = p$. If P possesses a minimal nonabelian maximal subgroup H, then P has exactly $p+1$ abelian maximal subgroups.*

PROOF. By Exercise P.10 in [3], $P = H\mathrm{C}_P(H)$ and in our case $\mathrm{C}_P(H)$ is abelian so that $\mathrm{C}_P(H) = \mathrm{Z}(P)$. But then all $p+1$ maximal subgroups of P containing $\mathrm{Z}(P)$ are abelian and so we are done (see Exercise 1.6(a) in [1]). □

THEOREM 5. *Let G be a p-group, $p > 2$, with $\mathrm{d}(G) = 3$ which has exactly one maximal subgroup M which is neither abelian nor minimal nonabelian. Suppose that G has no abelian maximal subgroups. Then we have $\Phi(G) = \mathrm{Z}(G)$, $\Omega_1(G) = G' \cong \mathrm{E}_{p^3}$, $|M'| = p$, $\mathrm{d}(M) \geq 3$ and we have one of the following possibilities:*

(a) *$G = \langle a, b, c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1,\, a^p = z,\, b^p = y,\, c^p = x,\, [a, b] = z,\, [a, c] = yz^\beta,\, [b, c] = x^{-1}y^\delta z^\eta,\, [x, b] = [x, a] = [y, a] = [y, c] = [z, b] = [z, c] = 1\rangle$, where β, δ, η are integers mod p, $\eta \not\equiv 0$ (mod p) and $(\beta - \delta)^2 + 4\eta$ is not a square in GF(p). We have $|G| = p^6$, $\Omega_1(G) = G' = \Phi(G) = \mathrm{Z}(G) = \langle x, y, z\rangle \cong \mathrm{E}_{p^3}$ and so G is a special p-group. Also, $M = \langle a, b\rangle G'$ and all other maximal subgroups of G are non-metacyclic minimal nonabelian.*

(b) *$G = \langle a, b, c \mid a^{p^2} = b^{p^2} = c^{p^n} = 1,\, a^p = x,\, b^p = y,\, c^{p^{n-1}} = z,\, [a, b] = z,\, [c, a] = x^\alpha y^\eta z^\beta,\, [c, b] = x^\zeta y^\gamma z^\delta,\, [x, b] = [x, c] = [y, a] = [y, c] = [z, a] = [z, b] = 1\rangle$, where $n \geq 3$, $\alpha, \beta, \gamma, \delta, \eta, \zeta$ are integers mod p, $\eta \not\equiv 0$ (mod p), $\zeta \not\equiv 0$ (mod p), and $(\alpha - \gamma)^2 + 4\eta\zeta$ is not a square in GF(p). We have $|G| = p^{n+4}$, $\Omega_1(G) = G' = \langle x, y, z\rangle \cong \mathrm{E}_{p^3}$ and*

$\Phi(G) = \mathrm{Z}(G) = \langle c^p, x, y\rangle$ is abelian of type (p^{n-1}, p, p). Also, $M = \langle a, b\rangle\Phi(G)$ and all other maximal subgroups of G are non-metacyclic minimal nonabelian.

Proof. Let $\Gamma_1 = \{H_1, H_2, ..., H_{p^2+p}, M\}$ be the set of maximal subgroups of G, where H_i $(i = 1, ..., p^2 + p)$ are minimal nonabelian and M is neither abelian nor minimal nonabelian. By Lemma 2, $\Phi(G) = \mathrm{Z}(G) = \mathrm{Z}(H_i) = \mathrm{Z}(M)$, $\mathrm{d}(M) \geq 3$, $|M'| = p$, and $|G'| > p$. By a result of A. Mann (see Exercise 1.69(a) in [1]), $|G : (H_1'H_2')| \leq p$ and so $p^2 \leq |G'| \leq p^3$.

Suppose that for some $H_i \neq H_j$ we have $H_i' = H_j'$ or $H_i' = M'$. Then, by the above result of A. Mann, $|G'| \leq p^2$ and so $|G'| = p^2$. If $G' \cong \mathrm{C}_{p^2}$, then the nonabelian group $G/\Omega_1(G')$ has p^2+p+1 abelian maximal subgroups, which is a contradiction by Exercise 1.6(a) in [1]. Hence $G' \cong \mathrm{E}_{p^2}$. Let X be any fixed subgroup of order p in G'. Then there is a minimal nonabelian maximal subgroup H_i ($i \in \{1, ..., p^2+p\}$) such that $H_i' \neq X$ so that H_i/X is minimal nonabelian. By Lemma 4, G/X has exactly $p+1$ abelian maximal subgroups. Hence there are exactly $p+1$ maximal subgroups of G whose commutator subgroup is equal X. But G' has $p+1$ subgroups of order p and so G must have $(p+1)^2 = p^2+2p+1$ maximal subgroups, a contradiction.

We have proved that $\{H_1', H_2', ..., H_{p^2+p}', M'\}$ is the set of p^2+p+1 pairwise distinct subgroups of order p in G' and so, in particular, $G' \cong \mathrm{E}_{p^3}$. Assume that there is an element $t \in G - G'$ of order p. Since G/G' is abelian of rank 3, $G/(G' \times \langle t\rangle)$ is noncyclic. But then there is a maximal subgroup Y of G containing $G' \times \langle t\rangle$ which is minimal nonabelian, a contradiction (with the structure of Y). We have proved that $\Omega_1(G) = G' \cong \mathrm{E}_{p^3}$ and all minimal nonabelian maximal subgroups of G are non-metacyclic.

Set $T/G' = \Omega_1(G/G') \cong \mathrm{E}_{p^3}$. If G/T is noncyclic, then there is a maximal subgroup K of G containing T which is minimal nonabelian. Since $K > T$, T is abelian of type (p^2, p^2, p^2). But $K' < G'$ and so K' is not a maximal cyclic subgroup in K, contrary to the fact that K is non-metacyclic minimal nonabelian. We have proved that G/T is cyclic.

(i) First assume that $T = G$, i.e., $G/G' \cong \mathrm{E}_{p^3}$ and so in this case G is a special group of order p^6 with $G' = \Omega_1(G) \cong \mathrm{E}_{p^3}$.

We determine the structure of M. We have $M = G' * S$, where $S = \langle a, b\rangle$ is minimal nonabelian and $G' \cap S = \Phi(S) < G'$ since $\mathrm{d}(M) \geq 3$. Set $S' = M' = \langle z\rangle \cong \mathrm{C}_p$. If $\Phi(S) = \langle z\rangle$, then $\langle z\rangle$ is a unique subgroup of order p in S which implies that S would be cyclic, a contradiction. Hence $\Phi(S) = \Omega_1(S) \cong \mathrm{E}_{p^2}$ and so S is metacyclic of order p^4 and exponent p^2. We may choose $a, b \in S - G'$ so that $a^p = z$, $b^p = y \in \Phi(S) - \langle z\rangle$ and $[a, b] = z$. Since $\mho_1(M) = \mho_1(S) = \langle y, z\rangle \geq M' = \langle z\rangle$, it follows that M is a powerful group. By Proposition 26.10 in [1], each element in $\langle y, z\rangle$ is a p-th power of an element in M. Let c be an element in $G - M$. Suppose that $c^p \in \langle y, z\rangle$. Then there is $m \in M$ such that $m^p = c^{-p}$. We get $(mc)^p = m^p c^p [c, m]^{\binom{p}{2}} = 1$, contrary to the fact that $\Omega_1(G) = G'$. Hence $c^p = x \in G' - \langle y, z\rangle$ and so $G' = \langle x, y, z\rangle$ and $G = \langle a, b, c\rangle$. All $p^2 + p$ maximal subgroups of G which are distinct from M must be minimal nonabelian.

We have $\Phi(\langle a, c\rangle) = \langle z, x, [a, c]\rangle = G'$ and so $[a, c] = y^r y'$, where $r \not\equiv 0 \pmod p$ and $y' \in \langle x, z\rangle$. Replacing c with $c' = c^{r'}$, where $r' \not\equiv 0 \pmod p$ is such that $rr' \equiv 1 \pmod p$, we get

$$[a, c'] = [a, c^{r'}] = [a, c]^{r'} = y^{rr'}(y')^{r'} = y(y')^{r'},$$

where $c' \in G - M$, $(y')^{r'} \in \langle x, z\rangle$ and $(c')^p = (c^p)^{r'} = x^{r'} = x' \in G - \langle y, z\rangle$. Writing again c and x instead of c', x', respectively, we see that we may assume from the start $[a, c] = yy''$ and $c^p = x$, where $y'' \in \langle x, z\rangle$ and so we may set $[a, c] = x^\alpha y z^\beta$ for some integers α, β mod p. From $\Phi(\langle b, c\rangle) = \langle y, x, [b, c]\rangle = G'$ follows that $[b, c] = x^\gamma y^\delta z^\eta$, where γ, δ, η are some integers mod p with $\eta \not\equiv 0 \pmod p$.

Maximal subgroups of G containing $\langle c\rangle$ are $\langle a, c\rangle G'$ and $\langle a^i b, c\rangle G'$. Therefore we must have for all integers i mod p:

$$\Phi(\langle a^i b, c\rangle) = \langle yz^i, x, [a^i b, c] = x^{\alpha i + \gamma} y^{i+\delta} z^{\beta i + \eta}\rangle = G',$$

which is equivalent with:

$$\begin{vmatrix} 0 & 1 & i \\ 1 & 0 & 0 \\ \alpha i + \gamma & i + \delta & \beta i + \eta \end{vmatrix} = i^2 + (\delta - \beta)i - \eta \not\equiv 0 \, (\mathrm{mod}\ p).$$

Hence the quadratic congruence $i^2 + (\delta - \beta)i - \eta \equiv 0 \pmod p$ should not have any solution in i which is equivalent with the requirement that $(\beta - \delta)^2 + 4\eta$ is not a square in $\mathrm{GF}(p)$.

We have to examine p^2 maximal subgroups $\langle c^j a, c^k b\rangle G'$ of G (j, k are integers mod p) which do not contain $\langle c\rangle$. For all $k \not\equiv 0 \pmod p$ we have:

$$\Phi(\langle a, c^k b\rangle) = \langle z, x^k y, [a, c^k b] = x^{k\alpha} y^k z^{k\beta + 1}\rangle = G',$$

which is equivalent with:

$$\begin{vmatrix} 0 & 0 & 1 \\ k & 1 & 0 \\ k\alpha & k & k\beta + 1 \end{vmatrix} = k^2 - k\alpha = k(k - \alpha) \not\equiv 0 \, (\mathrm{mod}\ p).$$

Hence we must have $\alpha \equiv 0 \pmod p$ and so $[a, c] = yz^\beta$.

For all $j \not\equiv 0 \pmod p$ we have:

$$\Phi(\langle c^j a, b\rangle) = \langle x^j z, y, [c^j a, b] = x^{-\gamma j} y^{-\delta j} z^{-\eta j + 1}\rangle = G',$$

which is equivalent with:

$$\begin{vmatrix} j & 0 & 1 \\ 0 & 1 & 0 \\ -\gamma j & -\delta j & -\eta j + 1 \end{vmatrix} = j(-\eta j + \gamma + 1) \not\equiv 0 \, (\mathrm{mod}\ p).$$

Since $\eta \not\equiv 0 \pmod p$, we must have $\gamma \equiv -1 \pmod p$ and so $[b,c] = x^{-1}y^{\delta}z^{\eta}$. We have obtained the groups of order p^6 stated in part (a) of our theorem.

It remains to check that all maximal subgroups $\langle c^j a, c^k b\rangle G'$ are minimal nonabelian unless $j \equiv k \equiv 0 \pmod p$ in which case $\langle a,b\rangle G' = M$. Indeed,

$$\Phi(\langle c^j a, c^k b\rangle) = \langle x^j z, x^k y, [c^j a, c^k b] = x^j y^{-\delta j + k} z^{-\eta j + \beta k + 1}\rangle < G'$$

if and only if

$$\begin{vmatrix} j & 0 & 1 \\ k & 1 & 0 \\ j & -\delta j + k & -\eta j + \beta k + 1 \end{vmatrix} = k^2 + k(\beta j - \delta j) - \eta j^2 \equiv 0 \pmod p.$$

The quadratic congruence in k:

$$(*) \qquad k^2 + kj(\beta - \delta) - \eta j^2 \equiv 0 \pmod p,$$

(where $j \in \mathrm{GF}(p)$ is fixed) has a solution in k if and only if the discriminant:

$$j^2(\beta-\delta)^2 + 4\eta j^2 = ((\beta-\delta)^2 + 4\eta)j^2$$

is a square in $\mathrm{GF}(p)$. But we know that $(\beta-\delta)^2 + 4\eta$ is not a square in $\mathrm{GF}(p)$ and so we must have $j \equiv 0 \pmod p$. From $(*)$ we get then $k^2 \equiv 0 \pmod p$ and so also $k \equiv 0 \pmod p$ and we are done.

(ii) Now assume that $T < G$, where $T/G' = \Omega_1(G/G') \cong \mathrm{E}_{p^3}$ and T/G' is cyclic. Hence G/G' is abelian of type (p^m, p, p), $m \geq 2$, and the unique maximal subgroup of G containing T is obviously equal M. There are normal subgroups U and V of G such that $G = UV$, $U \cap V = G'$, $U/G' \cong \mathrm{E}_{p^2}$ and $V/G' \cong \mathrm{C}_{p^m}$, $m \geq 2$. Let c be an element in $V - G'$ such that $\langle c\rangle$ covers V/G'. We have $\mathrm{o}(c) = p^n$, $n \geq 3$, where $n = m+1$ (noting that $\Omega_1(G) = G'$). Set $c^{p^{n-1}} = z$, where $z \in G'$. Then $M = \langle c^p\rangle U$, $\Phi(G) = \mathrm{Z}(G) = G'\langle c^p\rangle$ is abelian of type (p^{n-1}, p, p) and $|G| = p^{n+4}$. Let $a, b \in U - G'$ be such that $U = G'\langle a, b\rangle$, where a^p, $b^p \in G'$ and $G = \langle a,b,c\rangle$. Since each minimal nonabelian maximal subgroup H_i of G $(i = 1, ..., p^2+p)$ is non-metacyclic and contains $\Phi(G)$ and $\langle z\rangle$ is not a maximal cyclic subgroup in $\Phi(G)$, it follows that $H_i' \neq \langle z\rangle$ (for all i) and so $M' = \langle z\rangle$. Therefore $1 \neq [a,b] \in \langle z\rangle$ and so we may assume $[a,b] = z$.

Now, $G/\langle z\rangle$ has the unique abelian maximal subgroup $M/\langle z\rangle$ and all other maximal subgroups of $G/\langle z\rangle$ are minimal nonabelian. Hence $G/\langle z\rangle$ is an A_2-group with $\mathrm{d}(G/\langle z\rangle) = 3$ and order $p^{n+3} > p^4$ (since $n \geq 3$), $G'/\langle z\rangle \cong \mathrm{E}_{p^2}$, $G'/\langle z\rangle \leq \mathrm{Z}(G/\langle z\rangle)$ and $G/\langle z\rangle$ has a normal elementary abelian subgroup $\langle G', \Omega_2(\langle c\rangle)\rangle/\langle z\rangle$ of order p^3. Hence $G/\langle z\rangle$ is an A_2-group of Proposition 71.4(b) in [2] which implies that $\Omega_1(G/\langle z\rangle) = \langle G', \Omega_2(\langle c\rangle)\rangle/\langle z\rangle$. Set $a^p = x$ and $b^p = y$ and consider the abelian group $M/\langle z\rangle$. If the abelian subgroup $U/\langle z\rangle$ of order p^4 and exponent $\leq p^2$ has rank > 2, then $\Omega_1(U/\langle z\rangle) > G'/\langle z\rangle$ which contradicts the above fact. Hence $U/\langle z\rangle \cong \mathrm{C}_{p^2} \times \mathrm{C}_{p^2}$ which implies $G' = \langle x, y, z\rangle$.

Since $\Phi(\langle c, a\rangle) = \langle c^p, x, [c, a]\rangle = \Phi(G)$ and $\Phi(\langle c, b\rangle) = \langle c^p, y, [c, b]\rangle = \Phi(G)$, we must have $[c, a] = x^\alpha y^\eta z^\beta$, $[c, b] = x^\zeta y^\gamma z^\delta$ for some integers $\alpha, \beta, \gamma, \delta, \eta, \zeta$ mod p with $\eta \not\equiv 0 \pmod p$ and $\zeta \not\equiv 0 \pmod p$.

Maximal subgroups of G containing $\langle c\rangle$ are $\langle a, c\rangle\Phi(G)$ and $\langle a^i b, c\rangle\Phi(G)$. Therefore we must have for all integers i mod p:

$$\Phi(\langle c, a^i b\rangle) = \langle c^p, x^i y, [c, a^i b] = x^{\alpha i+\zeta} y^{\eta i+\gamma} z^{\beta i+\delta}\rangle = \Phi(G),$$

which is equivalent with:

$$\begin{vmatrix} 0 & 0 & 1 \\ i & 1 & 0 \\ \alpha i + \zeta & \eta i + \gamma & \beta i + \delta \end{vmatrix} = \eta i^2 + (\gamma - \alpha)i - \zeta \not\equiv 0 \,(\mathrm{mod}\ p).$$

Hence the quadratic congruence $\eta i^2 + (\gamma-\alpha)i - \zeta \equiv 0 \pmod p$ should not have any solution in i which is equivalent with the requirement that $(\gamma - \alpha)^2 + 4\eta\zeta$ is not a square in $\mathrm{GF}(p)$. We have obtained the groups stated in part (b) of our theorem.

It remains to check that all maximal subgroups $\langle c^j a, c^k b\rangle\Phi(G)$ are minimal nonabelian unless $j \equiv k \equiv 0 \pmod p$ in which case $\langle a, b\rangle\Phi(G) = M$. Note that $\Phi(G) = \langle c^p\rangle \times \langle x\rangle \times \langle y\rangle$ and $\Phi(\Phi(G)) = \langle c^{p^2}\rangle \geq \langle z\rangle$, where $\Phi(G)/\Phi(\Phi(G)) \cong \mathrm{E}_{p^3}$. We have

$$\Phi(\langle c^j a, c^k b\rangle) = \langle c^{pj} x, c^{pk} y, [c^j a, c^k b] = x^{\zeta j - \alpha k} y^{\gamma j - \eta k} z^{\delta j - \beta k + 1}\rangle < \Phi(G)$$

if and only if

$$\begin{vmatrix} j & 1 & 0 \\ k & 0 & 1 \\ 0 & \zeta j - \alpha k & \gamma j - \eta k \end{vmatrix} = \eta k^2 + (\alpha - \gamma)jk - \zeta j^2 \equiv 0 \,(\mathrm{mod}\ p).$$

The quadratic congruence in k:

$$(**) \qquad \eta k^2 + (\alpha - \gamma)jk - \zeta j^2 \equiv 0 \,(\mathrm{mod}\ p),$$

(where $j \in \mathrm{GF}(p)$ is fixed) has a solution in k if and only if the discriminant:

$$j^2(\alpha - \gamma)^2 + 4\eta\zeta j^2 = ((\alpha - \gamma)^2 + 4\eta\zeta)j^2$$

is a square in $\mathrm{GF}(p)$. But we know that $(\alpha - \gamma)^2 + 4\eta\zeta$ is not a square in $\mathrm{GF}(p)$ and so we must have $j \equiv 0 \pmod p$. From $(**)$ we get then $\eta k^2 \equiv 0 \pmod p$ and so (noting that $\eta \not\equiv 0 \pmod p$) $k \equiv 0 \pmod p$ and we are done. Our theorem is proved. □

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Received: 26.1.2010.
Revised: 5.2.2010.