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RANK ONE REDUCIBILITY FOR UNITARY GROUPS

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Abstract. Let (G, G') denote a dual reductive pair consisting of two unitary groups over a nonarchimedean local field of characteristic zero. We relate the reducibility of the parabolically induced representations of these two groups if the inducing data is cuspidal and related to each other by theta correspondence. We calculate theta lifts of the irreducible subquotients of these parabolically induced representations. To obtain these results, we explicitly calculate filtration of Jacquet modules of the appropriate Weil representation (as Kudla did for the orthogonal–symplectic dual pairs), but keeping in mind the explicit splittings of covers of these two unitary groups, also obtained by Kudla.

1. Introduction

In this paper we study the relation between reducibilities of the parabolically induced representations of two unitary groups constituting a dual reductive pair in a symplectic group over F, where F is a non-archimedean field of characteristic zero. In more words, let G'_n be the unitary group preserving a skew–hermitian form on the vector space W_n over E, a quadratic extension of F, of the Witt index equal to n. On the other hand, we look at the tower of hermitian vector spaces, where the unitary group of the vector space V_l on the l-th level (i.e., with the Witt index l) is denoted by G_l. The pair (G_l, G'_n) constitutes a dual reductive pair in the symplectic group (over F) $Sp(V_l \otimes W_n)$. Let σ be an irreducible supercuspidal representation of G'_n, and let r be the smallest index for which σ appears in the theta correspondence with the representations of the group G_r, i.e., for which there is a non-zero G'_n intertwining map from $\omega_{r,n,\psi}$ to σ. Here

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$\omega_{r,n,\psi}$ denotes the Weil representation of $\widetilde{Sp(V_r \otimes W_n)}$, an (infinite) cover of $Sp(V_l \otimes W_r)$, corresponding to the additive character ψ of F ([18]), pulled back as a representation of $G_r \times G'_n$ (since this pair splits in $\widetilde{Sp(V_r \otimes W_n)}$, we can indeed view it as a representation of $G_l \times G'_n$ and not of their covers). The biggest quotient of $\omega_{r,n,\psi}$ on which G'_n acts as a multiple of σ is of the form $\sigma \otimes \Theta(\sigma, r)$, where $\Theta(\sigma, r)$ is a smooth, finite length representation of G_r (for general irreducible representation σ); for σ supercuspidal (as in our case) it is known ([13], Théorème principal, p. 69) that $\Theta(\sigma, r)$ is an irreducible cuspidal representation of G_r. We denote it by τ. Let ρ be an irreducible cuspidal representation of $GL(j, E)$. Note that $GL(j, E) \times G'_n$ is isomorphic to a Levi subgroup of a maximal parabolic subgroup P'_j of G'_{j+n}; the analogous statement is true for G_{j+r}, where the corresponding maximal parabolic subgroup is denoted by P_j. So, we want to relate the reducibility of the representations $\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho \otimes \sigma)$ with the reducibility of the representation $\mathrm{Ind}_{P_j}^{G_{r+j}}(\rho \otimes \tau)$, using theta correspondence and to describe the first non–vanishing theta–lifts of each irreducible subquotient of these representations. The first work in this direction was [14], where a situation of a dual pair consisting of an even-orthogonal and a symplectic group was studied. After that, in the joint work with Goran Muić ([6]), we studied the representations of the same form, but we considered dual pairs consisting of odd-orthogonal and symplectic groups, so the result was about relating representations of metaplectic group (since the symplectic member in the dual pair in this situation does not split in the double cover of the "big" symplectic group) and the representations of odd–orthogonal groups.

The main idea of this work (as well as [14] and [6]) was using information on theta correspondence from the filtration of Jacquet modules of the representation $\omega_{r,n,\psi}$. In the case of symplectic–orthogonal dual pairs this filtration is explicitly calculated in [10]. There is also a calculation of this filtration for the general type I reductive dual pairs in [13], but in terms of their covers. We calculate this filtration for the unitary dual pairs, following the procedure in [10], but now having in mind explicitly described splittings for unitary dual pairs, calculated in [11]. So, we calculated the filtration of Jacquet modules of the representation $\omega_{r+j,n+j,\psi}$, where these Jacquet modules are viewed as representations of $P_j \times G'_{n+j}$ or $G_{r+j} \times P'_j$ and not of their covers.

When this calculation is obtained, for most representations ρ of $GL(j, E)$ (described above) the lifts and reducibility of representations $\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho \otimes \sigma)$ and $\mathrm{Ind}_{P_j}^{G_{r+j}}(\rho \otimes \tau)$ are described in Theorem 4.4. While this case is very similar to Theorem 3.5 in [6], there are more exceptional cases than in the case of symplectic–odd -orthogonal dual pair of [6] (the fourth section there).

One of the exceptional cases here (Proposition 5.6) is very similar to Theorem 4.4 in [6], but cases covered in Propositions 5.2 and 5.3 here do not have a direct analogon in [6].

We hope that the results obtained in this work on theta correspondence for unitary dual pairs would find an application not only in the local representation theory, but also in the theory of automorphic forms.

For the convenience of the readers, we describe the main results of this paper. N_E denotes the composition of the reduced norm on $GL_j(E)$ with the norm on F^*. We have an explicit description of the filtration of Jacquet module of the representation $\omega_{r,n,\psi}$:

THEOREM. Let W_n and V_l be a skew–hermitian and hermitian space, respectively. Then, the normalized Jacquet module $R_{P'_j}(\omega_{l,n})$ has the following $G_l \times M'_j$–invariant filtration:

$$R_{P'_j}(\omega_{n,l}) = \tau_j'^{(0)} \supset \tau_j'^{(1)} \supset \cdots \supset \tau_j'^{(r)} \supset \{0\}.$$

Here $r = \min(l, j)$, but we need only to consider the case $l \geq j$ so we continue to assume $r = j$. The successive quotients $\tau'_{jk} = \tau_j'^{(k)}/\tau_j'^{(k+1)}$ are described as follows:

$$\tau'_{jk} \cong \mathrm{Ind}_{P_k \times R_{jk} \times G'_{n-j}}^{G_l \times GL_j \times G'_{n-j}} \beta_{jk} \otimes \Sigma_k \otimes \omega_{l-k,n-j},$$

where β_{jk} is a character on $GL_k(E) \times GL_{j-k}(E) \times GL_k(E) \subset P_k \times R_{jk}$ defined as follows

$$\beta_{jk}(a, g', g) = N_E(a)^{\frac{m_l-k}{2}} N_E(g')^{\frac{m_l-k-t_n+j}{2}} \beta_V(g') N_E(g)^{\frac{m_l-k}{2}} \beta_V(g) \beta_{W_0^j}(g^{-*}),$$

and Σ_k is the usual action of $GL_k(E) \times GL_k(E)$ on $S(GL_k(E))$ given by $\Sigma_k(a, g)\phi(h) = \phi(a^* hg)$. Here R_{jk} is a maximal parabolic subgroup of $GL_j(E)$ with the Levi subgroup isomorphic to $GL_{j-k}(E) \times GL_k(E)$; $\beta_V(g')$ and $\beta_V(g)$ correspond to the embedding of $a \in GL_j(E) \subset G(W'_j \oplus W''_j)$ into $Sp(V \otimes_E \widetilde{(W'_j} \oplus W''_j))$, and $\beta_{W_0^j}$ corresponds to the embedding of $a \in GL_k(E) \subset G(V'_k \oplus V''_k)$ into $Sp((V'_k \widetilde{\oplus V''_k}) \otimes_E W_0^j)$.

Then, we were able to relate the reducibilities of the induced representations described above, in the following four results; firstly we cover the main case:

THEOREM. *Let $m_r = \dim_E V_r$, where V_r is a hermitian space, and let G_r be the corresponding unitary group. Let $t_n = \dim_E W_n$, where W_n is a skew–hermitian space, and G'_n is a unitary group of that space. Let σ be a cuspidal representation of G'_n whose first non-zero lift in the hermitian power containing V_r is cuspidal representation τ of G_r. Let ρ be an irreducible cuspidal representation of $GL_j(E)$. Then, if $\rho \notin \{N_E^{\pm\frac{m_r-t_n+1}{2}}\xi, N_E^{\pm\frac{m_r-t_n-1}{2}}\xi\}$,* *the representation* $\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho\otimes\sigma)$ *reduces if and only if the representation* $\mathrm{Ind}_{P_j}^{G_{r+j}}(\xi'\xi^{-1}\rho\otimes\tau)$ *reduces. Here* ξ *and* ξ' *are characters of* E^*, *whose restrictions to* F^* *are* $\epsilon_{E/F}^{m_r}$ *and* $\epsilon_{E/F}^{t_n}$, *respectively, and where* $\epsilon_{E/F}(x)=(x,\Delta)_F$ *is a quadratic character related to the extension* E/F. *In the case of irreducibility we have* $\Theta(\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho\otimes\sigma),r+j)=\mathrm{Ind}_{P_j}^{G_{r+j}}(\xi'\xi^{-1}\rho\otimes\tau)$ *(and vice versa). In the case of reducibility, the representation* $\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho\otimes\sigma)$ *has two irreducible subquotients, say,* π_1 *and* π_2, *satisfying*

$$0\to\pi_1\to\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho\otimes\sigma)\to\pi_2\to 0.$$

Then, $\Theta(\pi_i,r+j)\neq 0$, *and the following holds*

$$0\to\Theta(\pi_1,r+j)\to\mathrm{Ind}_{P_j}^{G_{r+j}}(\xi'\xi^{-1}\rho\otimes\tau)\to\Theta(\pi_2,r+j)\to 0.$$

The analogous statement holds for the theta lifts of the irreducible subquotients of $\mathrm{Ind}_{P_j}^{G_{r+j}}(\xi'\xi^{-1}\rho\otimes\tau)$.

Then, we have couple of exceptional cases:

THEOREM. Assume that $m_r\neq t_n$, so that the representations

$$N_E^{\frac{m_r-t_n-1}{2}}\xi'\rtimes\tau \text{ and } N_E^{-\frac{m_r-t_n+1}{2}}\xi\rtimes\sigma\cong N_E^{\frac{m_r-t_n+1}{2}}\xi\rtimes\sigma$$

are irreducible. If we additionally assume that $m_r-t_n\neq -1$, then the representation $\Theta(N_E^{-\frac{m_r-t_n+1}{2}}\xi\rtimes\sigma,r+1)$ is non–zero, it has a unique irreducible quotient, namely $\Theta(\sigma,r+1)$. Moreover, then we have $\Theta(\Theta(\sigma,r+1),n+1)=N_E^{-\frac{m_r-t_n+1}{2}}\xi\rtimes\sigma$. Also, we have $\Theta(\pi_2,n+1)=0$.

Totally symmetrically, we have

THEOREM. Assume that $m_r\neq t_n$ so that the representation $N_E^{\frac{m_r-t_n-1}{2}}\xi'\rtimes\tau$ is irreducible. Assume further that $m_r-t_n\neq 1$. Then, $\Theta(N_E^{\frac{m_r-t_n-1}{2}}\xi'\rtimes\tau,n+1)$ has a unique quotient, namely $\Theta(\tau,n+1)$. Moreover, we have $\Theta(\Theta(\tau,n+1),r+1)=N_E^{\frac{m_r-t_n-1}{2}}\xi'\rtimes\tau$. Also, we have $\Theta(\pi_1,r+1)=0$.

The last exceptional case is the following:

THEOREM. Assume that $m_r=t_n$. Then,

(i) $\Theta(\Theta(\tau,n+1),r+1)=\xi'N_E^{\frac{1}{2}}\rtimes\tau$, $\Theta(\Theta(\sigma,r+1),n+1)=\xi N_E^{\frac{1}{2}}\rtimes\sigma$.
(ii) We have $\Theta(\pi_1,r)=\Theta(\pi_2,n)=0$ and $\Theta(\pi_1,r{+}2)\neq 0$, $\Theta(\pi_2,n{+}2)\neq 0$. One of the following two situations occurs:
-$\Theta(\pi_1,r+1)\neq 0$ and every irreducible quotient of $\Theta(\pi_1,r+1)$ is π_2, and vice versa, $\Theta(\pi_2,n+1)\neq 0$, and every irreducible quotient of $\Theta(\pi_2,n+1)$ is π_1.

-$\Theta(\pi_1, r+1) = 0 = \Theta(\pi_2, n+1)$. Then, every irreducible quotient of $\Theta(\pi_1, r+2)$ is a unique (tempered) common irreducible subquotient of $N_E^{\frac{1}{2}}\xi' \rtimes \Theta(\sigma, r+1)$ and $\xi' St_{GL(2,E)} \rtimes \sigma$. In the same way, every irreducible quotient of $\Theta(\pi_2, n+2)$ is a unique tempered common irreducible subquotient of $N_E^{\frac{1}{2}}\xi \rtimes \Theta(\tau, n+1)$ and $\xi St_{GL(2,E)} \rtimes \tau$.

Now we briefly describe the content of this paper: in the Preliminaries section we recall of the unitary groups which we study, together with the way in which they form a reductive dual pair in a certain symplectic group. Then we describe cocycle defining the cover of this symplectic group, and write down the explicit splittings of the covers of groups in this dual pair, but under condition that the skew–hermitian unitary group is split reductive group. This constraint turns out to be of no importance later (Remark after Theorem 3.1). In the third section we calculate the filtration of the normalized Jacquet module $R_{P'_j}(\omega_{n,l,\psi})$ ($R_{P_j}(\omega_{l,n,\psi})$, respectively). In the fourth section, using the filtration of the previous section, we calculate certain isotypic components in the Jacquet module of $R_{P'_j}(\omega_{n,l,\psi})$ and $R_{P_j}(\omega_{l,n,\psi})$ (Proposition 4.3) and we obtain Theorem 4.4 where most of the cases of the relation between the representations $\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho \otimes \sigma)$ and $\mathrm{Ind}_{P_j}^{G_{r+j}}(\rho \otimes \tau)$ are covered. All of the remaining cases of ρ are covered in the fifth section.

2. Preliminaries

Let F be a non–archimedean field of characteristic zero. We fix a nontrivial additive character ψ of F. By γ_F we denote the Weil invariant acting on the characters of second degree (on F). It assumes values in the group of the eighth roots of 1, if we consider it as a one–variable function, and in the group of the fourth roots of 1 if we consider it as a two–variable function ([10, p. 231]). Let E be a quadratic extension of F and let τ be the non–trivial Galois automorphism. Let $W \cong E^{2n}$ (row vectors) be a vector space over E of dimension $2n$ with skew–hermitian form given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1^t y_2^\tau - y_1^t x_2^\tau,$$

and let

$$G'_n = G(W) = \{g \in GL_{2n}(E) : \langle w_1 g, w_2 g \rangle = \langle w_1, w_2 \rangle, \forall w_1, w_2 \in W\}$$

be the isometry group of W. More generally, W is a left vector space over a division algebra, so that the linear operators act on it from right, but since we only treat the quadratic field case, this is not of big importance. W has an obvious complete polarization $W = X + Y$, where $X = \{(x, 0) : x \in E^n\}$ and $Y = \{(0, y) : y \in E^n\}$. Let $(V, (\cdot, \cdot))$ be a vector space (right, if we wish) of dimension m over E with a non–degenerate hermitian form and let $G = G(V)$

be the isometry group of V. If we denote by $\mathrm{tr} : E \to F$ the reduced trace, then

$$\mathbf{W} = V \otimes_E W,\ \langle\langle \cdot, \cdot \rangle\rangle = \frac{1}{2}\mathrm{tr}\,((\cdot, \cdot) \otimes \langle \cdot, \cdot \rangle^{\tau})$$

is a symplectic vector space over F of dimension $4mn$. Then, there is a natural embedding $i : G'_n \times G \to Sp(\mathbf{W})$, so that (G'_n, G) is a dual reductive pair in $Sp(\mathbf{W})$.

We introduce the metaplectic group $Mp(\mathbf{W})$ as $\mathbb{C}^1$– extension

$$1 \to \mathbb{C}^1 \to Mp(\mathbf{W}) \to Sp(\mathbf{W}) \to 1.$$

This extension (not the usual two–fold central extension of the symplectic group) is better suited for our purposes since some subgroups of $Sp(\mathbf{W})$ split in this $Mp(\mathbf{W})$, and do not split in the two–fold central extension. We recall that the metaplectic group is equipped with the natural representation (the Weil representation) depending on the fixed additive character ψ ([13, Chapter 2]).

To describe the cocycle in the metaplectic group we need a notion of the Leray invariant. Now we follow closely the exposition in [11]. Let $\Omega = \Omega(\mathbf{W})$ denote the set of Lagrangiens of $\mathbf{W}$, i.e., the set of maximal isotropic (with respect to $\langle\langle \cdot, \cdot \rangle\rangle$) planes in $\mathbf{W}$. The symplectic group acts transitively on Ω and on the set of pairs $U_1, U_2 \in \Omega$ which are transverse ($U_1 \cap U_2 = \{0\}$). For any $U \in \Omega(\mathbf{W})$, by $P_U \subset Sp(\mathbf{W})$ we denote the stabilizer of U which is a maximal parabolic subgroup in $Sp(\mathbf{W})$, and by $N_U = \{g \in P_U : g|_U = \mathrm{id}\}$ its unipotent radical. To a given ordered triple $U_1, U_2, U_3 \in \Omega$ which are pairwise transverse there is associated n–dimensional F vector space $L = L(U_1, U_2, U_3)$ with the symmetric, non–degenerate, F–bilinear form $(\cdot, \cdot)_F$ which gives rise to a quadratic form on $L = L(U_1, U_2, U_3)$. In more words, (for the transverse triple) there exists a unique element $g \in N_{U_1}$ such that $U_2 g = U_3$, and, with respect to the complete polarization $\mathbf{W} = U_2 + U_1$, the matrix of the element g looks like $g = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, where $g \in \mathrm{Hom}(U_2, U_1)$. Since U_3 and U_2 are transverse, b is an isomorphism. We put $L(U_1, U_2, U_3) = U_2$ with a (non–degenerate) quadratic form defined by $q(x) = \frac{1}{2}\langle\langle x, xb \rangle\rangle$. So the Leray invariant attached to that triple is this quadratic space. For the triple (U_1, U_2, U_3) in which we do not assume that the isotropic subspaces are transverse in pairs, the definition of the Leray invariant is a bit more involved, and can be found in [16], or in [9], p. 12.

We have the following theorem of Rao and Perrin ([16],[15]).

THEOREM 2.1. *For any fixed* $\mathbf{Y} \in \Omega(\mathbf{W})$ *there is an isomorphism*

$$Mp(\mathbf{W}) \cong Sp(\mathbf{W}) \times \mathbb{C}^1$$

where

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 c_{\mathbf{Y}}(g_1, g_2))$$

with the cocycle $c_{\mathbf{Y}}$ *given by*

$$c_{\mathbf{Y}}(g_1,g_2)=\gamma_F(\frac{1}{2}\circ L(\mathbf{Y},\mathbf{Y}g_2^{-1},\mathbf{Y}g_1)).$$

To define an explicit splittings for the dual pair (G'_n,G), Kudla (in [11]) defined the Leray invariant for the hermitian spaces over E (originally it is defined for the symmetric, or better skew–symmetric spaces over F). The definitions are similar and we recall that W is a split skew–hermitian space over E, and $\Omega(W)$ denotes the set of maximal isotropic subspaces in W (with respect to $\langle\cdot,\cdot\rangle$). So, for the triple $U_1,U_2,U_3\in\Omega(W)$ the Leray invariant is defined analogously, so that $L_E(U_1,U_2,U_3)$ is a hermitian form over E of rank $n-r$ (where $r=\dim_E R=\dim_E((U_1\cap U_2)+(U_1\cap U_3)+(U_3\cap U_1))$, for U_1,U_2,U_3 in the general position) (we refer to [11], p. 367 to see that the obtained form $(\cdot,\cdot)_{L_E}$ on U_2 is indeed hermitian). Also, the Rao's function $x(g)$ (related to the Bruhat decomposition) has to be carried over from the skew–symmetric to the skew–hermitian case. Let $g\in G'_n$ be expressed as $g=p_1\tau_S p_2$, where $p_1,p_2\in P_Y$, where a parabolic subgroup P_Y is associated with the fixed complete polarization of $W=E^{2n}=X+Y$ and τ_S as described on p. 370 of [11]. We define $x(g)$ modulo NE^* as element in E^* given by $\det(p_1p_2|_Y)$. Since we have that $L_E(U_1,U_2,U_3)$ is a (left) vector space over E, by tensoring with the hermitian space V as above, we get a map

$$\begin{aligned}\mu_V:\{&\text{hermitian forms over } E \text{ of rank } k\ \}\to\\ &\{\text{symmetric forms over } F \text{ of rank } 2mk\}\end{aligned}$$

given by

$$L\mapsto V\otimes_E L,\ (\cdot,\cdot)_{\mu_V(L)}=\frac{1}{2}\mathrm{tr}\left((\cdot,\cdot)_V\otimes(\cdot,\cdot)_L^\tau\right).$$

Of course, we also have a map $R_V:\Omega(W)\to\Omega(\mathbf{W})$ given by $R_VU=V\otimes_E U$. For the construction of the exact splitting, we need the following Proposition (Proposition 0.1 form [11]).

Proposition 2.2. 1. *R_V is compatible with the Leray invariant, i.e.,*

$$\mu_V(L_E(U_1,U_2,U_3))=L(R_VU_1,R_VU_2,R_VU_3),$$

2. *for any $Y\in\Omega(W)$, let $\mathbf{Y}=R_VY$. Then*

$$c_{\mathbf{Y}}(i_V(g_1),i_V(g_2))=\gamma_F(\frac{1}{2}\psi\circ\mu_V(L_E(Y,Yg_2^{-1},Yg_1))).$$

Now we can state the form of the exact splitting ([11, Theorem 3.1 and Corollary 3.3]).

Theorem 2.3. *Assume that W and V are as above, with* $\dim_E V=m$. *Fix $Y\in\Omega(W)$ and let $R_VY=\mathbf{Y}$. For a fixed additive character ψ of F, let $\eta=\frac{1}{2}\psi$. Choose a character ξ of E^* whose restriction to F^* is $\epsilon_{E/F}^m$, where*

$\epsilon_{E/F}(x) = (x, \Delta)_F$ is the quadratic character corresponding to the extension E/F. For $g \in P\tau_j P \subset G'_n = G(W)$, let $x(g)$ be as defined above and let

$$\beta_V(g) = \xi(x(g))\gamma_F(\eta \circ RV)^{-j}.$$

Then,

$$c_{\mathbf{Y}}(i_V(g_1), i_V(g_2)) = \beta_V(g_1g_2)\beta_V(g_1)^{-1}\beta_V(g_2)^{-1},$$

so that the map $\tilde{i_V}(g) = (i_V(g), \beta_V(g))$ defines a splitting of the restriction to G'_n of the metaplectic cover. Here RV denotes, for V hermitian, of dimension m over E, the underlying $2m$–dimensional F–vector space with quadratic form $\frac{1}{2}\mathrm{tr}$, so that $\gamma_F(\eta \circ RV) = (\Delta, \det(V))_F\gamma_F(-\Delta, \eta)^m\gamma_F(-1, \eta)^{-m}$.

Recall that complete polarization $W = X + Y$ gives rise to a complete polarization $\mathbf{W} = V \otimes X + V \otimes Y$. Rao defines unitary operators on $S(V \otimes X) = S(V^n)$(row vectors of length n) which give rise to the Schrödinger model of the Weil representation ω_ψ (corresponding to the fixed additive character ψ of F) of $Mp(\mathbf{W})$. For a vector space X, from now on, $S(X)$ denotes the space of Schwartz functions on X.

Corollary 2.4. *The image of the group $G = G(V)$ under the embedding i_W lies in the Levi factor of the parabolic subgroup of $Sp(\mathbf{W})$ which stabilizes $V \otimes Y$. Since Rao's cocycle for $Sp(\mathbf{W})$ is trivial on this subgroup there is a natural splitting $G \to Mp(\mathbf{W})$ given by $h \mapsto (h, 1)$, and the resulting action of group $G = G(V)$ on $S(V^n)$ is just $\omega(h)\phi(x) = \phi(h^{-1}x)$.*

3. Filtration of Jacquet modules

We represent our skew–hermitian space W over E as a direct sum in the following way: $W = W_n = \mathrm{span}_E\{e_1, e_2, \ldots e_n, e'_1, \ldots, e'_n\} \oplus W_0$ where W_0 is anisotropic, and the vectors $\{e_1, e_2, \ldots e_n\}$ form a basis for a maximal isotropic subspace of W, and the rest of them for another maximal isotropic subspace. We also assume that this basis is chosen in such way that the vectors satisfy $\langle e_i, e'_j\rangle = \delta_{ij}$. For $j \in \{1, 2, \ldots, n\}$, let $W'_j = \mathrm{span}_E\{e_1, \ldots, e_j\}$ and $W''_j = \mathrm{span}_E\{e'_1, \ldots, e'_j\}$, with $W_0^j = (W'_j + W''_j)^\perp$ so that there is a decomposition $W = W'_j + W_0^j + W''_j$. Of course, $W_0^j \cong W_{n-j}$. From now on, $G'_n = G(W_n)$ denotes the unitary group attached to the skew–hermitian space W_n. Note that, contrary to the situation in the previous section, we do not assume that W_n is split, i.e., we allow $W_0 \neq 0$. Let P'_j be a parabolic subgroup of G'_n stabilizing W''_j. Then P'_j has a Levi decomposition $P'_j = M'_jN'_j$ with $M'_j \cong GL_j(E) \times G'_{n-j}$. Let V_l be a non–degenerate hermitian space (as in the previous section) of the split rank l. We denote by $G_l = G(V_l)$, i.e., unitary group preserving the hermitian form on V_l. In the same way as for the skew–hermitian space, we introduce vectors $\{v_1, \ldots, v_l\}$ which span one maximal isotropic subspace of V_l, and the vectors $\{v'_1, \ldots, v'_l\}$ which also span a disjoint maximal isotropic subspace. We assume that these vectors satisfy an analogous relations as the vectors e_i, e'_j. In the same way, we have a decomposition $V_l = V'_k + V_0^k + V''_k$ with $V_0^k \cong V_{l-k}$. We analogously define a maximal parabolic subgroup P_k of G_l attached to this decomposition. Let $m_l = \dim_E(V_l)$ and $t_n = \dim_E(W_n)$. We note that, even for general W_n (not necessarily split) there is a splitting i_{V_l} from G'_n to $Mp(\mathbf{W}) = Mp(V_l \otimes_E W_n)$, analogously there is a splitting i_{W_n} from G_l to $Mp(\mathbf{W}) = Mp(V_l \otimes_E W_n)$ obtained dually (there is a simple way of turning skew hermitian space into hermitian, and vice versa -[8]). The explicit formula for i_{W_n} in the general case is very involved ([11], Proposition 4.1 and Proposition 4.6), but in the rest of the paper we need explicitly the description of i_{W_n} (i_{V_l}, respectively) only when W_n (V_l, respectively) is split. If we denote by ω_ψ the Weil representation of $Mp(\mathbf{W}) = Mp(V_l \otimes_E W_n)$ (as in the previous section) with respect to the character ψ of F, we denote by $\omega_{l,n} = (\tilde{i_V}, \tilde{i_W})^*(\omega_\psi)$ the representation of $G_l \times G'_n$. From now on, we suppress ψ from the notation. In this section we explicitly calculate the filtration of the Jacquet module $R_{P'_j}(\omega_{l,n})$ (i.e., Jacquet module of the representation $\omega_{l,n}$ with respect to the parabolic subgroup P'_j). A general form of this filtration in terms of covering groups (of the groups in the dual pair) is known ([13]); we write down (a very similar) proof for the expression of this filtration for the unitary groups, but keeping in mind the explicit splittings constructed in [11] (and thus obtain the results for the unitary groups in the dual pair, and not the covering groups). Because of the completeness, we write down the whole proof. We follow the argument of Kudla in [10], but adjusting it when needed. For a quadratic matrix $A \in M_n(E)$, A^* denotes the matrix which is obtained by transposing a matrix A, and then letting the non-trivial Galois element $\tau \in \mathrm{Gal}(E/F)$ act on each matrix element.

THEOREM 3.1. *Let W_n and V_l be a skew–hermitian and hermitian space, respectively, as described above. Then, the normalized Jacquet module $R_{P'_j}(\omega_{l,n})$ has the following $G_l \times M'_j$–invariant filtration:*

$$R_{P'_j}(\omega_{n,l}) = \tau_j'^{(0)} \supset \tau_j'^{(1)} \supset \cdots \supset \tau_j'^{(r)} \supset \{0\}.$$

Here $r = \min(l, j)$, but we need only to consider the case $l \geq j$ so we continue to assume $r = j$. The successive quotients $\tau'_{jk} = \tau_j'^{(k)} / \tau_j'^{(k+1)}$ are described as follows:

$$\tau'_{jk} \cong \mathrm{Ind}_{P_k \times R_{jk} \times G'_{n-j}}^{G_l \times GL_j \times G'_{n-j}} \beta_{jk} \otimes \Sigma_k \otimes \omega_{l-k,n-j},$$

where β_{jk} is a character on $GL_k(E) \times GL_{j-k}(E) \times GL_k(E) \subset P_k \times R_{jk}$ defined as follows

$$\beta_{jk}(a, g', g) = N_E(a)^{\frac{m_l-k}{2}} N_E(g')^{\frac{m_l-k-t_n+j}{2}} \beta_V(g') N_E(g)^{\frac{m_l-k}{2}} \beta_V(g) \beta_{W_0^j}(g^{-*}),$$

and Σ_k is the usual action of $GL_k(E) \times GL_k(E)$ on $S(GL_k(E))$ given by $\Sigma_k(a, g)\phi(h) = \phi(a^* h g)$. Here R_{jk} is a maximal parabolic subgroup of

$GL_j(E)$ with the Levi subgroup isomorphic to $GL_{j-k}(E) \times GL_k(E)$; $\beta_V(g')$ and $\beta_V(g)$ correspond to the embedding of $a \in GL_j(E) \subset G(W_j' \oplus W_j'')$ into $Sp(V \otimes_E \widetilde{(W_j' \oplus W_j'')})$, and $\beta_{W_0^j}$ corresponds to the embedding of $a \in GL_k(E) \subset G\widetilde{(V_k' \oplus V_k'')}$ into $Sp((V_k' \oplus V_k'') \otimes_E W_0^j)$.

REMARK 3.2. 1. We note that for the filtration above (apart from the further explicit calculation of $\omega_{l-k,n-j}$ which is not pursued in this theorem) we do not need the full formula for the explicit splitting $\beta_V : G_n' \to Mp(V_l \otimes_E W_n)$. Namely, β_V stands for the embedding of $G(W_j' \oplus W_j'')$ (which is split so the assumptions of Theorem 2.3 are satisfied) in $Mp(V_l \otimes_E (W_j' + W_j''))$.

2. Let ξ be a character of E^* whose restriction to F^* is $\epsilon_{E/F}^{m_l}$, where $\epsilon_{E/F}(x) = (x, \Delta)_F$ is the quadratic character related to the extension E/F and ξ' a character of E^* whose restriction to F^* is $\epsilon_{E/F}^{t_n}$. We can easily see, using Theorem 2.3 and notation there, that $\beta_V(g) = \xi(\det g)$ and $\beta_{W_0^j}(a) = \xi'(\det a)$. All the results here depend on the choice of splitting, namely on ξ and ξ'.

PROOF. Let S denote the model of the representation $\omega_{l,n}$. A direct sum $W = W_j' + W_0^j + W_j''$, when tensored by V_l, gives rise to the direct sum $\mathbf{W} = \mathbf{W}_j' + \mathbf{W}_0^j + \mathbf{W}_j''$. We know that we can realize S as a mixed model, $S \cong S(V_l^j) \otimes S^0$, where S^0 is a model for $\omega_{l,n-j}$, and $V_l^j \cong \mathbf{W}_j' = V \otimes W_j'$. Every element of $V_l \otimes W_j'$ can be written as $\overline{v_1} \otimes e_1 + \cdots + \overline{v_j} \otimes e_j$, where $\overline{v_i} \in V$, $i = 1, \ldots, j$, so the map $\overline{v_1} \otimes e_1 + \cdots + \overline{v_j} \otimes e_j \mapsto (\overline{v_1}, \ldots, \overline{v_j})$ fixes this isomorphism. If there exists some non–degenerate pairing between vector spaces X' and some X'^*, and the same for Y' and Y'^*, then to $f \in \mathrm{Hom}(X', Y')$ we can attach $f^* \in \mathrm{Hom}(Y'^*, X'^*)$ in an obvious way. In future, we identify $W_j'^*$ with W_j'' through $\langle \cdot, \cdot \rangle$. Every $n \in N_j'$ (the unipotent radical of the parabolic subgroup P_j' of the unitary group G_n') can be written (in a unique way) as $n = n_1(s)n_2(h)$, as in [13], p. 24, where

$$n_1(s) = \begin{bmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with $s \in \mathrm{Hom}_E(W_j', W_j'')$, $s^* = -s$, and

$$n_2(h) = \begin{bmatrix} 1 & h & -hh^*/2 \\ 0 & 1 & -h^* \\ 0 & 0 & 1 \end{bmatrix},$$

with $-h^* \in \mathrm{Hom}_E(W_0^j, W_j'')$. All the $n_1(s)$, $s \in \mathrm{Hom}_E(W_j', W_j'')$ form a subgroup of N_1 of N_j', which is, when $W_0 \neq 0$, a commutator subgroup of N_j'. We want to calculate the space of coinvariants of $\omega_{l,n}$ with respect to

N'_j. First we calculate the space of coinvariants $(\omega_{l,n})_{N_1}$. The latter space has a nice description through a certain filtration, because of the quite general arguments ([13, pp. 72 and 74]) since the group N_1 is abelian and acts on $S(V_l^j)\otimes S^0$ through a character:

$$\omega_{l,n}(n_1(s))\phi(v\otimes w)=\psi(\frac{1}{4}\mathrm{tr}_{E/F}((v,v)\langle sw,w\rangle^\tau))\phi(v\otimes w),$$

if $w\in W'_j, v\in V_l, \phi\in S(V^j)\otimes S^0$. So, there is an isomorphism of $G_l\times P'_j$–modules

$$S_{N_1}\xrightarrow{\sim} T=S(X_0)\otimes S^0,$$

where $X_0=\{(\overline{v_1},\overline{v_2},\ldots,\overline{v_j}):\mathrm{span}_E\{\overline{v_1},\ldots,\overline{v_j}\}\text{ is isotropic}\}$. This isomorphism is given by the restriction to the (closed) subspace X_0 of V^j. We have $X_0=\cup X_{0k}$, where, in this disjoint union,

$$X_{0k}=\{(\overline{v_1},\overline{v_2},\ldots,\overline{v_j})\in X_0:\dim_E\mathrm{span}_E\{\overline{v_1},\ldots,\overline{v_j}\}=k\}.$$

This decomposition leads to a $G_l\times P'_j$–invariant filtration

$$T=T^{(0)}\supset T^{(1)}\supset\cdots\supset T^{(j)}\supset\{0\},$$

where the successive quotients $T_k=T^{(k)}/T^{(k+1)}$ have the form $T_k\cong S(X_{0k})\otimes S_0$. Now we want to identify how $M'_j\times G_l$ acts on T_k, and then take into the consideration the action of N'_j. Using the formulas for the unitary operators acting on $S(V_l^j)$ in the mixed model ([13], p. 41) and the splitting of Theorem 2.3, we get

$$\omega_{l,n}(a)\phi(v\otimes x)=\beta_V(a)N_E(a)^{m/2}\phi(v\otimes xa), \tag{3.1}$$

where $\phi\in S(V^j)=S(V\otimes W'_j),\ v\otimes x\in X_{0k},\ a\in GL_j(E)$, and we recall that N_E is a composition of the reduced norm on $GL_j(E)$ with the norm map on F^*. We denote the character $\beta_V N_E^{m/2}$ of $GL_j(E)$ appearing above by χ.

As for the action of $G'_{n-j}\subset M'_j$, we note that $x(1,h)=x(h)$, where x is the function from Theorem 2.3. Here $x(h)$ is obtained when we view h as an element of G'_{n-j}, and $(1,h)$ we view as an element of G'_n via an obvious map $G'_j\times G'_{n-j}\to G'_n$. This map is similar to the one in the symplectic case, described in detail in [7], Section 3. The same holds for the proof that $x(1,h)=x(h)$. This also means that $\beta_V(h)=\beta_V((1,h))$. This ensures that, for $h\in G'_{n-j}$ we have

$$\omega_{l,n}(h)\phi(v\otimes x)=\omega^0(\tilde{i_V}(h))(\phi(v\otimes x)), \tag{3.2}$$

where ω^0 is the Weil representation on S^0, and, on the right–hand side of the relation above, $\tilde{i_V}:G'_{n-j}\to Sp(V\otimes W_0^j)$ is the splitting analogous to the one defined in the previous section (but the target space is smaller).

For the action of the group G_l we get:

$$\omega_{l,n}(g)\phi(v\otimes x)=\omega^0(\tilde{i}_{W_0^j}(g))\phi(g^{-1}v\otimes x). \tag{3.3}$$

If we denote by $K = M'_j \times G_l = GL_j(E) \times G_l \times G'_{n-j}$, from equations (3.1),(3.2),(3.3) ($a \in GL_j(E), g \in G_l, h \in G'_{n-j}$) we get

$$\omega_{n,l}|_K(a,g,h) = N_E(a)^{m/2}\beta_V(a)\mu \otimes \omega^0,$$

where μ is the natural action of $GL_j(E)\times G_l$ on $S(X_{0k})$ induced by translation on X_{0k}.

Now, following Kudla ([10]) we fix element $x_0 \in X_{0k}$, where $x_0 = (\dots, v'_1, v'_2, \dots, v'_k)$, and $v_1,' \dots, v'_l$ are as at the beggining of this section. Note that X_{0k} is exactly an orbit of x_0 for the $G_l \times GL_j(E)$–action described above. Assume that G'_{n-j} acts trivially on X_{0k}. We denote by H the stabilizer of x_0 in K with respect to this action. We define a representation τ of $H \times N'_j$ in S^0 in the following way: $\tau = \chi\omega^0$, where χ (and representation ω^0) act (as representations of H) in the way described above. On the other hand, we define $\tau|_{N'_j}$ in the following way: if $n = n_1(s)n_2(h)$ in the way described above, then $\tau(n) = \rho_0((x_0h, 0))$, where ρ_0 is the usual action of the Heisenberg group $\mathbf{W}_0^j = V_l \otimes W_0^j$ in S^0. Now the following steps are straightforward:

1^{st} *step*

$$\omega|_K \cong \mathrm{ind}_H^K(\tau|_H).$$

Of course, on the left–hand side we mean the action of K on T_k, and on the right–hand side we have the non-normalized induction, and the isomorphism is obtained through the mapping $T : \omega|_K \to \mathrm{ind}_H^K(\tau|_H)$ given by $T\phi(g) = (\chi\omega^0)(g)\phi(g^{-1}x_0)$, $g \in K$. The proof that this mapping is indeed K–intertwining and its surjectivity is straightforward, and for the injectivity we just note that X_{0k} is the orbit of x_0 under the $GL_j(E) \times G_l$–action.

2^{nd} *step*

$$\mathrm{ind}_H^K(\tau|_H) \cong (\mathrm{ind}_{H\ltimes N'_j}^{K\ltimes N'_j}(\tau))|_K.$$

Here the isomorphism is obtained through the mapping $T : \mathrm{ind}_H^K(\tau|_H) \to (\mathrm{ind}_{H\ltimes N'_j}^{K\ltimes N'_j}(\tau))|_K$ given by $T(f)(k,n) = \tau(1,n)f(k)$, $k \in K, n \in N'_j$. The proof that $T(f) \in (\mathrm{ind}_{H\ltimes N'_j}^{K\ltimes N'_j}(\tau))|_K$ and that T is K–intertwining is immediate.

3^{rd} *step*

$$\omega \cong \mathrm{ind}_{H\ltimes N'_j}^{K\ltimes N'_j}(\tau).$$

Of course, on the left–hand side we continue to assume the action of the Weil representation of $K \ltimes N'_j$ on T_k. According to the 2^{nd} step, we are left to verify that the mapping $T_1 : S(X_{0k}) \otimes S^0 \to \mathrm{ind}_{H\ltimes N'_j}^{K\ltimes N'_j}(\tau)$ given by

$$(T_1\phi)(k,n) = \tau(1,n)(\xi\omega^0)(k)\phi(k^{-1}x_0)$$

is N'_j–intertwining. This follows when we track down the definitions (right multiplications on W'_j and left on V), but keeping in mind the conjugation relations for the elements $n_1(s)$ and $n_2(h)$ of N'_j ([13, p. 25]).

Now we want to determine the space of coinvariants. We do that in the following lemma.

LEMMA 3.3. *We have*

$$(\mathrm{ind}_{H\ltimes N_j'}^{K\ltimes N_j'}(\tau))_{N_j'} \cong \mathrm{ind}_H^K(\tau_{N_j'}).$$

PROOF. We denote by $S^0[N_j'] = \mathrm{span}\{\tau(n)v - v : n \in N_j',\ v \in S^0\}$. The isomorphism of this lemma is constructed through the homomorphism $T : \mathrm{ind}_{H\ltimes N_j'}^{K\ltimes N_j'}(\tau) \to \mathrm{ind}_H^K(\tau_{N_j'})$ given by $T(F)(k) = F(k,1) + S^0[N_j']$. It is straightforward to check that this is K–intertwining. Only thing to check is to find out that the kernel of this map is precisely $\mathrm{ind}_{H\ltimes N_j'}^{K\ltimes N_j'}(\tau)[N_j']$. One inclusion (that the latter set is in the kernel) is trivial; for the other we proceed like in the proof of Lemma 3.4.

$\square$

Next question is how to describe $\tau_{N_j'}$ more precisely. Since τ acts on the representation space of $\omega_{l,n-j}$, (and this is S^0) we use the mixed model to represent it, now using the decomposition $V_l = V_k' \oplus V_k^0 \oplus V_k''$. By tensoring with W_0^j, we obtain

$$\mathbf{W}^0 = V_k' \otimes W_0^j \oplus V_k^0 \otimes W_0^j \oplus V_k'' \otimes W_0^j,$$

so that we have $S^0 \cong S((W_0^j)^k) \otimes S^{00}$, where (ω_{00}, S^{00}) is a model for $\omega_{l-k,n-j}$. In this model, we can describe ρ_0 (which appears for us in the description of $\tau(n)$, $n \in N_j'$). Using formula for the mixed model of the representation of the Heisenberg group and an explicit formula for the action of Heisenberg group ([13], p. 30), we see that for $\phi \in S((W_0^j)^k) \otimes S^{00}$, $z \in (W_0^j)^k$, $h \in \mathrm{Hom}_E(W_j', W_0^j)$ we have

$$\rho_0((x_0h, 0))\phi(z) = \psi(\langle\langle z, x_0h\rangle\rangle)\phi(z). \tag{3.4}$$

This formula leads to the description of $S^0_{N_j'}$.

LEMMA 3.4. *There is a natural homomorphism*

$$S^0_{N_j'} \cong S^{00},$$

given by the homomorphism $S^0 \to S^{00}$, $\phi \mapsto \phi(0)$, *where* $0 \in (W_0^j)^k = V_k' \otimes W_0^j$.

PROOF. If we denote the map $\phi \mapsto \phi(0)$ by T, it is obvious that this map is surjective, and that $S^0[N_j'] \subset \mathrm{Ker}T$. To show the other inclusion, we assume that $\phi \in S^0$ is such that $\phi(0) = 0$. Fix $z_0 \neq 0$ in formula (3.4). Then, the mapping $h \mapsto \psi(\langle\langle z_0, x_0h\rangle\rangle)$ is an additive smooth character, say ψ_{z_0}, on $\mathrm{Hom}_E(W_j', W_0^j)$. This character is non–trivial. Using matrix realization of

h using skew–hermitian basis for W_j', and for W_0^j and corresponding tensor expression for z_0, we easily obtain the following: since $\phi(0) = 0$, there is a neighborhood A of 0 described, for some $\delta > 0$, with $A = \{z \in V_k' \otimes W_0^j : \|z\| \leq \delta\}$, where $\|\cdot\|$ is appropriate non–archimedean norm on $V_k' \otimes W_0$, such that $\phi(z) = 0$, for $z \in A$. Here the norm $\|\cdot\|$ is taken component-wise, as a max-norm of the components of a matrix when when we expand z as tensor using skew–hermitian basis for W_0^j, and for V_k'. Also, since ψ is smooth, there is $\delta_1 > 0$, such that there exists $\alpha_0 \in F$, $|\alpha|_F = \delta_1$, and $\psi(\alpha_0) \neq 1$.

Note that we can also introduce an appropriate analogous non–archimedean norm on $\mathrm{Hom}_E(W_j', W_0^j)$, so that a basis of neighborhood of zero there is formed from the open compact subgroups of the form

$$N_\epsilon = \{h \in \mathrm{Hom}_E(W_j', W_0^j) : \|h\| \leq \epsilon\}.$$

On the other hand, if $\|z_0\| > \delta$, we can easily, using matrix description of the elements of $\mathrm{Hom}_E(W_j', W_0^j)$, find an element $h \in \mathrm{Hom}_E(W_j', W_0^j)$ such that $\psi_{z_0}(h) \neq 1$ and $\|h\| \leq \delta_1/\delta$. This guarantees

$$\int_{N_{\delta_1/\delta}} \psi_{z_0}(h)\mathrm{d}h = 0 = \int_{N_{\delta_1/\delta}} \rho_0((x_0 h, 0))\phi(z_0)\mathrm{d}h$$

by formula (3.4). On the other hand, the last expression is also valid for $\|z_0\| \leq \delta$, since then $\phi(z_0) = 0$. Since the last expression is valid for every z_0, we have $\int_{N_{\delta_1/\delta}} \rho_0((x_0 h, 0))\phi \mathrm{d}h = 0$ so, by ([5], p. 33) $\phi \in S^0[N_j']$ (here we just have to adjust measures on N_j' and on the set $\{n_2(h) : h \in \mathrm{Hom}_E(W_j', W_0^j)\}$ and note that τ is a representation of N_j' only through $\mathrm{Hom}_E(W_j', W_0^j)$, as the defining formula for $\tau(n)$ shows. ∎

Now, to describe $(\omega_{K \ltimes N_j'})_{N_j'} \cong \mathrm{ind}_H^K(\tau_{N_j'})$ (the former representation acting on T_k) in more familiar terms, we want to express the latter representation as a representation of $K \cong GL_j(E) \times G_{n-j}' \times G_l$ induced from a parabolic subgroup. We do that by embedding H in a parabolic subgroup, extending τ on this subgroup, and then use the transitivity of induction. First, by the natural action of $GL_j(E) \times G_l$ on $X_{0k} \subset V_l \otimes W_j'$, we see that the stabilizer of $x_0 = \sum_{i=1}^k v_i' \otimes e_{j-k+i}$, has to preserve $\mathrm{span}_E\{e_{j-k+1}, \ldots, e_j\}$ and $\mathrm{span}_E\{v_1', \ldots, v_k'\}$, so $H \subset P_k \otimes R_{jk} \otimes G_{n-j}'$, where P_k is a maximal parabolic subgroup of G_l, stabilizing $\mathrm{span}_E\{v_1', \ldots, v_k'\}$, and R_{jk} maximal parabolic subgroup of $GL_j(E)$ stabilizing the last k vectors of this fixed basis of W_j'. Moreover, since the identification of the standard Levi subgroup of P_k with $GL_k(E) \times G_{l-k}$ was via the action on the vectors $\{v_1, \ldots, v_j\}$ (and not $\{v_1', \ldots, v_j'\}$), we can explicitly describe H as

$$H = \{(p, g, g') \in P_k \times R_{jk} \times G_{n-j}' : \mathrm{pr}(p) = \mathrm{pr}(g)^{-*}\}.$$

Here the first pr stands for the projection on $GL_k(E)$–part of the Levi subgroup isomorphic to $GL_k(E) \times G_{l-k}$ of P_k, and the second pr for the projection on $GL_k(E)$–part of the Levi subgroup of $GL_j(W_j') \cong GL_j(E)$ isomorphic to $GL_{j-k}(E) \times GL_k(E)$.

We note that the explicit description of the action of H on S^{00} is given as follows: for $v \in S^{00}$, let $\phi \in S((W_0^j)^k) \otimes S^{00}$ be such that $\phi(0) = v$. Then, $h \cdot v = (h\phi)(0)$, where the action of H on S^0 is, of course, given by $\tau|_H$. This gives us, for $v \in S^{00}$, the following descriptions of $\tau_{N_j'}$ on certain subgroups of H :

- $a \in GL_k(E)(\subset P_k)$, $\tau_{N_j'}(a)v = \beta_{W_0^j}(a)' N_E(a)^{-\frac{\dim_E W_0^j}{2}} v$,
- $g \in G_{l-k}(\subset P_k)$, $\tau_{N_j'}(g)v = \omega^{00}(g, \beta''_{W_0^j}(g))v$,
- $a \in GL_j(E)(\subset G_n')$, $\tau_{N_j'}(a)v = \beta_V(a) N_E(a)^{m/2} v$,
- $h_{n-j} \in G'_{n-j}$, $\tau_{N_j'}(h_{n-j}) = \omega^{00}(\beta_{V_k^0}(h_{n-j}), h_{n-j})(v)$.

Here $\beta'_{W_0^j}$ corresponds to the embedding $G(V_k' \oplus V_k'') \otimes id \hookrightarrow \mathrm{Sp}((V_k' \oplus V_k'') \otimes W_0^j)$, $\beta''_{W_0^j}$ to the embedding $G(V_k^0) \otimes id \hookrightarrow \mathrm{Sp}(V_k^0 \otimes W_0^j)$, analogously for $\beta_{V_k^0}$. $\tau_{N_j'}$ acts trivially on N_j' (of course) and on $N_k \subset P_k$, as can be checked in ([13], p. 41 and 42).

On the other hand, a system of representatives of $H \setminus P_k \times R_{jk} \times G'_{n-j}$ is given by the set $\{(x^*, 1, 1) \in P_k \times R_{jk} \times G'_{n-j} : x \in GL_k(E)\}$. As mentioned above, we want to describe

$$\mathrm{ind}_H^K(\tau_{N_j'}) \cong \mathrm{ind}_{P_k \times R_{jk} \times G'_{n-j}}^{G_l \times GL_j(E) \times G'_{n-j}}(\mathrm{ind}_H^{P_k \times R_{jk} \times G'_{n-j}}(\tau_{N_j'})).$$

By the restriction to the above mentioned set of representatives, we can realize $\pi_{jk} = \mathrm{ind}_H^{P_k \times R_{jk} \times G'_{n-j}}(\tau_{N_j'})$ on the set $S(GL_k(E)) \otimes S^{00}$, indeed, the isomorphism $T : \mathrm{ind}_H^{P_k \times R_{jk} \times G'_{n-j}}(\tau_{N_j'}) \to S(GL_k(E)) \otimes S^{00}$ is given by $Tf(x) = f((x^*, 1, 1))$, where $x \in GL_k(E)$. In this way, we carry over the action of $P_k \times R_{jk} \times G'_{n-j}$ on $S(GL_k(E)) \otimes S^{00}$. To get the final formulas, we describe this action in detail; for $\phi \in S(GL_k(E)) \otimes S^{00}$, $x \in GL_k(E)$ we have:

- for $(a, g) \in GL_k(E) \times GL_k(E) \subset P_k \times R_{jk}$,
$$\pi_{jk}(a, g)\phi(x) = N_E(g)^{m/2 + \frac{\dim_E W_0}{2}} \beta_V(g) \beta'_{W_0^j}(g^{-*})' \phi(a^* x g),$$
- for $g_{l-k} \in G_{l-k}(\subset P_k)$,
$$\pi_{jk}(g_{l-k})\phi(x) = \omega^{00}(g_{l-k}, \beta''_{W_0^j}(g_{l-k}))\phi(x),$$
- for $g' \in GL_{j-k}(E)(\subset R_{jk})$
$$\pi_{jk}(g')\phi(x) = N_E(g')^{m/2} \beta_V(g')\phi(x),$$

- for $h_{n-j} \in G'_{n-j}$

$$\pi_{jk}(h_{n-j})\phi(x) = \omega^{00}(\beta_{V_k^0}(h_{n-j}), h_{n-j})\phi(x).$$

The unipotent radicals of P_k and R_{jk} act trivially. Finally, to get normalized parabolic induction, we have to take into the account appropriate modular functions. We have $N_E(a)^{m-k}$, for $a \in GL_k(E)$ as a modular function of P_k in G_l (but note that we have used the "lower" GL–block for the identification, so we should take $N_E(a)^{-(m-k)}$), $N_E(g')^k \otimes N_E(g)^{-(j-k)}$ for $(g', g) \in GL_{j-k}(E) \times GL_k(E) \subset R_{jk}$. We also take into the account the normalization of the Jacquet functor, from the beginning, i.e., we originally wanted to calculate the normalized Jacquet functor of $R_{P'_j}(\omega_{l,n})$, so we have to multiply the final result by $\delta_{P'_j}^{-\frac{1}{2}}$, where $\delta_{P'_j}(a) = N_E(a)^{\dim_E W_n - j}$, $a \in GL_j(E)$. $\square$

We need the previous theorem in the following form.

Corollary 3.5. *Let $W = W_{n+j}$ be a skew–hermitian vector space of split-rank $n+j$ and $V = V_{l+j}$ a hermitian vector space of split rank $l+j$. Then, the normalized Jacquet module $R_{P'_j}(\omega_{l+j,n+j})$ has the following filtration*

$$R_{P'_j}(\omega_{l+j,n+j}) = \tau_j'^{(0)} \supset \tau_j'^{(1)} \supset \cdots \supset \tau_j'^{(j)} \supset \{0\},$$

where the successive quotients $\tau'_{jk} = \tau_j'^{(k)}/\tau_j'^{(k+1)}$ are described as follows:

$$\tau'_{jk} \cong \mathrm{Ind}_{P_k \times R_{jk} \times G'_n}^{G_{l+j} \times GL_j \times G'_n}(\beta_{jk} \otimes \Sigma'_k \otimes \omega_{l+j-k,n}),$$

where β_{jk} is a character on $GL_k \times GL_{j-k} \times GL_k \subset P_k \times R_{jk}$ defined as follows

$$\beta_{jk}(a, g', g) = N_E(g')^{\frac{m_l - t_n}{2} + \frac{j-k}{2}} \xi(\det g')\xi(\det g)\xi'(\det g).$$

Here Σ'_k is a twist of the usual action of $GL_k(E) \times GL_k(E)$ on $S(GL_k(E))$ given by

$$\Sigma'_k(a, g)f(h) = N_E(a)^{-(\frac{m_l}{2} + j - \frac{k}{2})} N_E(g)^{\frac{m_l}{2} + j - \frac{k}{2}} f(a^{-1}hg)$$

(hence the change of sign in the exponent of $N_E(a)$ in comparison to the previous theorem). Specifically, the subrepresentation equals

$$\tau'_{jj} \cong \mathrm{Ind}_{P_j \times GL_j \times G'_n}^{G_{l+j} \times GL_j \times G'_n}(\xi\xi'|_{GL_j} \Sigma'_j \otimes \omega_{l,n}),$$

and quotient equals $\tau'_{j0} \cong N_E^{\frac{m_l - t_n}{2} + \frac{j}{2}}(\xi \circ \det) \otimes \omega_{l+j,n}$.

We know state the analogous corollary for $R_{P_j}(\omega_{n+j,l+j})$, where know P_j is a maximal parabolic subgroup of $G(V_{l+j})$ with a Levi subgroup isomorphic to $GL(j, E) \times G(V_l)$. The proof of this proposition is analogous to the proof of Theorem 3.1.

Proposition 3.6. *Let $W = W_{n+j}$ be a skew–Hermitian vector space of split-rank $n+j$ over E and $V = V_{l+j}$ a hermitian vector space of split rank $l+j$. Then, the normalized Jacquet module $R_{P_j}(\omega_{l+j,n+j})$ has the following filtration*

$$R_{P_j}(\omega_{n+j,l+j}) = \tau_j^{(0)} \supset \tau_j^{(1)} \supset \cdots \supset \tau_j^{(j)} \supset \{0\},$$

where the successive quotients $\tau_{jk} = \tau_j^{(k)}/\tau_j^{(k+1)}$ are described as follows:

$$\tau_{jk} \cong \mathrm{Ind}_{R_{jk}\times G_l \times P'_k}^{GL_j \times G_l \times G_{n+j}}(\gamma_{jk} \otimes \Sigma'_k \otimes \omega_{l,n+j-k}),$$

where R_{jk} is a parabolic subgroup of $GL_j(\subset M_j)$ with Levi subgroup isomorphic to $GL_{j-k} \times GL_k$ and γ_{jk} is a character of $GL_{j-k} \times GL_k \times GL_k \subset R_{jk} \times P'_k$ given by

$$\gamma_{jk}(g, a, g') = N_E(g)^{-\frac{m_l - t_n + k - j}{2}} \xi'(\det g)\xi'(\det g')\xi(\det g'),$$

and Σ'_k is a twist of the usual action of $GL_k(E) \times GL_k(E)$ on $S(GL_k(E))$ given by $\Sigma'_k(a, g')f(h) = N_E(a)^{-\frac{m_l+k}{2}} N_E(g')^{\frac{m_l+k}{2}} f(a^{-1}hg')$. Specifically, the subrepresentation equals

$$\tau_{jj} = \mathrm{Ind}_{GL_j \times G_l \times P'_j}^{GL_j \times G_l \times G_{n+j}}(\xi\xi'|_{GL_j}\Sigma'_j \otimes \omega_{l,n})$$

and the quotient equals $\tau_{j0} = N_E^{-\frac{m_l - t_n - j}{2}}(\xi' \circ \det) \otimes \omega_{l,n+j}$.

4. Theta correspondence and isotypic components

We have defined, for unitary groups G'_n and G_l as above, the pull-back of the representation ω_ψ of the metaplectic group $\widetilde{Sp(V_l \otimes_E W_n)}$ to the product $G_l \times G'_n$, using splittings from the previous sections, and we denoted this representation by $\omega_{l,n}$. We note that this representation depends on the additive character ψ and on the choice of characters ξ and ξ' defined in Remark after Theorem 3.1. For an irreducible, smooth representation π_1 of G'_n, let $\Theta(\pi_1, l)$ be a smooth representation of G_l given as the full lift of π_1 to the l–level of the hermitian tower (in question), i.e., the biggest quotient of $\omega_{l,n}$ on which G'_n acts as a multiple of π_1. It is of the form $\pi_1 \otimes \Theta(\pi_1, l)$, as a representation of $G'_n \times G_l$ ([9], p. 33, [13], p. 45). Analogously, for an irreducible, smooth representation π_2 of G_l, let $\Theta(\pi_2, n)$ be a smooth representation of G'_n given as the full lift of π_2 to the n–level of the skew–hermitian tower.

We fix some notation throughout this section. Let σ be an smooth, irreducible, cuspidal representation of G'_n and let $\Theta(\sigma, r)$ be the first (full) nontrivial lift of σ in the hermitian tower. Then, $\Theta(\sigma, r)$ is an irreducible cuspidal representation of G_r and we will denote it by τ. Let ρ denote an irreducible cuspidal representation of $GL_j(E)$. For the calculation of the certain isotypic components, we use the following well–known facts.

Recall that $M_j \cong GL_j(E) \times G_{l-j}$ ($M'_j \cong GL_j(E) \times G'_{n-j}$, respectively) is a Levi subgroup of a maximal parabolic subgroup of G_l (G'_n, respectively). As such, it has a character N_E (on $GL_j(E)$–part).

LEMMA 4.1 ([2, Proposition 26]). *Let π be an irreducible cuspidal representation of M_j, and let V be a smooth representation of M_j. Then, there exist two subrepresentations of V, say $V(\pi)$ and $V(\pi)^{\perp}$, such that we have*

$$V = V(\pi) \oplus V(\pi)^{\perp},$$

and all the subquotients of $V(\pi)$ are isomorphic to πN_E^s, for some $s \in \mathbb{C}$ and $V(\pi)^{\perp}$ does not have an irreducible subquotient isomorphic to some πN_E^s; $s \in \mathbb{C}$. The analogous decomposition holds for the representations of M'_j.

We know give the statement of the second Frobenius reciprocity (the original Bernstein argument appeared in [3]; there is an alternative proof due to Bushnell ([4])).

LEMMA 4.2. *Let G be G'_n or G_l. Let $P = MN$ be a standard parabolic subgroup of G and let $\overline{P} = M\overline{N}$ be the opposite parabolic subgroup. Assume π is a smooth representation of M and Π is a smooth representation of G. Then, the following holds*

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G(\pi), \Pi) \cong \mathrm{Hom}_M(\pi, R_{\overline{P}}(\Pi)).$$

Let ρ be an irreducible cuspidal representation of $GL_j(E)$, and σ and τ irreducible cuspidal representations of G'_n and G_r, related by theta correspondence, as explained above. To finally relate the reducibility of the representation $\mathrm{Ind}_{P_j}^{G'_{n+j}}(\rho N_E^s \otimes \sigma)$ with the reducibility of the representation $\mathrm{Ind}_{P'_j}^{G_{r+j}}(\rho N_E^s \otimes \tau)$, we use the same basic approach as in [6]. Namely, we identify certain isotypic components in the filtration of $R_{P'_j}(\omega_{r+j,n+j})$, which enable us to relate the reducibilities in question in most cases, i.e., if ρN_E^s is not the one dimensional representation appearing as a GL–part of the quotient of the filtration $R_{P'_j}(\omega_{r+j,n+j})$ (Corollary 3.5). In general, if π is an irreducible smooth representation of some group G_1, and Π a smooth representation of $G_1 \times G_2$, then the isotypic component (a smooth representation of G_2) of π in Π is denoted by $\Theta(\pi_1, \Pi)$ (when it is obvious what are G_1 and G_2).

PROPOSITION 4.3. 1. *Assume that $j > 1$ and $s \in \mathbb{C}$. Then*

$$\mathrm{Hom}_{M'_j}(R_{P'_j}(\omega_{r+j,n+j})/\tau'_{jj}, \rho N_E^s \otimes \sigma) = 0$$

and

$$\mathrm{Hom}_{M_j}(R_{P_j}(\omega_{r+j,n+j})/\tau_{jj}, \rho N_E^s \otimes \tau) = 0.$$

2. *For a cuspidal representation $\rho \otimes \sigma$ (j can be equal to 1) we have*

$$\Theta(\rho \otimes \sigma, \tau'_{jj}) \cong \mathrm{Ind}_{P_j}^{G_{r+j}}(\xi\xi'\widetilde{\rho} \otimes \tau)$$

and

$$\Theta(\xi'\xi^{-1}\rho\otimes\tau,\tau_{jj})\cong \mathrm{Ind}_{P'_j}^{G'_{n+j}}(\xi^2\widetilde{\rho}\otimes\sigma).$$

3. *If* $\rho\neq N_E^{\frac{m_r-t_n+1}{2}}\xi$, *then* $\Theta(\rho\otimes\sigma,R_{P'_j}(\omega_{r+j,n+j}))\cong \mathrm{Ind}_{P_j}^{G_{r+j}}(\xi\xi'\widetilde{\rho}\otimes\tau)$, *and if* $\rho\neq N_E^{-\frac{m_r-t_n-1}{2}}\xi$, *then*

$$\Theta(\xi'\xi^{-1}\rho\otimes\tau,R_{P_j}(\omega_{r+j,n+j}))\cong \mathrm{Ind}_{P'_j}^{G'_{n+j}}(\xi^2\widetilde{\rho}\otimes\sigma).$$

PROOF. The exact splittings in Kudla's filtrations of Theorem 3.1 enables us to essentially use the splitting which cuspidal components induce in the category of smooth representations. So, using Lemma 4.1, Lemma 1.1 of [14], and Lemma 4.2, quite analogously to the proof of Proposition 3.4 of [6], we prove all the claims above. ∎

Now, we are able to state our main theorem which covers most of the cases of reducibility.

THEOREM 4.4. *Let* $m_r=\dim_E V_r$, *where* V_r *is a hermitian space, and let* G_r *be the corresponding unitary group. Let* $n_t=\dim_E W_n$, *where* W_n *is a skew–hermitian space, and* G'_n *is a unitary group of that space. Let* σ *be a cuspidal representation of* G'_n *whose first non-zero lift in the hermitian power containing* V_r *is a cuspidal representation* τ *of* G_r. *Let* ρ *be an irreducible cuspidal representation of* $GL_j(E)$. *Then, if* $\rho\notin\{N_E^{\pm\frac{m_r-t_n+1}{2}}\xi, N_E^{\pm\frac{m_r-t_n-1}{2}}\xi\}$, *the representation* $\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho\otimes\sigma)$ *reduces if and only if the representation* $\mathrm{Ind}_{P_j}^{G_{r+j}}(\xi'\xi^{-1}\rho\otimes\tau)$ *reduces. Here* ξ *and* ξ' *are characters of* E^*, *whose restriction to* F^* *are* $\epsilon_{E/F}^{m_r}$ *and* $\epsilon_{E/F}^{t_n}$, *respectively and where* $\epsilon_{E/F}(x)=(x,\Delta)_F$ *is the quadratic character of the extension* E/F. *In the case of irreducibility we have* $\Theta(\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho\otimes\sigma),r+j)=\mathrm{Ind}_{P_j}^{G_{r+j}}(\xi'\xi^{-1}\rho\otimes\tau)$ *(and vice versa). In the case of reducibility, the representation* $\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho\otimes\sigma)$ *has two irreducible subquotients, say,* π_1 *and* π_2, *satisfying*

$$0\to\pi_1\to\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho\otimes\sigma)\to\pi_2\to 0.$$

Then, $\Theta(\pi_i,r+j)\neq 0$, *and the following holds*

$$0\to\Theta(\pi_1,r+j)\to\mathrm{Ind}_{P_j}^{G_{r+j}}(\xi'\xi^{-1}\rho\otimes\tau)\to\Theta(\pi_2,r+j)\to 0.$$

The analogous statement holds for the theta lifts of the irreducible subquotients of $\mathrm{Ind}_{P_j}^{G_{r+j}}(\xi'\xi^{-1}\rho\otimes\tau)$.

PROOF. As soon as we have proved Proposition 4.3, the proof of this theorem is analogous to the proof of Theorem 2.1 in [14]. ∎

5. Exceptional cases

In this section we study the reducibility of the representation $\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho\otimes\sigma)$ if $\rho\in\{N_E^{\pm\frac{m_r-t_n+1}{2}}\xi, N_E^{\pm\frac{m_r-t_n-1}{2}}\xi\}$, and the structure of theta lifts of its irreducible subquotients. It is interesting to note that in the case of unitary groups these exceptional cases we have to cover, in some way, contain the exceptional cases of metaplectic and odd orthogonal group ([6]), but there are some cases which do not appear to be similar. By Theorem 4 on p. 69 of [13] (keeping in mind our splittings), we know that $\Theta(\tau,n+1)$ is irreducible representation of G'_{n+1}, and $\Theta(\sigma,r+1)$ is an irreducible representation of G'_{r+1}.

We use the classical notation for the parabolic induction for general linear and classical groups ([6, 19]): for a representation ρ of $GL_j(E)$ and a representation π of G'_n, we denote the representation $\mathrm{Ind}_{P'_j}^{G'_{n+j}}(\rho\otimes\pi)$ by $\rho\rtimes\pi$; analogously for the induced representations of G_r.

Lemma 5.1. *We have* $R_{P'_1}(\Theta(\tau,n+1))=N_E^{\frac{m_r-t_n-1}{2}}\xi\otimes\sigma$ *and* $R_{P_1}(\Theta(\sigma,r+1))=N_E^{-\frac{m_r-t_n+1}{2}}\xi'\otimes\tau$. *Moreover, the representations* $N_E^{\frac{m_r-t_n-1}{2}}\xi\rtimes\sigma$ *and* $N_E^{-\frac{m_r-t_n+1}{2}}\xi'\rtimes\tau$ *reduce, they are of length two and we have the following exact sequences*

$$0\to\Theta(\tau,n+1)\to N_E^{\frac{m_r-t_n-1}{2}}\xi\rtimes\sigma\to\pi_1\to 0,$$
$$0\to\Theta(\sigma,r+1)\to N_E^{-\frac{m_r-t_n+1}{2}}\xi'\rtimes\tau\to\pi_2\to 0,$$

where π_1 *and* π_2 *are some irreducible representations.*

Proof. There is an epimorphism $T:\omega_{r,n+1}\to\tau\otimes\Theta(\tau,n+1)$, which leads to epimorphism $R_{P'_1}(\omega_{r,n+1})\to\tau\otimes R_{P'_1}(\Theta(\tau,n+1))$. Now, we use the filtration of Corollary 3.5 to see that this filtration of $R_{P'_1}(\omega_{r,n+1})$ has two members, namely the quotient $\tau'_{10}=N_E^{\frac{m_r-t_n-1}{2}}\xi\otimes\omega_{r,n}$ and a subrepresentation $\tau'_{11}\cong\mathrm{Ind}_{P_1\times GL_1\times G'_n}^{GL_r\times GL_1\times G'_n}(\xi\xi'\Sigma'_1\otimes\omega_{r,n-1})$. The assumption that $T|_{\tau'_{11}}\neq 0$, (when the second Frobenius reciprocity is applied) leads to the contradiction with the fact that τ is cuspidal, so we get an epimorphism from τ'_{10} to $\tau\otimes R_{P'_1}(\Theta(\tau,n+1))$. Since we know all the relevant isotypic components of $\omega_{r,n}$ when we want some epimorphism to factor through τ, we get that there is an epimorphism from $N_E^{\frac{m_r-t_n-1}{2}}\xi\otimes\sigma$ to $R_{P'_1}(\Theta(\tau,n+1))$, which proves the first part of the claim. Situation with $R_{P_1}(\Theta(\sigma,r+1))$ is similar. Since the length of the relevant Jacquet module of $N_E^{\frac{m_r-t_n-1}{2}}\xi\rtimes\sigma$ ($N_E^{-\frac{m_r-t_n+1}{2}}\xi'\rtimes\tau$, respectively) is two, we get the claim of the lemma. ∎

Now, we want to have an analogon of Theorem 4.4, i.e.; we want to describe the lifts of the representations $\Theta(\sigma,r+1)$, $\Theta(\tau,n+1)$, π_1 and π_2. Since in the settings of unitary groups, which are connected algebraic reductive groups, the Silbereger's result of the uniqueness of the reducibility point of the parabolic induced representation in the generalized rank one case and inducing data cuspidal, we know that the representations $N_E^{-\frac{m_r-t_n+1}{2}}\xi \rtimes \sigma$ and $N_E^{\frac{m_r-t_n-1}{2}}\xi' \rtimes \tau$ are irreducible if $-\frac{m_r-t_n+1}{2} \neq \frac{m_r-t_n-1}{2}$, i.e., if $m_r \neq t_n$. So, we cover this situation first.

Proposition 5.2. *Assume that $m_r \neq t_n$, so that the representations*

$$N_E^{\frac{m_r-t_n-1}{2}}\xi' \rtimes \tau \text{ and } N_E^{-\frac{m_r-t_n+1}{2}}\xi \rtimes \sigma \cong N_E^{\frac{m_r-t_n+1}{2}}\xi \rtimes \sigma$$

are irreducible. If we additionally assume that $m_r - t_n \neq -1$, then the representation $\Theta(N_E^{-\frac{m_r-t_n+1}{2}}\xi \rtimes \sigma, r+1)$ is non–zero, it has a unique irreducible quotient, namely $\Theta(\sigma, r+1)$. Moreover, we then have $\Theta(\Theta(\sigma, r+1), n+1) = N_E^{-\frac{m_r-t_n+1}{2}}\xi \rtimes \sigma$. Also, we have $\Theta(\pi_2, n+1) = 0$.

Proof. We have

$$\begin{aligned}\mathrm{Hom}_{G'_{n+1}}&(\omega_{r+1,n+1}, N_E^{-\frac{m_r-t_n+1}{2}}\xi \rtimes \sigma)\\ &\cong \mathrm{Hom}_{M'_1}(R_{P'_1}(\omega_{r+1,n+1}), N_E^{-\frac{m_r-t_n+1}{2}}\xi \otimes \sigma).\end{aligned}$$

Observe that the isomorphism of vector spaces above is also an isomorphism of G_{r+1}–modules. Now we apply Proposition 4.3, the third part (to apply it, we need $m_r - t_n \neq -1$), to obtain that the last intertwining space in non-zero; moreover, $\Theta(N_E^{-\frac{m_r-t_n+1}{2}}\xi \otimes \sigma, R_{P'_1}(\omega_{r+1,n+1})) \cong N_E^{\frac{m_r-t_n+1}{2}}\xi' \rtimes \tau$. This means, taking the smooth part of the intertwining spaces above, that $\widetilde{\Theta(N_E^{-\frac{m_r-t_n+1}{2}}\xi \rtimes \sigma, r+1)} \cong \widetilde{N_E^{\frac{m_r-t_n+1}{2}}\xi' \rtimes \tau}$, so that $\Theta(N_E^{-\frac{m_r-t_n+1}{2}}\xi \rtimes \sigma, r+1)$ indeed has a unique irreducible quotient because $N_E^{\frac{m_r-t_n+1}{2}}\xi' \rtimes \tau$ does (and it is $\Theta(\sigma, r+1)$). On the other hand, let λ be an irreducible representation of G_{r+1}. We denote $\pi = N_E^{-\frac{m_r-t_n+1}{2}}\xi \rtimes \sigma$. Then

$$\mathrm{Hom}_{G_{r+1}\times G'_{n+1}}(\omega_{r+1,n+1}, \pi\otimes\lambda) \cong \mathrm{Hom}(R_{P'_1}(\omega_{r+1,n+1}), N_E^{-\frac{m_r-t_n+1}{2}}\xi\otimes\sigma\otimes\lambda),$$

and the last space in non-zero if λ is a quotient of $N_E^{\frac{m_r-t_n+1}{2}}\xi' \rtimes \tau$, i.e., if $\lambda = \Theta(\sigma, r+1)$ (again we used $m_r - t_n \neq -1$). This means that π is a quotient of $\Theta(\Theta(\sigma, r+1), n+1)$. On the other hand, there is an epimorphism $\omega_{r+1,n+1} \to \Theta(\sigma, r+1) \otimes \Theta(\Theta(\sigma, r+1), n+1)$, leading to the epimorphism

$$R_{P_1}(\omega_{r+1,n+1}) \to N_E^{-\frac{m_r-t_n+1}{2}}\xi' \otimes \tau \otimes \Theta(\Theta(\sigma, r+1), n+1),$$

and again, by Proposition 4.3 we see there is an epimorphism $\xi N_E^{\frac{m_r-t_n+1}{2}} \rtimes \sigma \to \Theta(\Theta(\sigma, r+1), n+1)$, and we conclude that $\Theta(\Theta(\sigma, r+1), n+1) = \xi N_E^{\frac{m_r-t_n+1}{2}} \rtimes \sigma$. Now we show that $\Theta(\pi_2, n+1) = 0$. If we assume that it is non-zero, we could use the analysis of its cuspidal support ([13, p. 69]) to see that $\Theta(\pi_2, n+1) = 0$, but there the arguments are in terms of covering groups, so to be clear, we prove our claim bearing in mind the splittings. First, we prove that $\Theta(\pi_2, n) = 0$. Assume that $\Theta(\pi_2, n) \neq 0$; then we have an (non-zero) epimorphism $\omega_{r+1,n} \to \pi_2 \otimes \Theta(\pi_2, n)$. We use the filtration of $R_{P_1}(\omega_{r+1,n})$ of Proposition 3.6 to conclude that the only option is $\Theta(\pi_2, n) = \sigma$. But this would imply that there is an epimorphism $\Theta(\sigma, r+1) \to \pi_2$, which is impossible, so $\Theta(\pi_2, n) = 0$. If we assume that $\Theta(\pi_2, n+1) \neq 0$, by λ_1 we denote an irreducible quotient of $\Theta(\pi_2, n+1)$, so that there is an epimorphism $T : R_{P_1'}(\omega_{r+1,n+1}) \to \pi_2 \otimes R_{P_1'}(\lambda_1)$. Now we use the filtration of $R_{P_1'}(\omega_{r+1,n+1})$. If $T|_{\tau_{11}'} = 0$, we may take that T is an epimorphism $T : N_E^{\frac{m_r - t_n + 1}{2}} \xi \otimes \omega_{r+1,n} \to \pi_2 \otimes R_{P_1'}(\lambda_1)$, meaning that $\Theta(\pi_2, n) \neq 0$, which is impossible. So, $T|_{\tau_{11}'} \neq 0$. Now, by applying the second Frobenius reciprocity to $T|_{\tau_{11}'}$ we get that there is a non-zero intertwining $N_E^{\frac{m_r - t_n + 1}{2}} \xi \otimes \sigma \to R_{P_1'}(\lambda_1)$, but this means $\lambda_1 = N_E^{\frac{m_r - t_n + 1}{2}} \xi \rtimes \sigma = \pi$. This means that $\mathrm{Hom}_{G_{r+1} \times G_{n+1}'}(\omega_{r+1,n+1}, \pi_2 \otimes \pi) \neq 0$, but this forces (by the argument from the beginning of the proof) $\pi_2 = \Theta(\sigma, r+1)$, which is impossible. $\square$

Totally symmetrically, we get the following proposition.

Proposition 5.3. *Assume that $m_r \neq t_n$ so that the representation $N_E^{\frac{m_r - t_n - 1}{2}} \xi' \rtimes \tau$ is irreducible. Assume further that $m_r - t_n \neq 1$. Then, $\Theta(N_E^{\frac{m_r - t_n - 1}{2}} \xi' \rtimes \tau, n+1)$ has a unique quotient, namely $\Theta(\tau, n+1)$. Moreover, we have $\Theta(\Theta(\tau, n+1), r+1) = N_E^{\frac{m_r - t_n - 1}{2}} \xi' \rtimes \tau$. Also, we have $\Theta(\pi_1, r+1) = 0$.*

Now we analyze the rest of the special cases which appear in the two previous propositions. Assume first that $m_r - t_n = -1$. Note that then the assumptions of Proposition 5.3 are meet, so we know the lifts of $\Theta(\tau, n+1)$ and π_1 (at the $r+1$-th level). We describe the lifts of $\Theta(\sigma, r+1)$ and π_2 in this situation.

Proposition 5.4. *We keep the notation from the beginning of the section. Assume that $m_r - t_n = -1$. Then, the following holds*

1. $\Theta(\Theta(\sigma, r+1), n+2) \neq 0$, $\Theta(\pi_2, n+2) \neq 0$. *Moreover, we have:*
2. $\Theta(\Theta(\sigma, r+1), n+1) = \xi \rtimes \sigma$ *and* $\Theta(\pi_2, n+1) = 0$.
3. *Every irreducible quotient of $\Theta(\pi_2, n+2)$ equals $L(N_E^{\frac{1}{2}} \xi St_{GL(2,E)}; \sigma)$, where $L(N_E^{\frac{1}{2}} \xi St_{GL(2,E)}; \sigma)$ denotes the Langlands' quotient of a standard representation $N_E^{\frac{1}{2}} \xi St_{GL(2,E)} \rtimes \sigma$.*

Proof. To prove claim 1. we use the idea of descending in skew–hermitian tower, starting from some level n' where we are sure that $\Theta(\Theta(\sigma, r+$

$1), n') \neq 0$, for example, by the stable range condition. We start by proving the following claim:

Assume that $m_r - t_{n'} + 1 \neq 0$. Then, we have

$$\text{(5.1)}\ \Theta(\Theta(\sigma, r+1), n') \neq 0 \Longleftrightarrow R_{P_1'}(\Theta(\Theta(\sigma, r+1), n'+1))(N_E^{\frac{m_r - t_{n'}+1}{2}}\xi) \neq 0.$$

Here, for a smooth, finite length representation π (of some group) and character χ of the center of that group, $\pi(\chi)$ denotes the summand of π corresponding to the generalized central character χ. We now prove this claim. Assume first that $\Theta(\Theta(\sigma, r+1), n') \neq 0$. So, there is an $G_{r+1} \times G'_{n'}$ epimorphism $\omega_{r+1,n'} \rightarrow \Theta(\sigma, r+1) \otimes \Theta(\Theta(\sigma, r+1), n')$. On the other hand, using Kudla's filtration of Corollary 3.5, we know that there is $G_{r+1} \times GL_1 \times G_{n'}$ epimorphism $R_{P_1'}(\omega_{r+1,n'+1}) \rightarrow N_E^{\frac{m_r - t_{n'}+1}{2}}\xi \otimes \omega_{r+1,n'}$, so also $G_{r+1} \times GL_1 \times G_{n'}$ epimorphism

$$R_{P_1'}(\omega_{r+1,n'+1}) \rightarrow N_E^{\frac{m_r - t_{n'}+1}{2}}\xi \otimes \Theta(\sigma, r+1) \otimes \Theta(\Theta(\sigma, r+1), n').$$

Using Frobenius isomorphism , we get that there is a non-trivial intertwining

$$\Theta(\Theta(\sigma, r+1), n'+1) \rightarrow N_E^{\frac{m_r - t_{n'}+1}{2}}\xi \rtimes \Theta(\Theta(\sigma, r+1), n'),$$

so again, using Frobenius isomorphism, we get that $R_{P_1'}(\Theta(\Theta(\sigma, r+1), n' + 1))(N_E^{\frac{m_r - t_{n'}+1}{2}}\xi) \neq 0$. We proved one direction of the claim (5.1). On the other hand, if we assume that the right -hand side of (5.1) holds, it especially means that $\Theta(\Theta(\sigma, r+1), n'+1) \neq 0$, and there exists a finite length representation τ_1 of $G'_{n'}$ such that there is an epimorphism

$$T : R_{P_1'}(\omega_{r+1,n'+1}) \rightarrow \Theta(\sigma, r+1) \otimes N_E^{\frac{m_r - t_{n'}+1}{2}}\xi \otimes \tau_1.$$

Now, again using Kudla's filtration of $R_{P_1'}(\omega_{r+1,n'+1})$ we have: assume that $T|_{\tau'_{11}} \neq 0$. Now we use the second Frobenius reciprocity to see that if $m_r - t_{n'} + 1 \neq 0$, we get a contradiction. So, we have to have $T|_{\tau'_{11}} = 0$, which gives us an epimorphism $N_E^{\frac{m_r - t_{n'}+1}{2}} \otimes \omega_{r+1,n'} \rightarrow \Theta(\sigma, r+1) \otimes N_E^{\frac{m_r - t_{n'}+1}{2}}\xi \otimes \tau_1$, so that $\Theta(\Theta(\sigma, r+1), n') \neq 0$. We proved claim (5.1). Now, we prove that $\Theta(\Theta(\sigma, r+1), n+2) \neq 0$. If n' is such that $\Theta(\Theta(\sigma, r+1), n'+1) \neq 0$, such that $m_r - t_{n'} + 1 \neq 0$ (we take $n' \geq n+1$) we have an epimorpism $\omega_{r+1,n'+1} \rightarrow \Theta(\sigma, r+1) \otimes \Pi$, for some irreducible representation Π of $G'_{n'+1}$. This gives us a non-trivial intertwining belonging to

$$\begin{gathered}\mathrm{Hom}_{G_{r+1} \times G'_{n'+1}}(\omega_{r+1,n'+1}, \xi' \rtimes \tau \otimes \Pi) \cong \\ \mathrm{Hom}_{GL_1 \times G_r \times G_{n'+1}}(R_{P_1}(\omega_{r+1,n'+1}), \xi' \otimes \tau \otimes \Pi).\end{gathered}$$

By examining the filtration of $R_{P_1}(\omega_{r+1,n'+1})$, we see that for a nontrivial intertwining T belonging to the second intertwining space above, $T|_{\tau_{11}} \neq 0$, since $m_r - t_{n'} - 1 \neq 0$ (because $n' \geq n+1$). By examining $T|_{\tau_{11}}$, we get a nontrivial $G'_{n'+1}$ intertwining $\xi \rtimes \Theta(\tau, n') \to \Pi$. But, an easy argument gives us (because τ is cuspidal) $\Theta(\tau, n') \hookrightarrow \xi N_E^{\frac{m_r - t_{n'}+1}{2}} \rtimes \Theta(\tau, n'-1)$. We have

$$\Pi \hookrightarrow \xi \times \xi N_E^{\frac{m_r - t_{n'}+1}{2}} \rtimes \Theta(\tau, n'-1).$$

If $\frac{m_r - t_{n'}+1}{2} \notin \{-1, 1\}$ (this is satisfied if $n' \geq n+2$) we have

$$\Pi \hookrightarrow \xi N_E^{\frac{m_r - t_{n'}+1}{2}} \times \xi \rtimes \Theta(\tau, n'-1),$$

and we can conclude that $\Theta(\Theta(\sigma, r+1), n'+1)(\xi N_E^{\frac{m_r - t_{n'}+1}{2}}) \neq 0$ is fulfilled if $n' \geq n+2$, so, by claim (5.1) we have $\Theta(\Theta(\sigma, r+1), n') \neq 0$, and $\Theta(\Theta(\sigma, r+1), n+2) \neq 0$. The proof that $\Theta(\pi_2, n+2) \neq 0$ is totally analogous.

We now prove claim 2. Applying Proposition 4.3 we get

$$\Theta(\xi' \otimes \tau, R_{P_1}(\omega_{r+1,n+1})) = \xi \rtimes \sigma,$$

which is an irreducible representation. By Frobenius reciprocity, we have

$$\mathrm{Hom}_{G_{r+1} \times G'_{n+1}}(\omega_{r+1,n+1}, \xi' \rtimes \tau \otimes \xi \rtimes \sigma) \cong$$
$$\mathrm{Hom}_{P_1 \rtimes G'_{n+1}}(R_{P_1}(\omega_{r+1,n+1}), \xi' \otimes \tau \otimes \xi \rtimes \sigma) \cong \mathrm{Hom}_{G'_{n+1}}(\xi \rtimes \sigma, \xi \rtimes \sigma),$$

so that the dimension of the first intertwining space is one. On the other hand, the first intertwining space is isomorphic to

$$\mathrm{Hom}_{P'_1 \times G_{r+1}}(R_{P'_1}(\omega_{r+1,n+1}), \xi' \rtimes \tau \otimes \xi \otimes \sigma).$$

If we use the filtration of $R_{P'_1}(\omega_{r+1,n+1})$, we see that there is already non-zero intertwining from $\tau'_{10} \cong \xi \otimes \omega_{r+1,n} \to \xi' \rtimes \tau \otimes \xi \otimes \sigma$, where the image of this intertwining is precisely $\Theta(\sigma, r+1) \otimes \xi \otimes \sigma$. So, every (non-zero) intertwining operator from $\mathrm{Hom}_{P'_1 \times G_{r+1}}(R_{P'_1}(\omega_{r+1,n+1}), \xi' \rtimes \tau \otimes \xi \otimes \sigma)$ has image equal to $\Theta(\sigma, r+1) \otimes \xi \otimes \sigma$. From this easily follows that the image of a non-zero intertwining operator from $\mathrm{Hom}_{G_{r+1}, G'_{n+1}}(\omega_{r+1,n+1}, \xi' \rtimes \tau \otimes \xi \rtimes \sigma)$ is precisely $\Theta(\sigma, r+1) \otimes \xi \rtimes \sigma$. This guarantees that $\Theta(\Theta(\sigma, r+1), n+1) \neq 0$. We have an epimorphism

$$\omega_{r+1,n+1} \to \Theta(\sigma, r+1) \otimes \Theta(\Theta(\sigma, r+1), n+1),$$

and epimorphism

$$R_{P_1}(\omega_{r+1,n+1}) \to \xi' \otimes \tau \otimes \Theta(\Theta(\sigma, r+1), n+1),$$

meaning, by our previous considerations, that there is an epimorphism from $\xi \rtimes \sigma$ to $\Theta(\Theta(\sigma, r+1), n+1)$, so, actually, $\Theta(\Theta(\sigma, r+1), n+1) = \xi \rtimes \sigma$. If we assume that $\Theta(\pi_2, n+1) \neq 0$, by the same reasoning, we would get that $\Theta(\pi_2, n+1) = \xi \rtimes \sigma$. But, then, two epimorphisms $T_1 : \omega_{r+1,n+1} \to \Theta(\sigma, r+1) \otimes \xi \rtimes \sigma$ and $T_2 : \omega_{r+1,n+1} \to \pi_2 \otimes \xi \rtimes \sigma$ are linearly independent, which contradicts the fact that $\dim_{\mathbb{C}} \mathrm{Hom}_{G_{r+1} \times G'_{n+1}}(\omega_{r+1,n+1}, \xi' \rtimes \tau \otimes \xi \rtimes \sigma) = 1$ (with $\xi' \rtimes \tau = \Theta(\sigma, r+1) \oplus \pi_2$). We conclude $\Theta(\pi_2, n+1) = 0$. Now, let λ be an irreducible quotient of $\Theta(\pi_2, n+2)$. By the epimorphism $\omega_{r+1,n+2} \to$

$\pi_2 \otimes \lambda$, and by passing to the Jacquet module, there is an epimorphism $T : R_{P_1}(\omega_{r+1,n+2}) \to \xi' \otimes \tau \otimes \lambda$. Using filtration of $R_{P_1}(\omega_{r+1,n+2})$, we get that $T|_{\tau_{11}} \neq 0$. Now, the second Frobenius reciprocity applied to $T|_{\tau_{11}}$ gives us a non–trivial intertwining belonging to $\mathrm{Hom}_{G'_{n+2}}(\xi \rtimes \Theta(\tau, n+1), \lambda)$. In the appropriate Grothendieck group we have the following

$$N_E\xi \times \xi \rtimes \sigma = \xi \rtimes \Theta(\tau, n+1) + \xi \rtimes \pi_1 = \xi N_E^{\frac{1}{2}} St_{GL_2(E)} \rtimes \sigma + \xi N_E^{\frac{1}{2}} 1_{GL_2} \rtimes \sigma,$$

where $St_{GL_2(E)}$ denotes the Steinberg representation of $GL_2(E)$. Now, we use the formula for the calculation of the Jacquet modules of the induced representations due to Tadić ([17]), but in the context of unitary groups. Though formula is originally obtained for the odd orthogonal (split) and symplectic groups, it is easily seen that it is also valid for the unitary groups. This is explained in the first section of ([12]), and formula in question is (1.1) there. Note that the only difference in formula there and originally in ([17], Theorem 5.4, Theorem 6.5) is in the definition of M^* (which acts on the (virtual) representations of $GL_n(E)$) where instead of taking contragredient, for the unitary groups we use firstly conjugation (a nontrivial element of $\mathrm{Gal}_{E/F}$) and then taking contragredient (as explained before (1.1) in [12]).

We use this formula to obtain

$$R_{P'_1}(N_E\xi \times \xi \rtimes \sigma) = 2\xi \otimes N_E\xi \rtimes \sigma + N_E\xi \otimes \xi \rtimes \sigma + N_E^{-1}\xi \otimes \xi \rtimes \sigma. \tag{5.2}$$

This means that the length of representation $N_E\xi \times \xi \rtimes \sigma$ is at most six.

From this, we get that

(5.3)
$$R_P(N_E\xi \times \xi \rtimes \sigma) = 2N_E\xi \otimes \xi \otimes \sigma + 2N_E^{-1}\xi \otimes \xi \otimes \sigma + 2\xi \otimes N_E\xi \otimes \sigma + 2\xi \otimes N_E^{-1}\xi \otimes \sigma,$$

where P belongs to the smallest conjugacy class of parabolic subgroups of G'_{n+2} for which the corresponding Jacquet module of $N_E\xi \times \xi \rtimes \sigma$ is non-zero. On the other hand, it is obvious that, since $\xi \rtimes \pi_1$ has only tempered subquotients, $L(\xi N_E^{\frac{1}{2}} St_{GL(2,E)}; \sigma) + L(N_E\xi; \xi \rtimes \sigma) \leq \xi \rtimes \Theta(\tau, n+1)$, so the length of $\xi \rtimes \Theta(\tau, n+1)$ is at least two. Here we use $L(\xi N_E^{\frac{1}{2}} St_{GL(2,E)}; \sigma)$ ($L(N_E\xi; \xi \rtimes \sigma)$, respectively), to denote the Langlands quotients of the standard representation $\xi N_E^{\frac{1}{2}} St_{GL(2,E)} \rtimes \sigma$ ($N_E\xi \times \xi \rtimes \sigma$, respectively). But if we use Aubert duality ([1], Theorem 1.7 (2)) we see that the length of $\xi \rtimes \Theta(\tau, n+1)$ is the same as the length of $\xi \rtimes \pi_1$, i.e., at least two. But, each subquotient of $\xi \rtimes \pi_1$ is a tempered representation, having necessarily $\xi \otimes N_E\xi \otimes \sigma$ in it's appropriate Jacquet module. From (5.3), we see that there are at most two of them, so exactly two of them. This means that the length of representation $N_E\xi \times \xi \rtimes \sigma$ is four, and since $\Theta(\tau, n+1)$ is unitarizable, $\xi \rtimes \Theta(\tau, n+1) = L(\xi N_E^{\frac{1}{2}} St_{GL(2,E)}; \sigma) \oplus L(N_E\xi; \xi \rtimes \sigma)$. To prove that $\lambda = L(\xi N_E^{\frac{1}{2}} St_{GL(2,E)}; \sigma)$ observe that, by (5.2), the only irreducible subquotient π of $N_E\xi \times \xi \rtimes \sigma$ having an irreducible subquotient $\xi N_E^{-1} \otimes \xi \rtimes \sigma$

in its Jacquet module $R_{P_1'}(\pi)$, is exactly $L(N_E\xi;\xi \rtimes \sigma)$. If we assume that $\lambda = L(N_E\xi;\xi \rtimes \sigma)$, we would have $R_{P_1'}(\Theta(\pi_2, n+2))(\xi N_E^{-1}) \neq 0$, and this contradicts the condition (5.1) (expressed for π_2 instead of $\Theta(\sigma, r+1)$, but it still holds) for $n' = n+1$. We conclude $\lambda = L(\xi N_E^{\frac{1}{2}} St_{GL(2,E)};\sigma)$. □

Totally symmetrically, we get

PROPOSITION 5.5. *Assume that* $m_r - t_n = 1$. *Note that then* $\xi' \rtimes \tau$ *is irreducible. Then,* $\Theta(\Theta(\tau, n+1), r+1) = \xi' \rtimes \tau$, *and* $\Theta(\pi_1, r+1) = 0$. *Moreover,* $\Theta(\pi_1, r+2) \neq 0$, *and every irreducible quotient of* $\Theta(\pi_1, r+2)$ *equals* $L(N_E^{\frac{1}{2}}\xi' St_{GL(2,E)};\tau)$.

Now we just have to cover the case $m_r = t_n$.

Observe that in this situation, we have

$$0 \to \Theta(\tau, n+1) \to N_E^{-\frac{1}{2}}\xi \rtimes \sigma \to \pi_1 \to 0,$$

and

$$0 \to \Theta(\sigma, r+1) \to N_E^{-\frac{1}{2}}\xi' \rtimes \tau \to \pi_2 \to 0.$$

PROPOSITION 5.6. *Assume that* $m_r = t_n$. *Then,*

(i) $\Theta(\Theta(\tau, n+1), r+1) = \xi' N_E^{\frac{1}{2}} \rtimes \tau$, $\Theta(\Theta(\sigma, r+1), n+1) = \xi N_E^{\frac{1}{2}} \rtimes \sigma$.

(ii) *We have* $\Theta(\pi_1, r) = \Theta(\pi_2, n) = 0$ *and* $\Theta(\pi_1, r+2) \neq 0$, $\Theta(\pi_2, n+2) \neq 0$. *One of the following two situations occurs:*

-$\Theta(\pi_1, r+1) \neq 0$ *and every irreducible quotient of* $\Theta(\pi_1, r+1)$ *is* π_2, *and vice versa,* $\Theta(\pi_2, n+1) \neq 0$, *and every irreducible quotient of* $\Theta(\pi_2, n+1)$ *is* π_1.

-$\Theta(\pi_1, r+1) = 0 = \Theta(\pi_2, n+1)$. *Then, every irreducible quotient of* $\Theta(\pi_1, r+2)$ *is a unique (tempered) common irreducible subquotient of* $N_E^{\frac{1}{2}}\xi' \rtimes \Theta(\sigma, r+1)$ *and* $\xi' St_{GL(2,E)} \rtimes \tau$. *In the same way, every irreducible quotient of* $\Theta(\pi_2, n+2)$ *is a unique tempered common irreducible subquotient of* $N_E^{\frac{1}{2}}\xi \rtimes \Theta(\tau, n+1)$ *and* $\xi St_{GL(2,E)} \rtimes \sigma$.

REMARK 5.7. We can study lifts of the representations of the skew–hermitian group in "the other" hermitian tower (where the spaces have the same parity of dimension, but different determinant ([11, p. 374] and [8, p. 983])). We know that the Conservation Conjecture ([9, p. 76] and [8, Speculation 7.5]) holds for the supercuspidal representations. So, if r' denotes the level on which σ first appears in the correspondence with the representations of the unitary groups in the second hermitian tower, we have $m_r + m_{r'} = 2t_n + 2$, so that $m_{r'} = t_n + 2$. We denote the corresponding cuspidal representation of $G_{r'}^-$ by τ^- (the sign $^-$ emphasizes that we are in the other hermitian tower). We have the following exact sequence

$$0 \to \Theta(\tau^-, n+1) \to N_E^{\frac{1}{2}}\xi \rtimes \sigma \to \pi_1' \to 0.$$

This means that we can apply Proposition 5.3 to obtain $\Theta^-(\pi_1', r'+1) = 0$, and $\Theta^-(\Theta(\tau^-, n+1), r'+1) = N_E^{\frac{1}{2}}\xi' \rtimes \tau^-$. Here $\Theta^-(\cdot)$ denotes the (full) lift in the other hermitian tower. But we see that $\Theta(\tau^-, n+1) = \pi_1$ and $\pi_1' = \Theta(\tau, n+1)$. Now if we assume that the Conservation Conjecture holds for π_1, and if we denote the dimension of the first level of occurrence of π_1 in the first hermitian tower by $m_r(\pi_1)$ and in the second one by $m_{r'}(\pi_1)$, we have $m_r(\pi_1)+m_{r'}(\pi_1) = 2t_{n+1}+2 = 2t_n+6$, and we know that $m_{r'}(\pi_1) = m_{r'+1} = m_{r'}+2 = t_n+4$, so that $m_r(\pi_1) = t_n+2 = m_{r+1}$ and $\Theta(\pi_1, r+1) \neq 0$, so the first possibility in (ii) of the previous Proposition should occur.

Proof. The proof is totally analogous to the proof of Theorem 4.4 in [6], so we omit it. ∎

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